Perfect Fluids at the Threshold of Black Hole Formation

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Introduction

We investigate spherically-symmetric, general relativistic systems of collapsing perfect fluid distributions with primary interest in behavior observed at the threshold of black hole formation. Unlike other critical phenomena studies that use less realistic sources such as scalar fields and pressure-less dust, our ability to evolve baryonic matter in the hydrodynamic limit allows us to investigate how stellar objects can be driven toward the critical threshold. Particularly, we look at neutron star models that are driven to collapse in hopes of approximating failed supernovae and other sudden accretion events. The first method we employ to drive a star to collapse involves imparting the star with an initially in-going velocity profile, while the second one uses a shell of scalar field that falls onto the star and only interacts with the fluid through its effect on the spacetime. With the so-called velocity-induced initial data, we observe a phase space of dynamical scenarios in which both Type I and Type II critical behavior is observed. We observe and thoroughly study Type II behavior using this mechanism, however, we use the scalar field to examine Type I behavior. Finally, we see how an arbitrary distribution of scalar field reacts when in the neighborhood of a perfect fluid solution that has been tuned near the critical threshold.

Theory and Methods

For our neutron star model, we use spherically-symmetric hydrostatic solutions to Einstein's equations, known as Tolman-Oppenheimer-Volkoff (TOV) solutions. Initially, we assume that the fluid follows an isentropic, polytropic equation of state $P=K\rho_0^{\Gamma}$ while for $t\geqslant 0$ we assume a state equation of the form $P=(\Gamma-1)\rho_0\epsilon$. We use an adiabatic index of $\Gamma=2$ which approximates the stiffness of neutron stellar matter. The time-varying metric we use is the polar-areal metric:

$$ds^2=-\alpha(r,t)^2 dt^2+a(r,t)^2 dr^2+r^2 d\Omega^2$$

The system is evolved by solving the hydrodynamics equations of motion (local conservation of baryons and energy):

$$\partial_t q_k+\partial_r(r^2\alpha f_k/a)=S(q_j, g_{\mu\nu;\lambda})$$

while the geometry is calculated via the Hamiltonian constraint and Slicing condition for a(r,t) and $\alpha(r,t)$, respectively, at every time step. When used, the scalar field equations of motion we employ are the massless Einstein-Klein-Gordon (EMKG) equations, which are evolved by a procedure that calculates the updated scalar field functions and metric functions iteratively after the fluid fields have been updated.

Key features of our High-Resolution Shock-Capturing code are:

- approximate Roe solvers to calculate numerical flux:
  - Roe-type scheme;
  - Marquina flux method;
- Minmod, MC-limiters slope-limiters used;
- a variable mesh concentrates grid points about the origin and minimizes the effect of artificial floor or atmosphere that lies exterior to the star;
- mesh refinement automatically resolves dynamic changes in scale near the origin;

Driven Neutron Star Collapse

This work closely follows previous work accomplished by [Shapiro & Teukolsky 1980], [Gourgoulhon 1992], and [Novak 2001] which all performed numerical experiments in order to explore the threshold of black hole formation for neutron stars far from equilibrium. If there is a universal threshold which involves a neutron star of finite mass, then it provides an approximate lower bound for the mass of a nascent black hole born from failed supernovae and rapid accretion events.

We give the otherwise static and stable neutron star an initial in-going velocity profile, v(r,t=0), that resembles a cubic polynomial in r with overall amplitude v_{min}. By varying v_{min} and the central rest-mass density of the star, ρ_c , we make a phase space of dynamical outcomes shown in **Figure 2**. We categorize the outcomes using the following labels:

- Prompt Collapse (PC): entire star collapses to a black hole;
- Shock-Bounce-Collapse (SBC): some matter is ejected via a shock before black hole forms;
- Shock-Bounce-Dispersal (SBD): shock forms, nearly all matter disperses from origin;
- Shock-Bounce-Oscillation (SBO): shock forms, resultant star is "hot" and oscillates;
- Oscillation (O): no shock, star oscillates about equilibrium solution;

Results:

- $M_*>0.14\,M_{sun}$ to form a black hole;
- Type I behavior when $v_{min}<0.45, M_*>0.8\,M_{sun}$
- Type II behavior when $v_{min}>0.45, M_*<0.8\,M_{sun}$
- arbitrarily small black holes can be formed if v_{min} is abitrarily near Type II threshold;

