Center for Theoretical Astrophysics
University of Illinois at Urbana-Champaign

Improved Primitive Variable Calculation for Conservative GRMHD Codes

Scott C. Noble[1], Charles F. Gammie[1], Jonathan C. McKinney[3], Luca Del Zanna[4]

1 Physics Department, University of Illinois, Urbana, IL 61801, U.S.A. ,
3 Harvard-Smithsonian Center for Astrophysics, Cambridge, MA 02138, U.S.A.
4 Dipartimento di Astronomia e Scienza dello Spazio Universita degli Studi di Firenze, Firenze, Italy

Abstract

Conservative numerical methods, which take advantage of the conservation-law nature of the equations of motion to be solved, have been extraordinarily successful in evolving Newtonian and relativistic hydrodynamic systems. Such methods improve how well discontinuities are resolved while giving high-order accuracy in smooth regions. Recently, such techniques have been used to study magnetohydrodynamic flows in relativistic scenarios as well, specifically for accretion disk and jet evolutions. A major component of relativistic conservative schemes involves calculating the primitive variables (e.g. the pressure, rest-mass density and flow velocity) from the conserved variables (e.g. components of the stress-energy tensor). The problem ultimately amounts to solving five nonlinear algebraic equations for five unknown primitive variables. If implemented naively, a method that performs this calculation may be responsible for a large portion of a code's runtime and numerical error. Also, its robustness typically decides what region of phase space one can study (e.g. it may only be able to find solutions for flows with small Lorentz factors). Hence, it is critical that the primitive variables are calculated efficiently, robustly and accurately. We present six methods for finding the primitive variables, and contrast their performance with a number of benchmarks. Three new methods developed in this work are found to improve upon previous schemes in many aspects. These improved methods will allow us to more accurately and more efficiently evolve accretion disk models in the future with HARM.

Introduction

The dynamic equations governing ideal MHD systems in GR takes the general form:

$$\partial_t\left[\sqrt{\gamma}\,\boldsymbol{U}(\boldsymbol{P})\right]=-\partial_i\left[\sqrt{-g}\,\boldsymbol{F}^i(\boldsymbol{P})\right]+\boldsymbol{S}(\boldsymbol{P})$$

$$\boldsymbol{U}=\begin{bmatrix} D=-n_\mu u^\mu \rho \\ Q_\nu=-n_\mu T^\mu{}_\nu \\ B^i=-n_\mu{}^* F^{i\mu}\end{bmatrix} \qquad \boldsymbol{P}=\begin{bmatrix}\rho\\ u\\ u^i\\ B^i\end{bmatrix} \qquad (1)$$

Conserved Variables **Primitive Variables**

$\boldsymbol{F}^i$ = fluxes $\boldsymbol{S}$ = source functions

$$T_{\mu\nu}=(p+\rho+u+b^2)u_\mu u_\nu+(p+b^2/2)g_{\mu\nu}-b_\mu b_\nu$$

n^μ = timelike vector normal to spacelike hypersurfaces

The explicit dependence of **U** on **P** is given above. Obviously, the last three equations are trivial mappings, while the first 5 equations represent the real transformation. The inverse, **P(U)**, of these first 5 equations is not known and must be done numerically. Even though several different methods have been mentioned in the literature **[1-4]**, no quantitative comparisons have been made. We perform such comparisons between methods similar to the existing ones, and introduce new, improved schemes in **[5]**. We also summarize our findings in this poster.

