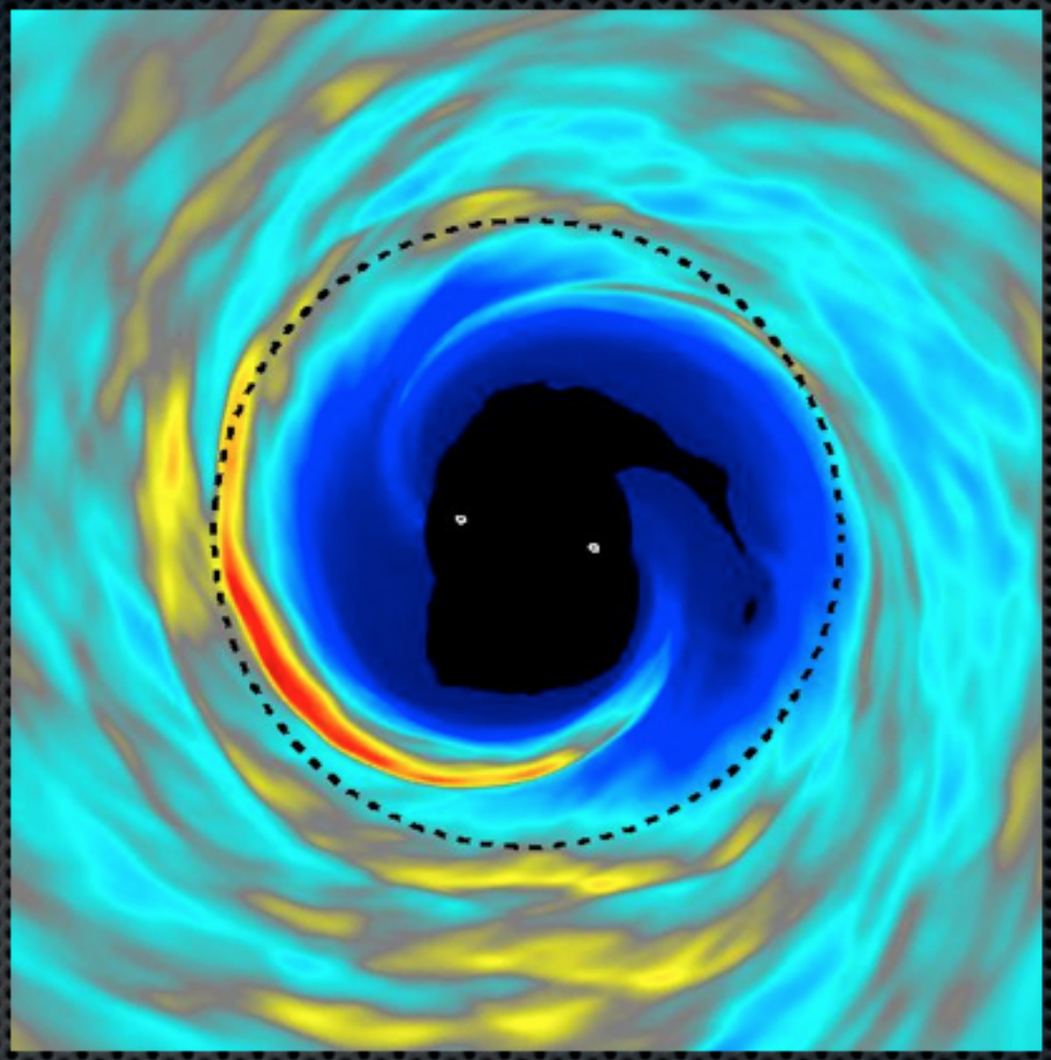
Magnetized Accretion onto Inspiraling Binary Black Holes