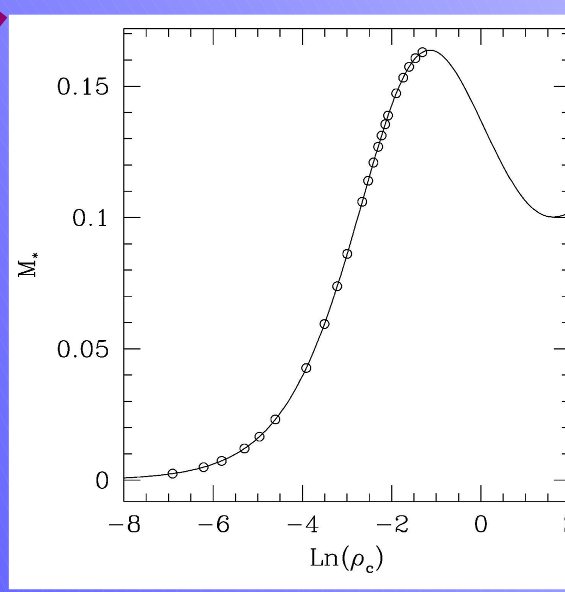


Figure 1: The TOV solutions used in the parameter survey (circles) shown on top of the continuum of TOV solutions (line).

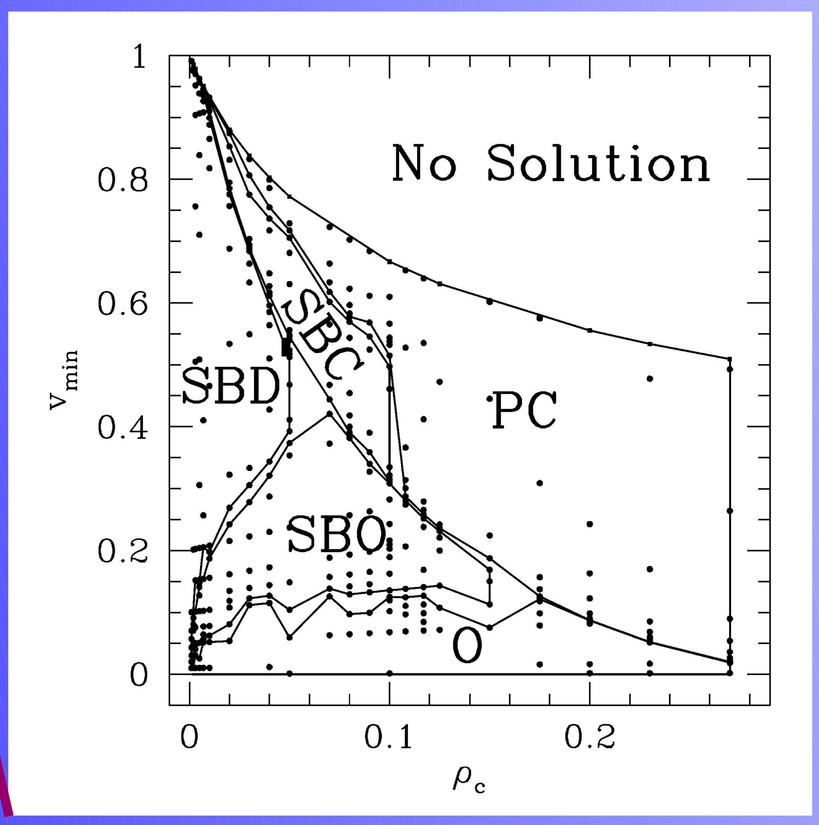


Figure 2: Phase space of outcomes for the velocity-driven neutron star collapse scenarios.

Type II Behavior

Critical phenomena in perfect fluid systems has been extensively studied using in the ultrarelativistic limit where P >> ρ_0. The only studies involving more realistic, ideal-gas fluids were done by [Neilsen and Choptuik 2000] for Γ = 1.4 and by [Novak 2001] for Γ = 2. The initial data sets used by Neilsen and Choptuik consisted of gaussian density distributions whose amplitudes were tuned to search for the critical solution, whereas Novak used TOV stars with added inward kinetic energy. Neilsen and Choptuik found that their ideal-gas scaling behavior matched closely to that for an ultrarelativistic fluid of Γ = 1.4. Novak, however, found scaling behavior different from that of a Γ = 2 ultrarelativistic fluid. Using similar initial data used by Novak, we find that scaling behavior near the critical point is in fact that expected from ultrarelativistic studies (**Figure 3**). The solutions near the threshold also closely resemble those for a Γ = 2 ultrarelativistic fluid.