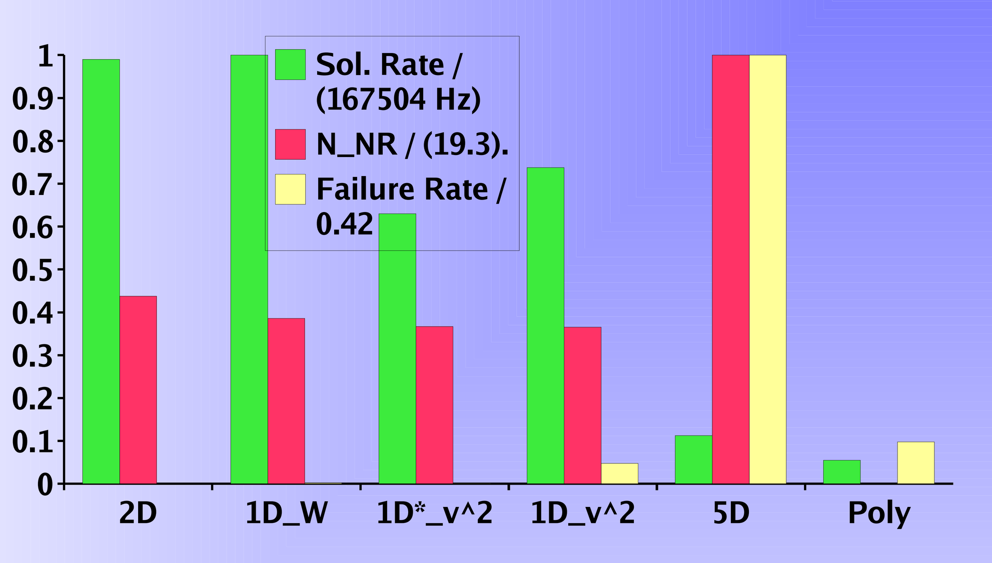


Figure 3: The solution rate (green), average number of NR iterations (red), failure rate (yellow) of each method for the parameter space survey.

Methods and Formulations

The numerical methods we have tested are described below. The majority of them entail an n-dimensional Newton-Raphson (NR) method used to find the root of a given set of algebraic residual functions. The residuals indicate when a given **P** yields a satisfactory numerical solution. **Table 1** describes the 6 methods. In many of the methods, we have used the following scalar quantities, derived from projections of conserved variables:

$$\tilde{Q}^\nu=\left(\delta^\nu{}_\mu+n^\nu n_\mu\right)Q^\mu \qquad v^\lambda=\frac{\tilde{u}^\lambda}{-n_\mu u^\mu}$$

$$W=p+\rho+u+b^2$$

$$\tilde{Q}^2=v^2\left(B^2+W\right)^2-\frac{\left(Q_\mu B^\mu\right)^2\left(B^2+2W\right)}{W^2} \qquad (2)$$

$$v^2=\frac{\tilde{Q}^2 W^2+\left(Q_\mu B^\mu\right)^2\left(B^2+2W\right)}{\left(B^2+W\right)^2 W^2} \qquad (3)$$

$$Q_\mu n^\mu=-\frac{B^2}{2}\left(1+v^2\right)+\frac{\left(Q_\mu B^\mu\right)^2}{2W^2}-W+p\left(W,v^2\right) \qquad (4)$$

Test 1: Parameter Space Survey

Using known conserved/primitive variable pairs, we test each method's accuracy exactly. The survey is taken with the following variables and ranges, which reflect the dynamical range often seen in our accretion disk simulations:

$$\log(\rho)\in[-7,1] \qquad \log(u)\in[-10,0] \qquad \log(\tilde{u}^2+1)\in[0.002,2.9]$$

$$\log(B^2)\in[-8,1] \qquad \cos\left(\frac{\tilde{u}_i B^i}{\sqrt{(\tilde{u}^2 B^2)}}\right)\in[-1,1]$$

For each parameter space point, ***P*** is specified and ***U***(***P***) is calculated. ***P*** is then offset by a random factor, used as a seed for each method, and a a resultant ***P*** is found per method. The accuracy of this result can then be compared to its known value. **Figure 1** shows each method's average accuracy in calculating *u* over the parameter space, while **Fig. 2** displays measures of each method's efficiency and robustness. One sees that the *2D*, $1D_W$ and $1D^*_{v^2}$ methods all excel in accuracy, rate of convergence and rate of success. Also, *2D* and $1D_W$ both are the fastest methods by far.

Figure 2: Accuracy of each method in calculating *u* on average over the entire parameter space modulo *u*. Points where any method failed were omitted from the averages.