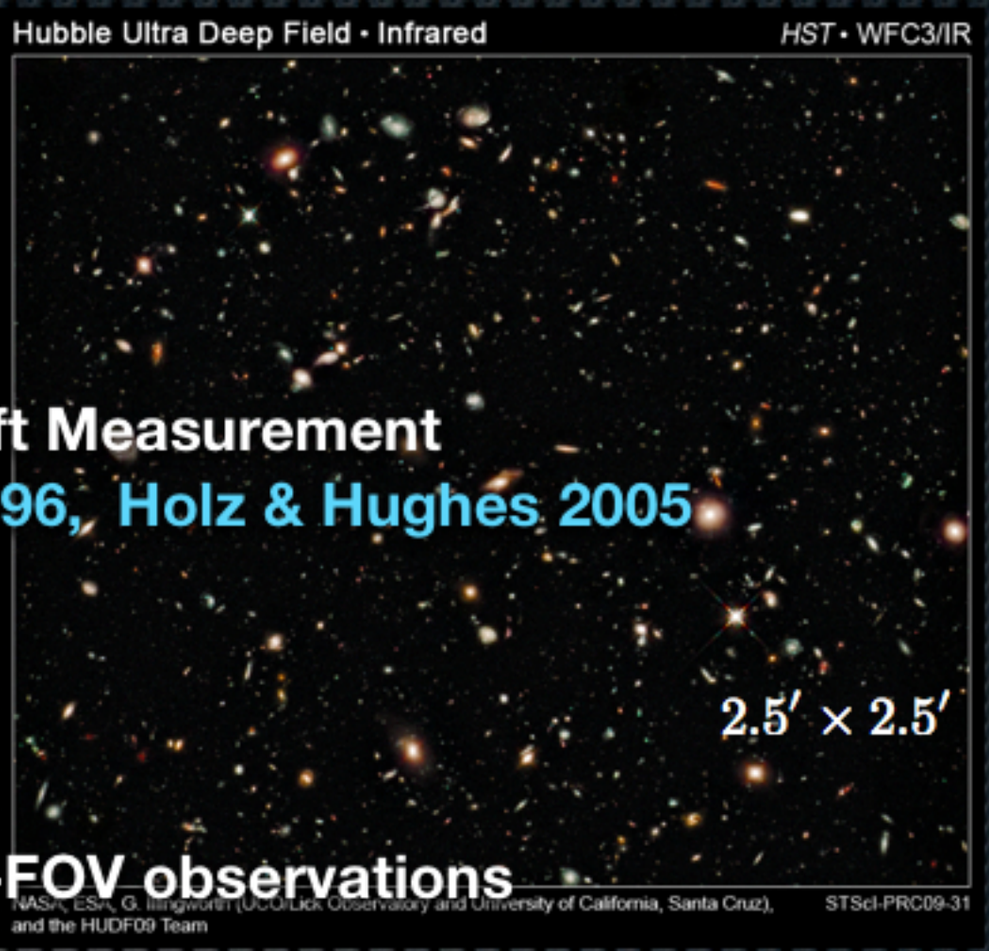
Scott Noble RIT

M. Campanelli, J. Krolik, B. Mundim, Johns Hopkins

H. Nakano, N. Yunes, Y. Zlochower Montana State U.

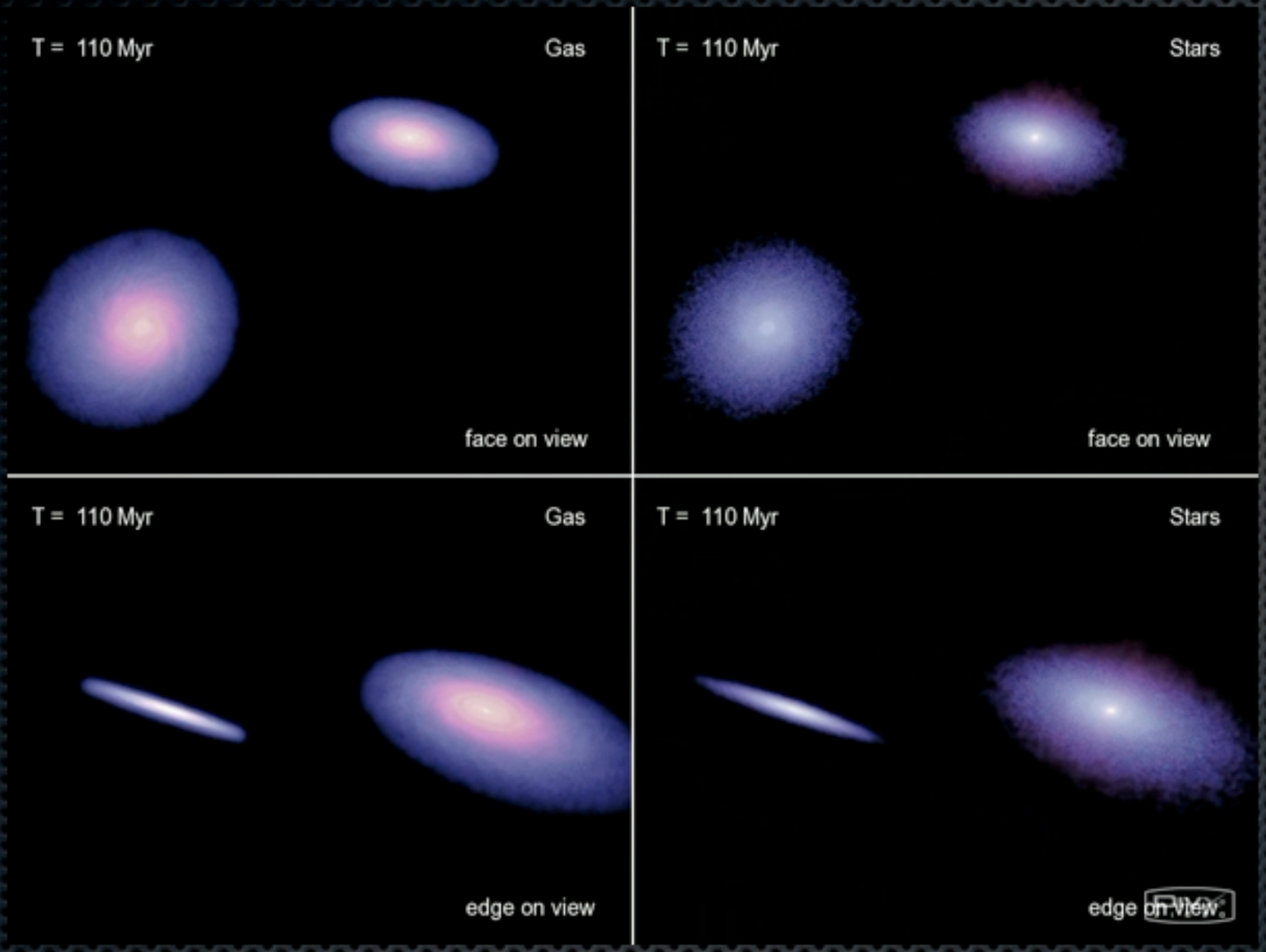
CITA * March 15, 2012

Motivation

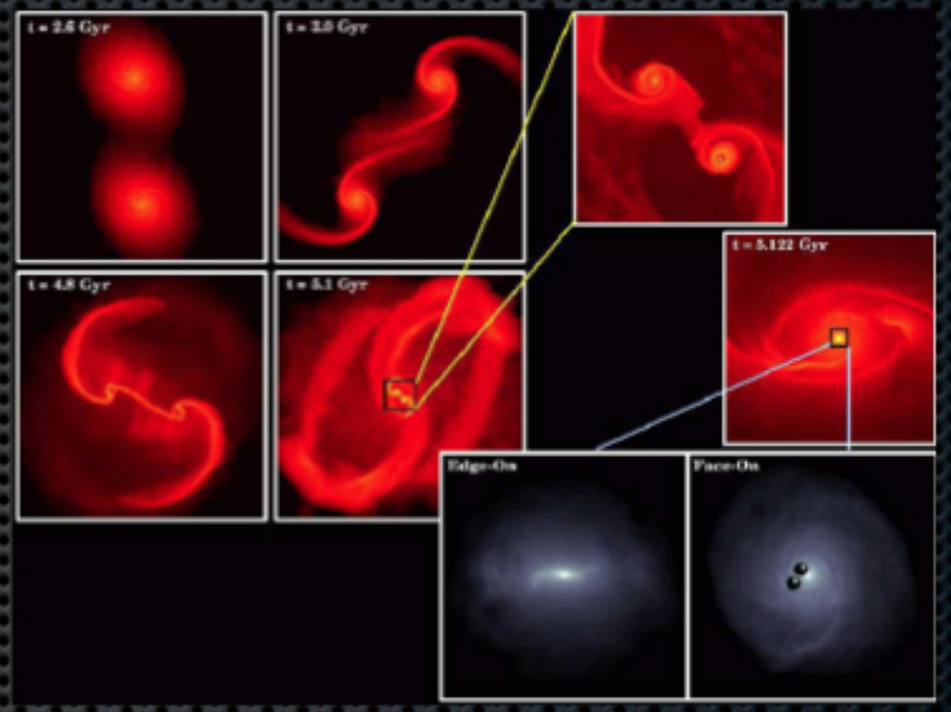


- "Standard Sirens": New Distance vs. Redshift Measurement
  - Schutz 1986, Chernoff+Finn 1993, Finn 1996, Holz & Hughes 2005
- Mutual Beneficial Localization
  - "LISA" localization days in advance:
  - EM localization using high-cadence, high-FOV observations
    - Need dynamical models to predict source variability accurately
    - GW constraints <--> EM constraints Lang & Hughes 2009 Kocsis et al 2007
- Science Impact:
  - SMBBH Merger Rates, SMBH Evolution/History
  - Plasma physics in limit of strong-field, dynamic GR
- Connecting robust theoretical predictions to event observations may be first evidence of BBH mergers (hopefully not!)
  - "LISA" expected to launch in...., ??
  - LIGO sensitive to smaller BBHs that are most likely in vacuum