$$\ln(T_{max})\propto -2\gamma\ln(p^*-p)$$

where $p=v_{min}$, p^*= estimate critical value for v_{min} ,
T_{max} = global maximum of the trace of the stress-energy tensor;

Results:

- can form arbitrarily small black holes;
- near-critical solutions tend toward a continuously self-similar (CSS) solution (**Figure 4**);
- ideal-gas near-critical solution tend to ultrarelativistic near-critical solution;
- ideal-gas critical scaling law closely matches ultrarelativistic critical scaling law;
- scaling behavior independent of initial data prescription, such as the initial TOV solution and functional form of the velocity profile;

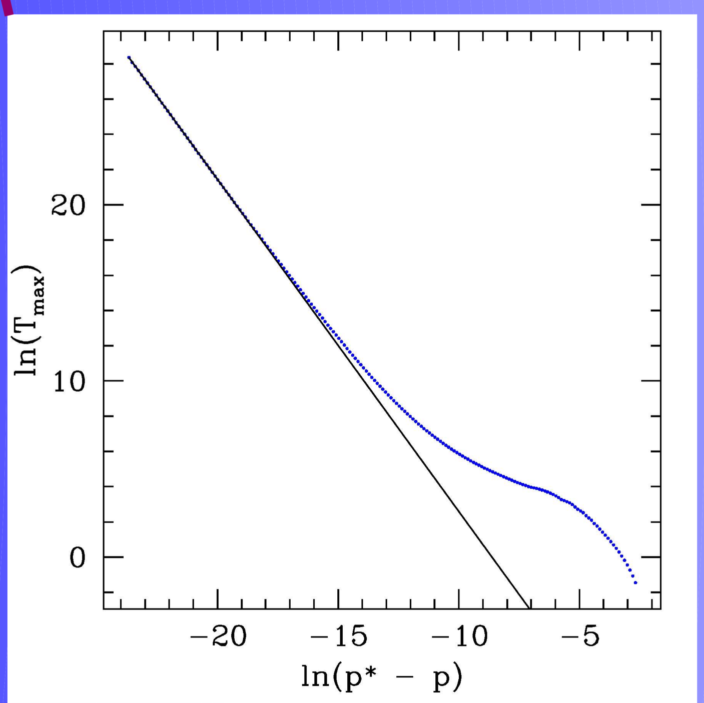


Figure 3: CSS scaling behavior for fluid of an ideal-gas EOS (blue) compared to the best linear fit for the most nearly critical points (black).. As one can see, the fluid asymptotes to the expected scaling law as the fluid becomes more "ultrarelativistic".

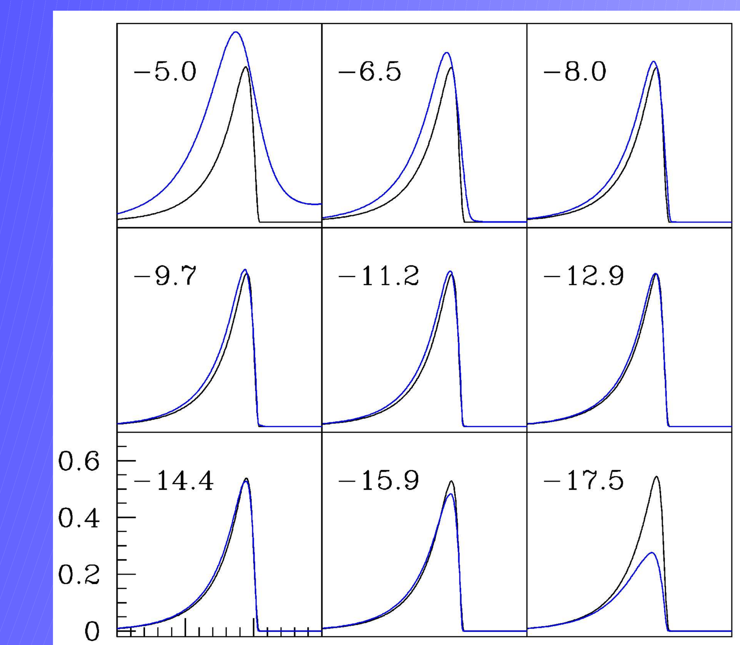


Figure 4: The quantity $4\pi r^2 a^2 \rho$ is plotted for the most nearly critical solutions for the ideal-gas fluid (blue) and the ultrarelativistic fluid (black). As the ideal-gas solution evolves, P grows and gradually diminishes the dynamical significance of ρ_0 thereby making the system effectively scale-free. In this limit, the ideal-gas EOM asymptote to the ultrarelativistic equations.