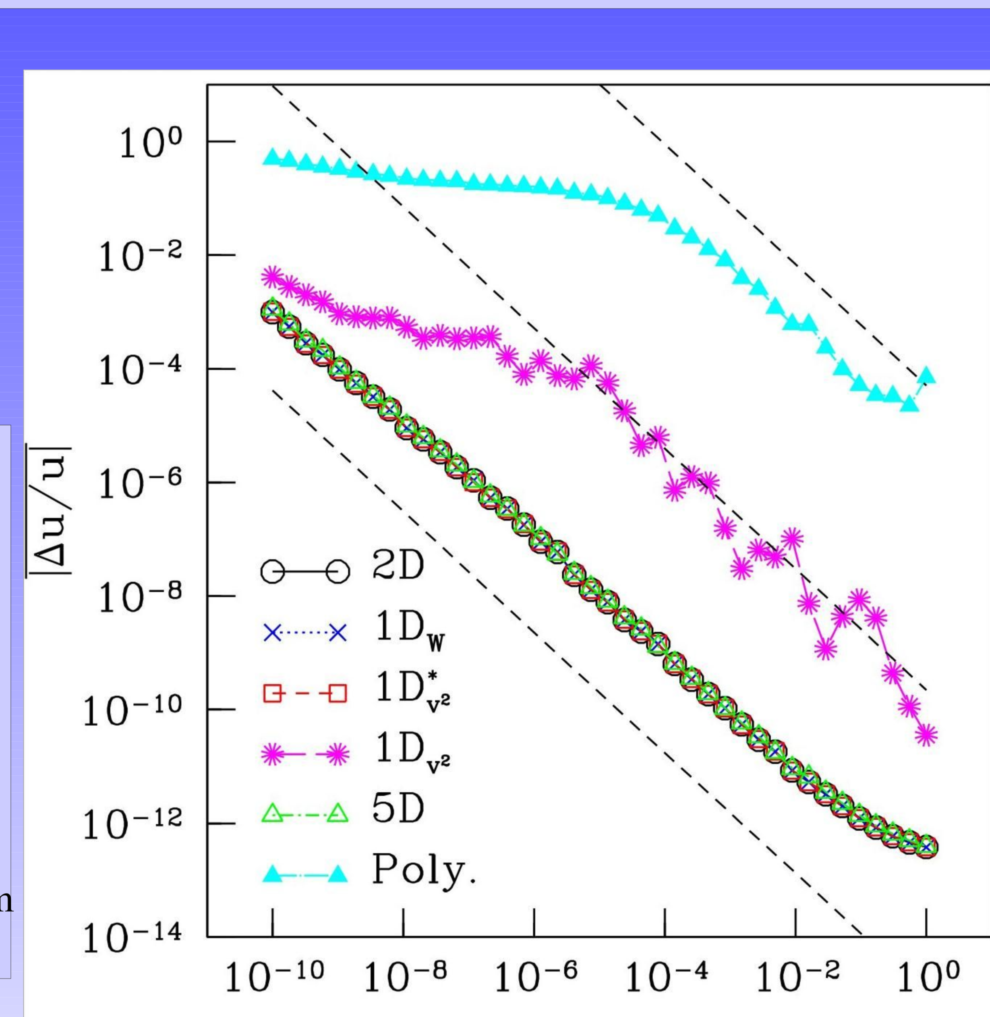


Table 1: Description of the 6 different methods for computing ***P(U)***.

Name	Var.'s	Methods and Equations	References
$2D$	v^2, W	Solve (2) and (4) simultaneously with 2D Newton-Raphson	[5,3]
$1D_W$	W	Substitute (3) for v^2 in (4) and solve for W with 1D Newton-Raphson	[5]
$1D_{v^2}$	v^2	Solve cubic (2) explicitly for W per Newton-Raphson iteration of (4) for v^2	[1,5]
$1D^*_{v^2}$	v^2	Use Newton-Raphson of (2) for W per Newton-Raphson iteration of (4) for v^2	[5]
$5D$	$\boldsymbol{P}$	Solve original equations (1) directly using 5D Newton-Raphson	[2,5]
Poly	W	Substituting (3) into (4) with Γ-law state equation $\rightarrow O(W^8)$ poly., solve with Laguerre's method	[5]