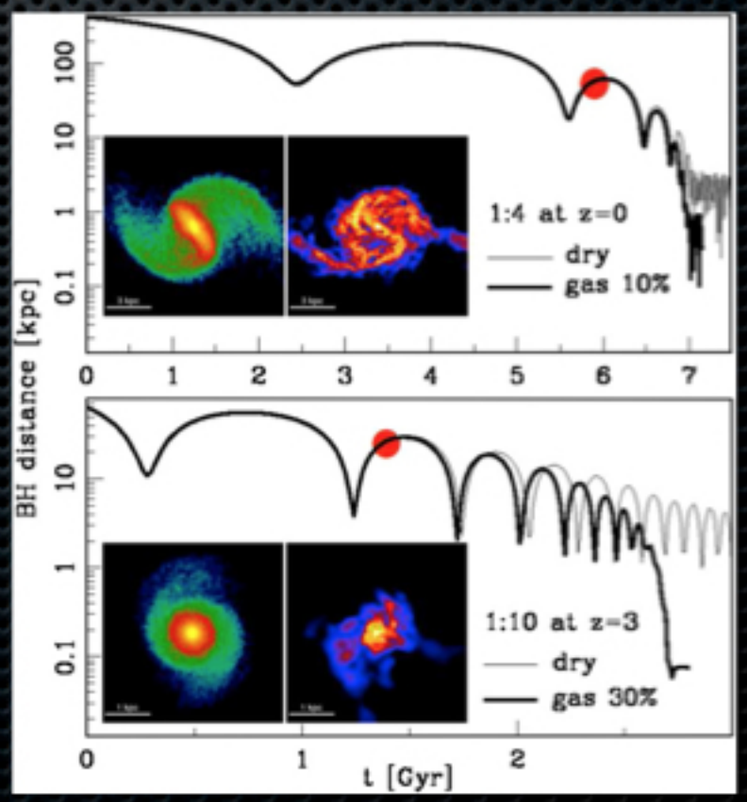
Pre-pre-merger



Hopkins, Hernquist, Springel et al.



Colpi, et al 2009



Sub-kpc Resolved Dual Nuclei

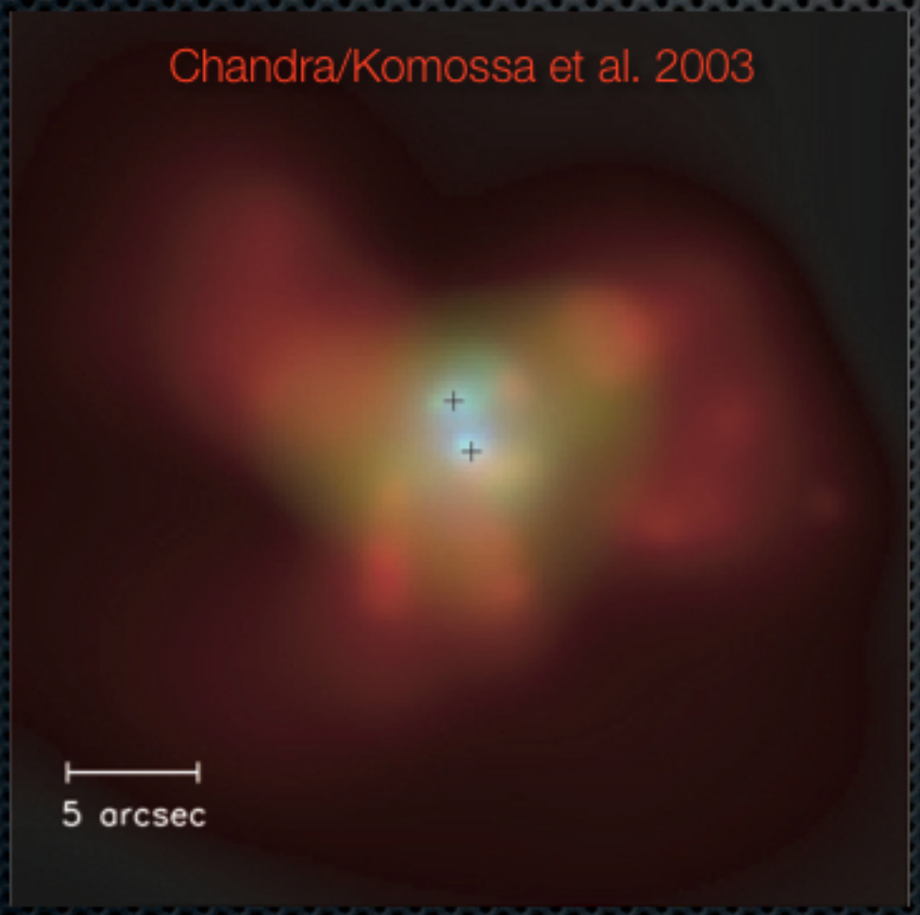
0402+379:

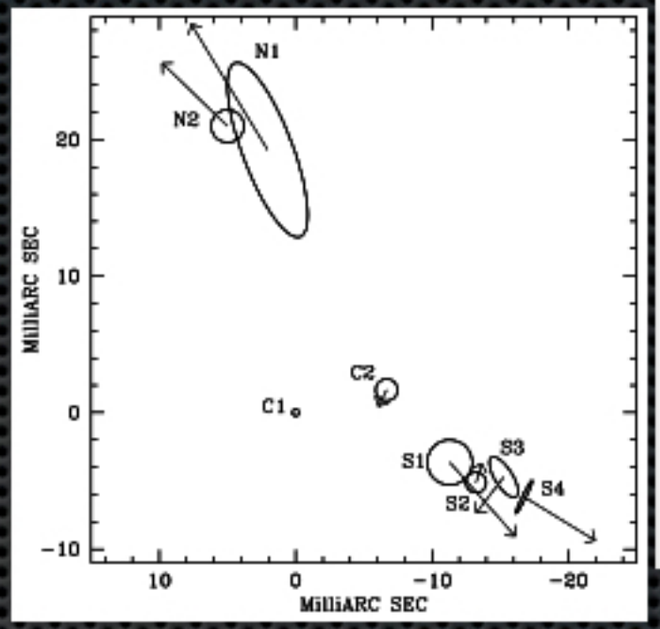
(Xu et al. 1994, Maness et al. 2004, Rodriguez et al. 2006):

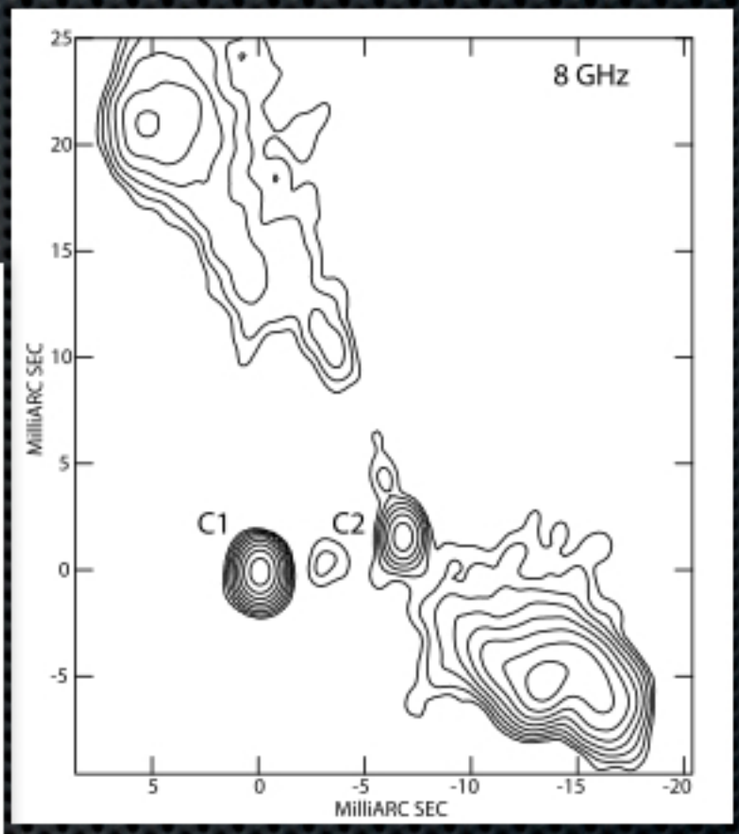
- Radio, Elliptical galaxy host
- $z = 0.055$, $d = 5$ pc $M \sim 10^8 M_\odot$

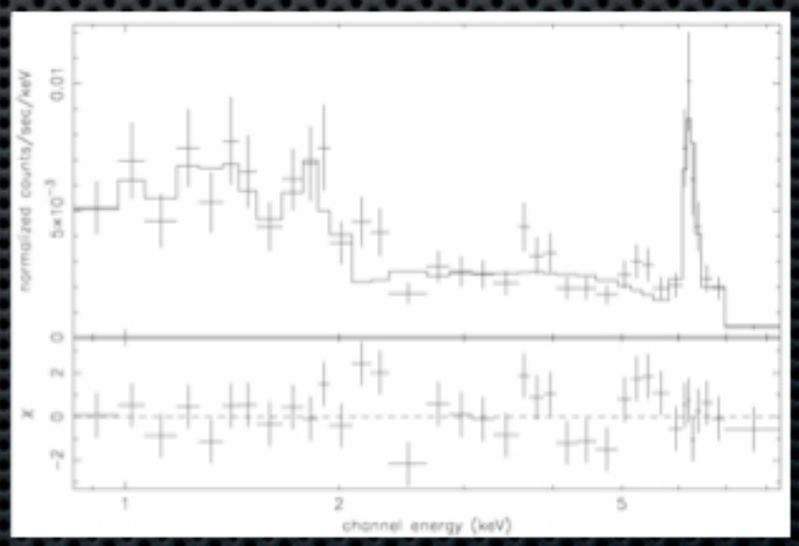
NGC 6240: (Komossa et al. 2003)

- Optical ID: (Fried & Schulz 1983)
- HST, Ultra-lum. IR galaxy host
- $z = 0.024$ $d = 0.5$ kpc