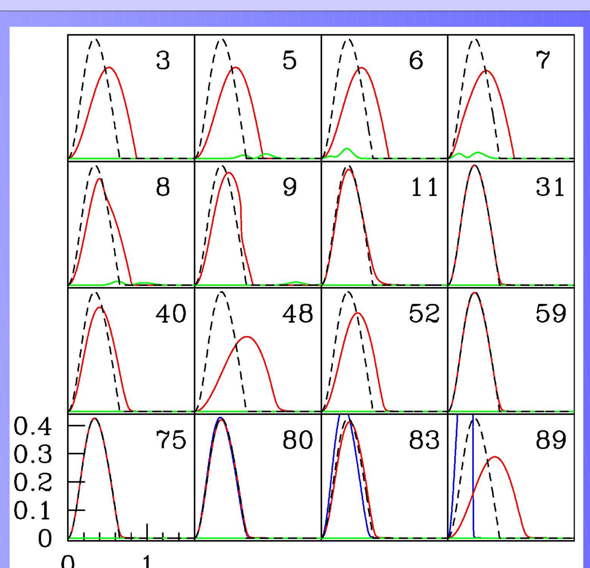
Type I Behavior

In order to study the Type I behavior of TOV stars, we use a minimally-coupled scalar field to act as the perturbation mechanism. If the scalar field pulse is not strong enough---e.g. near or below the threshold of black hole formation---then it merely passes through the star and disperses to infinity. For near-critical configurations, the gravitational interaction between the two sources drives the star to oscillate about an unstable, static TOV solution of mass comparable to that of the original star (**Figure 5**). Since the central density uniquely parameterizes the TOV solutions, we have found that the initial central densities of stars are one-to-one with the central densities of their respective nearly-static critical states (**Figure 6**). Also, as the initial data is tuned closer to the critical point for a given star, the longer the star emulates its associated unstable TOV star. If $\mathcal{T}(\rho_c, p)$ is the proper time interval during which a star of initial central density ρ_c emulates the unstable solution after being perturbed by a scalar field of initial amplitude p, then we find that

$$\mathcal{T}(\rho_c, p)\propto -[1/\omega(\rho_c)]\ln|p-p^*|$$

where $\omega(\rho_c)$ is the Lyapunov exponent of the unstable, critical solution for a star of initial central density ρ_c (**Figure 7**). It remains to be seen if the Lyapunov exponents calculated numerical here from our nonlinear evolutions correspond to the exponents of the unstable TOV solutions calculated using linear perturbation theory.

Figure 5: (left, below) Shown here are two sets of near-critical evolutions of a ρ_c=0.197 star perturbed by different scalar field pulses. dm_{fluid}/dr (dm_{scalar}/dr) is shown versus r at various times in red (green) for the marginally subcritical case and blue (cyan) for the supercritical ensemble. The two sets differ significantly only at late times. This particular star departs shortly from its unstable configuration at t=31, then returns at t=59, and departs for good at t=80.



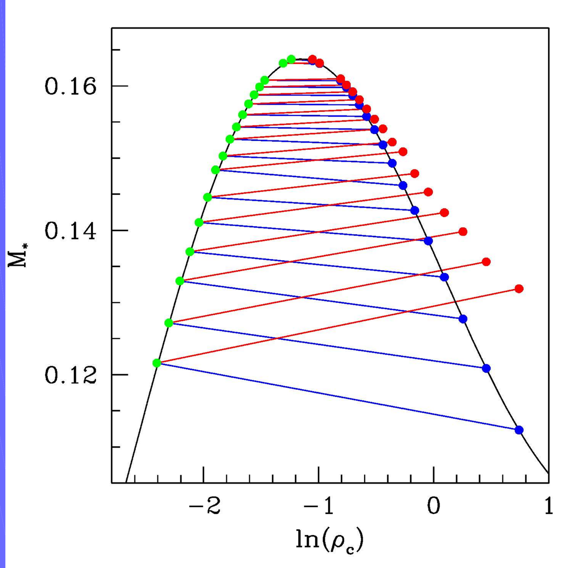


Figure 6: Masses of the initial stars (green), their respective time-averaged critical solutions (red), and the TOV solutions with ρ_c equal to the time-averaged $\rho_0(r=0,t)$ of their respective critical solutions.