Test 2: Accretion Disk Evolution

Our final test sets the best inversion methods against each other in a practical simulation: an accretion disk around a rotating black hole. We use Kerr Schild coordinates, with spin parameter *a=0.9375M*. Initially, the disk takes the shape of a Fishbone-Moncrief torus, which would be at equilibrium if it were not for the magnetic field. Its inner radius is at *r=6M*, pressure maximum at *r=12M*, and outer radius at *r=42M* (all along the equator) at *t=0*. The magnetic field is purely poloidal at the beginning, and is normalized so that the minimum ratio of gas pressure to magnetic pressure is 100. All runs used a resolution of 256x256 points. **Fig. 6** shows snapshots of the logarithm of the rest-mass density when the disk is in a steady-state; each snapshot is from a different run using a different inversion method. **Fig. 7** shows the accretion rates from these runs, while **Fig. 8** illustrates the efficiency and robustness of each method.

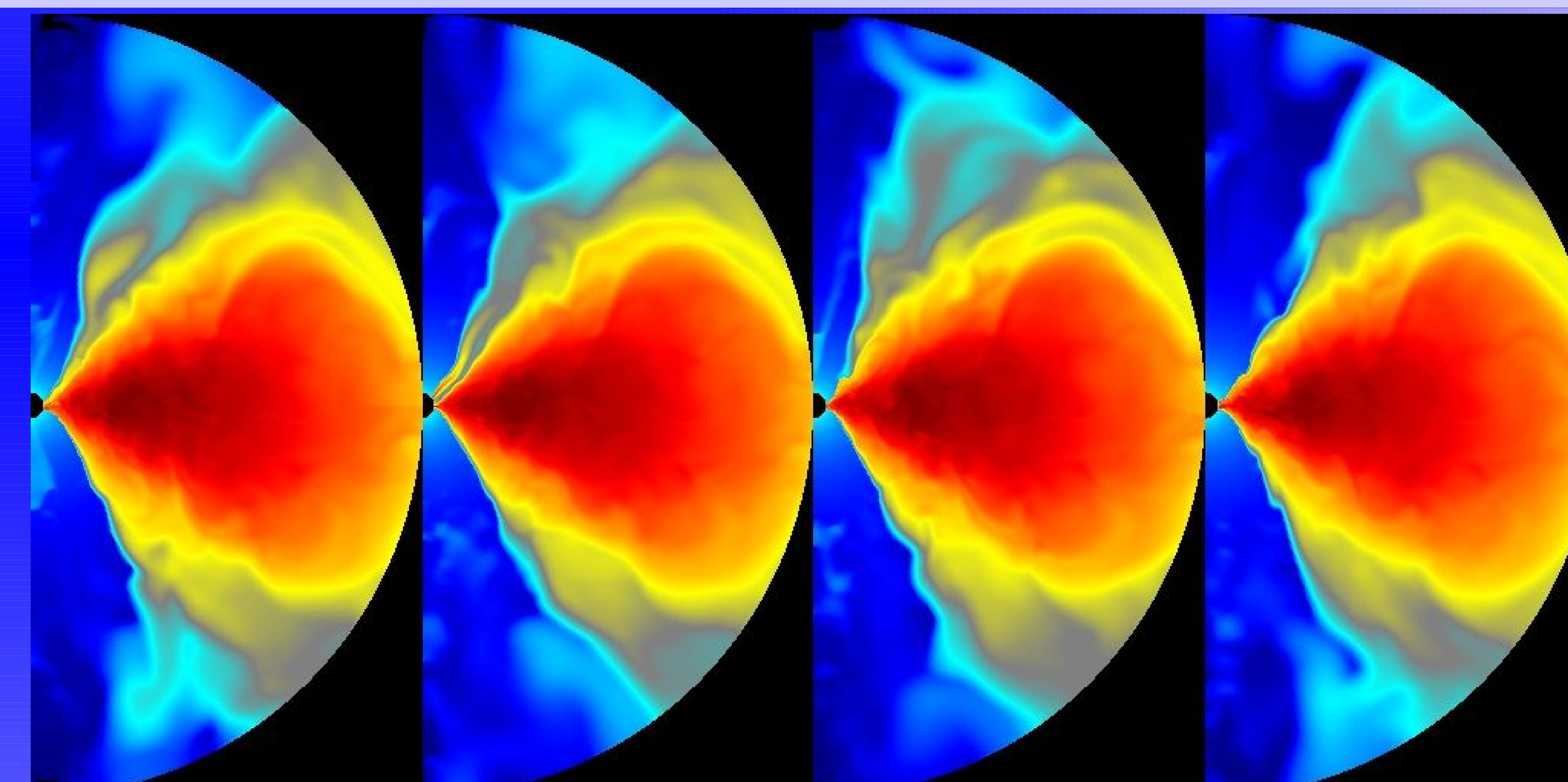
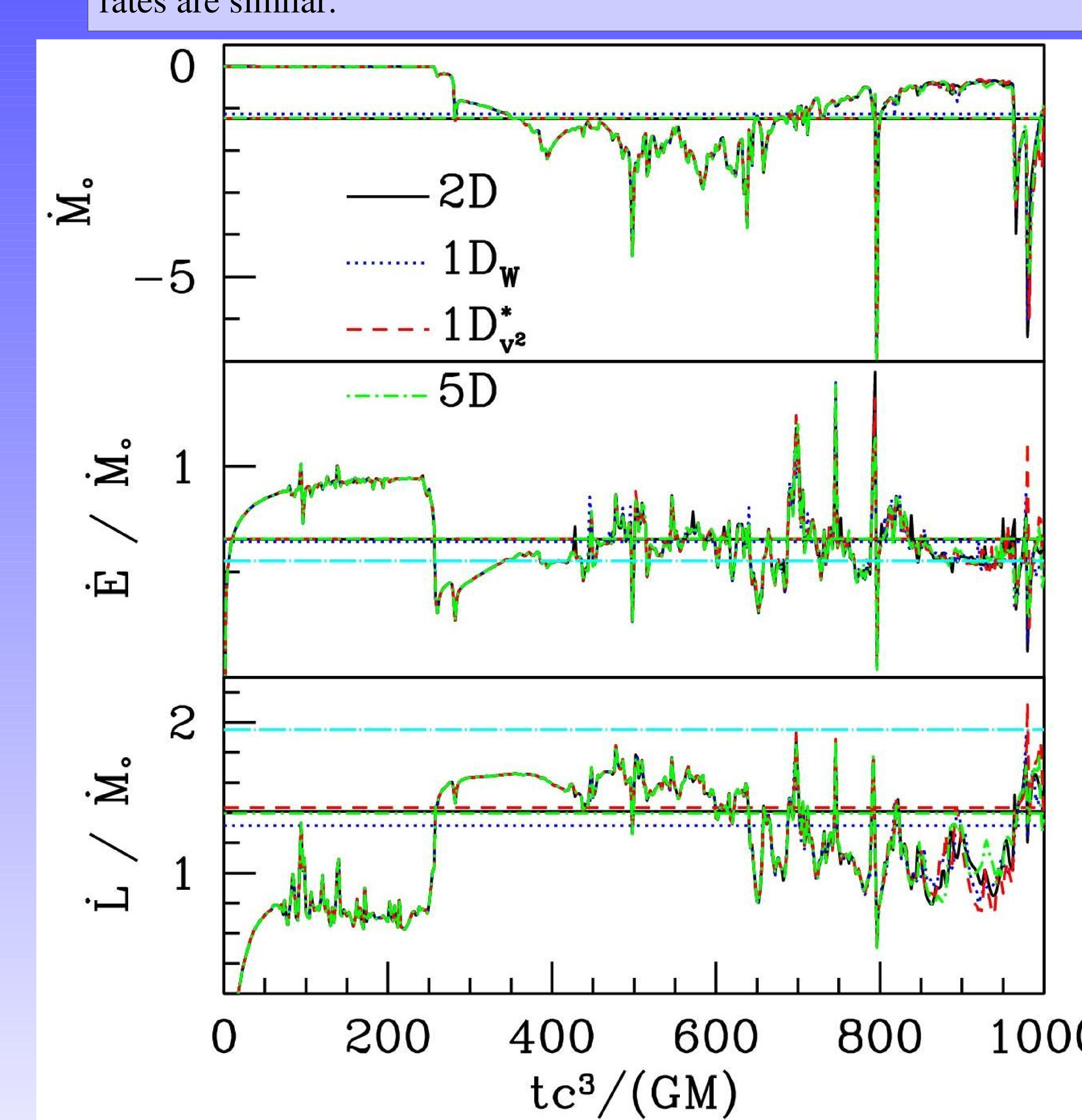