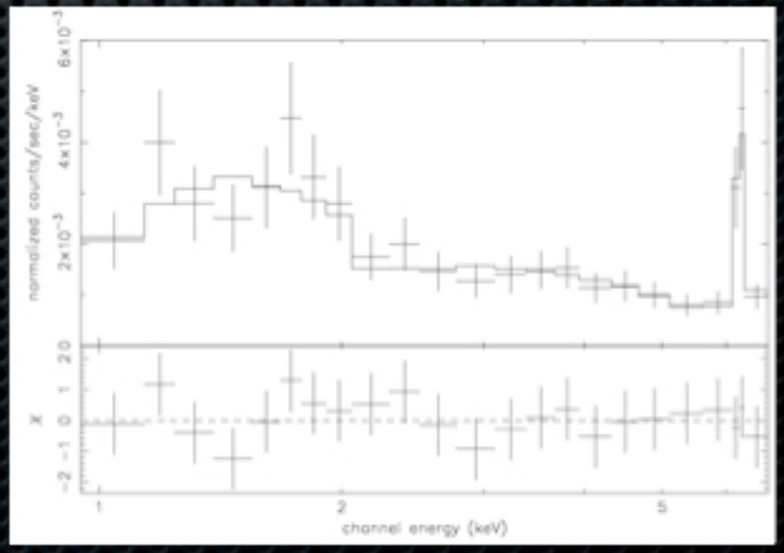








South



North

OJ287: Pre-minor-merger??

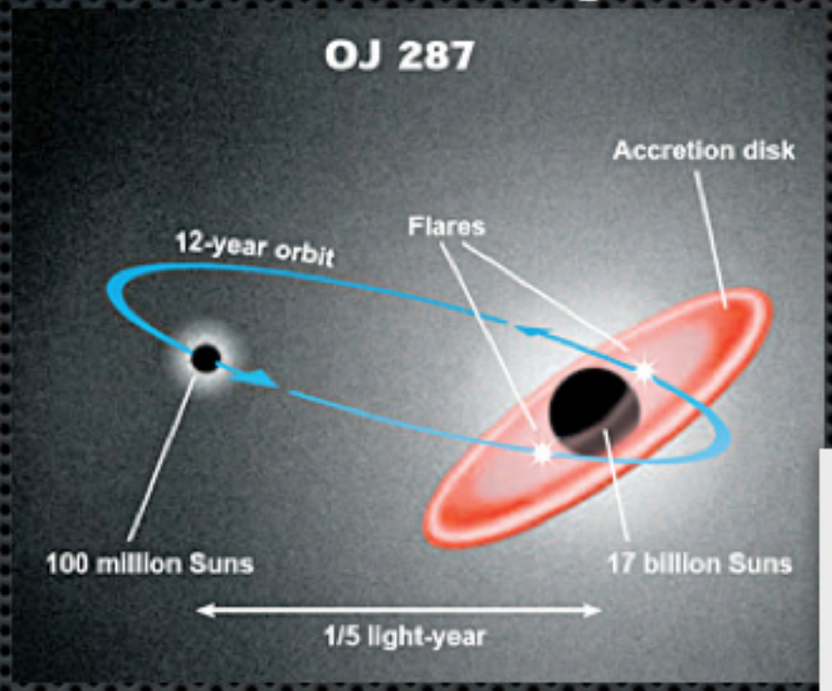
Lehto & Valtonen 1996:

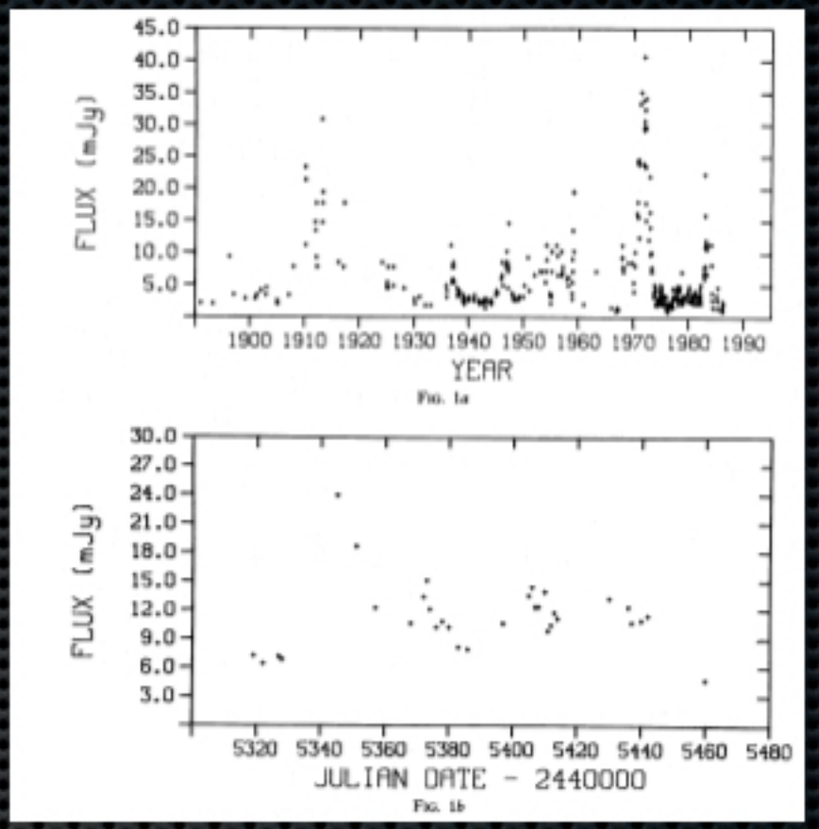
$$M_1 = 1.7 \times 10^{10} M_\odot \quad M_2 = 10^8 M_\odot$$