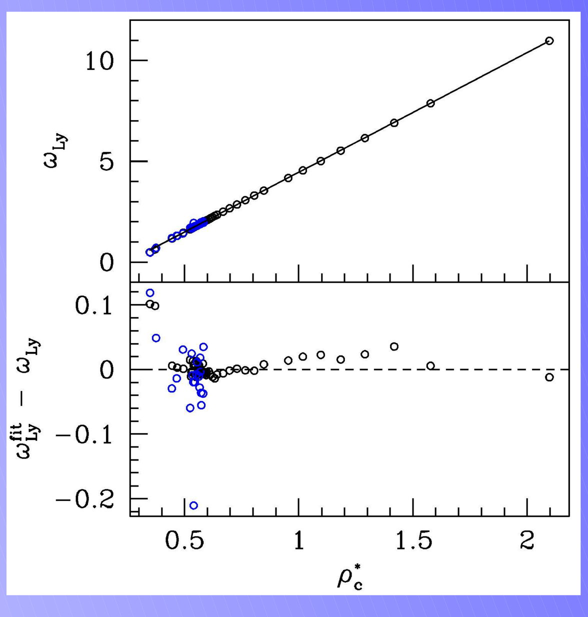


Figure 7: The top portion shows the Lyapunov exponents of critical solutions generated from several initial, stable TOV solutions, which are parameterized by ρ_c . On bottom, we show the differences between the exponents and points along a fitted line we made to the distribution.

EMKG Amplification by Fluid Critical Sol.

Starting from two gaussian profiles of ultrarelativistic fluid and scalar field, we tune the amplitude of the fluid distribution, τ_0 , about the threshold of black hole formation. If the scalar field pulse is initially near the bulk of the fluid, we find that the scalar field grows exponentially with respect to the self-similar time-coordinate T. Since these are numerical experiments, we can only tune τ_0 to within machine-precision of the critical value $\tau_0{}^*$. Hence, our near-critical solutions can only exist in the critical regime for a finite period ΔT. For smaller values of the scalar field amplitude, Φ_0, we find that the near-critical fluid solution behaves in its expected, CSS, way. As we increase Φ_0, we find that the global maximum the scalar field achieves---for solutions closest to the threshold---increases. Above a certain value of Φ_0, we find that the scalar field is amplified sufficiently so as to be the dynamically-dominant source. For these cases, we find that the fluid still dominates at early times, making the near-critical evolution during this period appear CSS. At later times, however, the scalar-field-dominated evolution resembles discretely self-similar (DSS) behavior, which is characteristic of the EMKG critical solution (**Figure 9**). By measuring the scaling behavior of T_{max} (**Figure 8**), we find that---in fact---the scaling behavior during the period dominated by the fluid (scalar field) is the same as one would expect without the scalar field (fluid).

This work yields analogous results to those found by Choptuik [Choptuik 2004] involving EMKG and Yang-Mills fields.