Figure 6: Snapshots of the log of the density at t=2000GM/c^3 in axisymmetry using different methods. From left to right: 2D, 1D_W, 1D*_v^2, 5D. The color map scales from dark blue (rho=4e-7) to dark red (rho=0.69).

Figure 7: Accretion rates for the methods of the rest-mass, total energy, and angular momentum over the period in which the runs' accretion rates are similar.



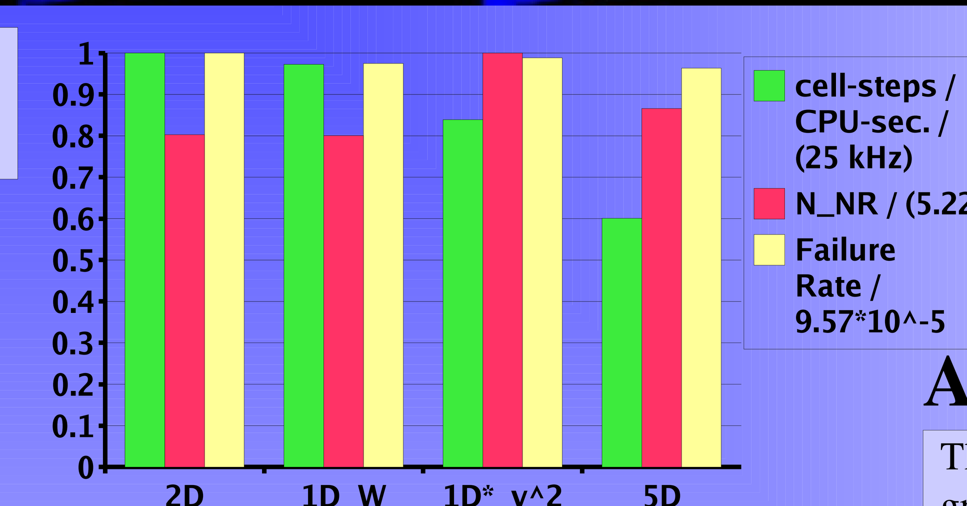


Figure 8: The solution rate (green), average number of NR iterations (red), failure rate (yellow) of each method for the accretion disk evolution.

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Conclusion

The most efficient methods are 2D and $1D_W$. The most robust method is the 2D method, closely followed by the $1D_W$ and $1D^*_{v^2}$ methods. The most accurate methods are the aforementioned and the 5D method, assuming that the guess for ***P*** is close enough to the solution. Hence, we recommend the 2D method for any conservative relativistic MHD code.

References

[1] Del Zanna, L., Bucciantini, N. & Londrillo, P. 2003, *A&A*, **400**, 397.
[2] Gammie, C. F., McKinney, J. C. & Toth, G. 2003, *ApJ*, **589**, 444.
[3] Koide, S., Nishikawa, K. I. & Mutel, R. L., 1996, *ApJ*, **463**, L71.
[4] Komissarov, S. S., 1999, *MNRAS*, **303**, 343.
[5] Noble, S. C., Gammie, C. F., McKinney, J. C., Del Zanna, L., *to be submitted*