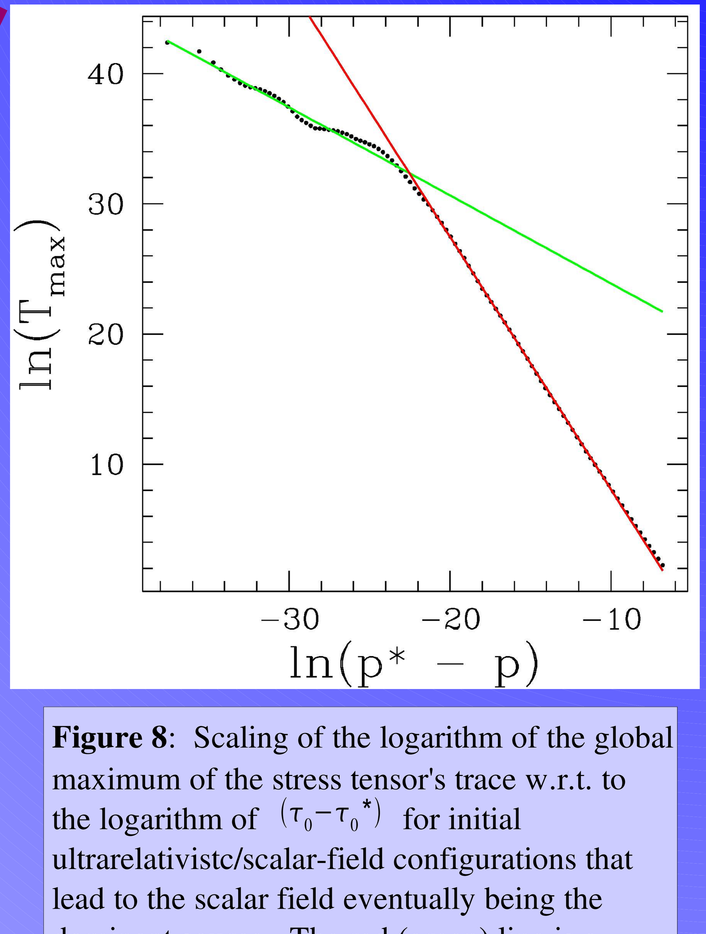


Figure 8: Scaling of the logarithm of the global maximum of the stress tensor's trace w.r.t. to the logarithm of $(\tau_0-\tau_0{}^*)$ for initial ultrarelativistc/scalar-field configurations that lead to the scalar field eventually being the dominant source. The red (green) line is a linear fit to that data for which the fluid (scalar field) is dominant.

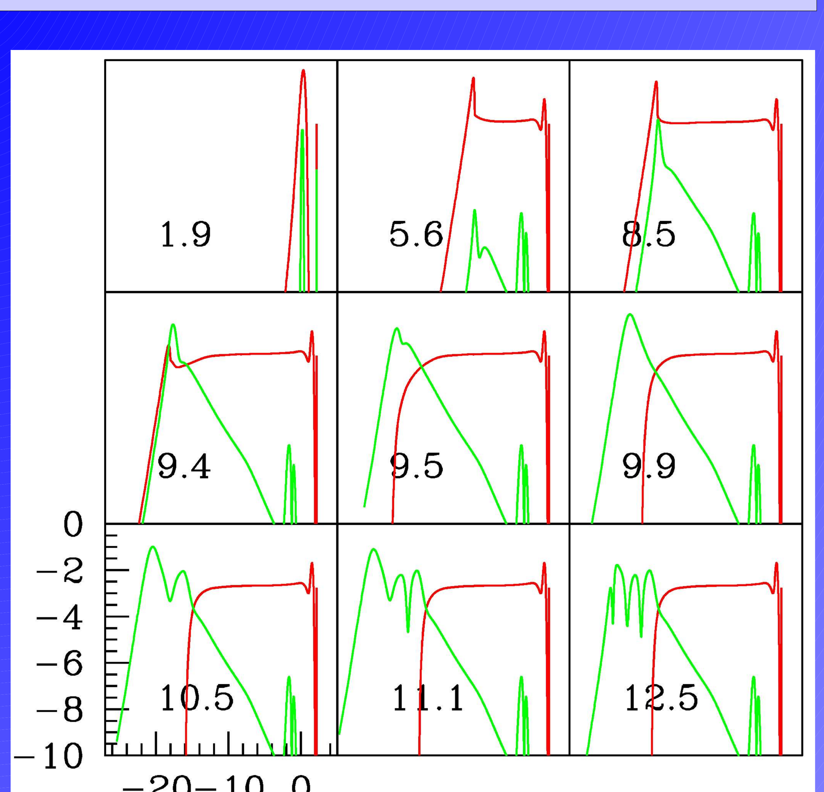


Figure 9: Snapshots of dm_{fluid}/dr (red) and dm_{scalar}/dr (green) versus Ln(r) at various times. The times shown in the lower-left corners of the frames are the frames' values of -Ln($\mathcal{T}^*$-$\mathcal{T}$), where $\mathcal{T}$ is the central proper time of a snapshot and $\mathcal{T}^*$ is the estimated accumulation time of the critical solution. CSS (DSS) behavior is clearly evident when the fluid (scalar field) is dominant.

Conclusions

By driving neutron stars toward black hole formation, we have uncovered a rich space of dynamical scenarios which tells where in parameter space the different types of critical behavior lie. In addition, we have presented evidence to support the universality conjecture of critical phenomena for perfect fluid systems. Using a scalar field instead of a ingoing velocity profile, we were able to elucidate the expected Type I behavior of perfect fluids with an intrinsic length scale. Finally, we have presented additional evidence of EMKG-field amplification in the presence of a critical solution.

Acknowledgements

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