$$T_{\rm orb} = 12.07{\rm yr} \quad T_{\rm precess} = 130{\rm yr} \quad T_{\rm merge} \simeq 10^4{\rm yr}$$

$$i_{\rm disk} = 4^\circ$$

$$e = 0.68$$





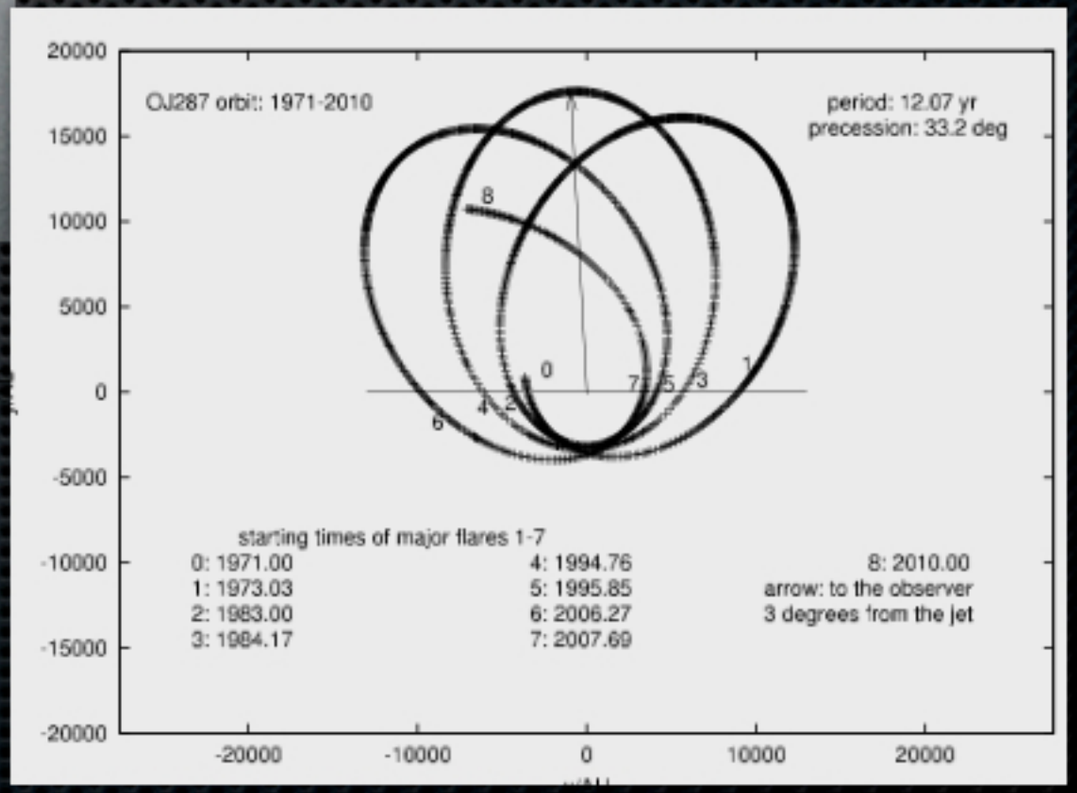
Sillanpaa et al 1988

Valtonen et al Nature 2008:

- 20 days earlier than expected
- Consistent to 10% predicted by radiation decay

Valtonen et al 2010:

Fit with 2.5PN expansion $\rightarrow a = 0.28 \pm 0.05 M$



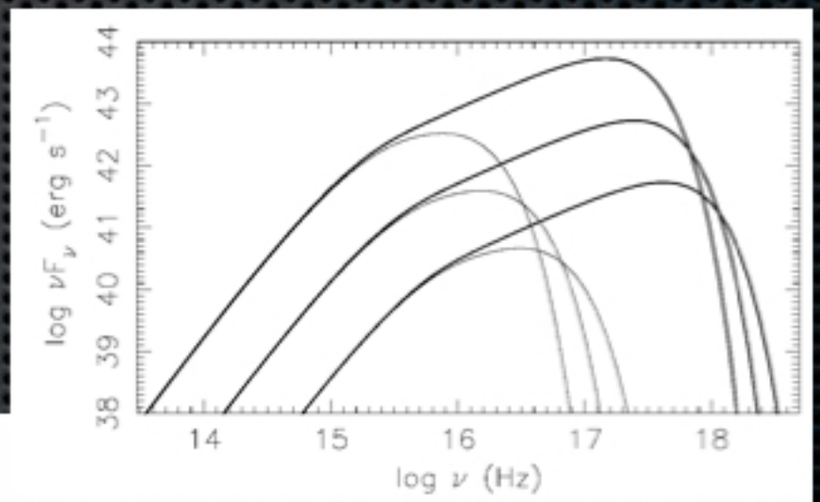
Post-merger

Mass Loss

Bode & Phinney 2007
O'Neill et al. 2009
Megevand et al 2009
Krolik 2010

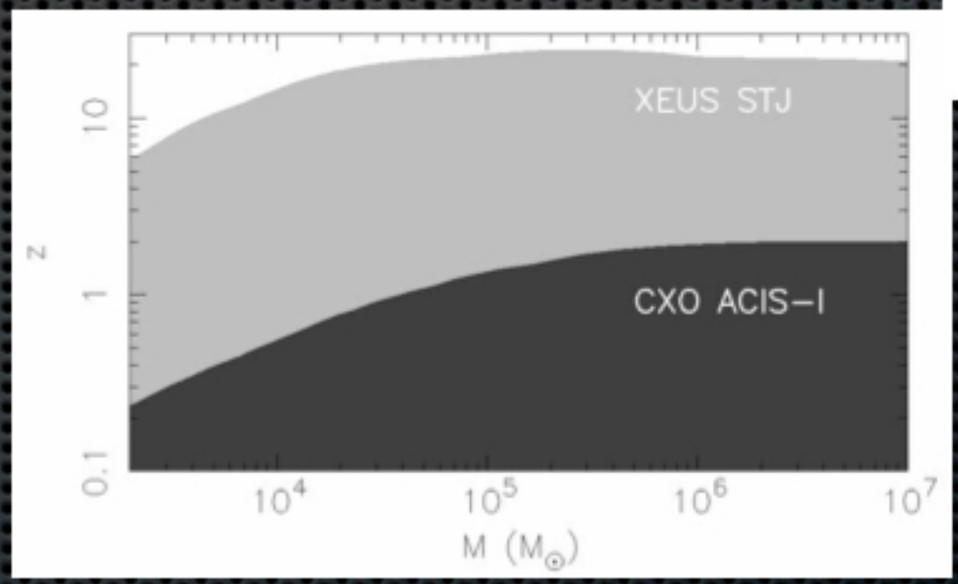
GW Dissipation in Disk

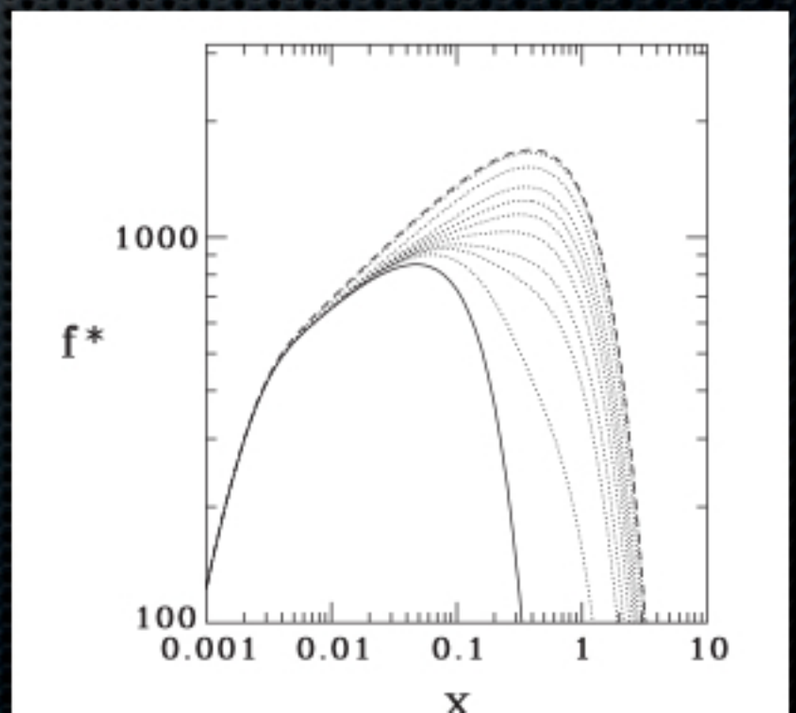
Kocsis & Loeb (2008)



Disk's Response to Recoil

Shields & Bonning 2008
Schnittman & Krolik 2008
Lippai et al 2008
Corrales et al 2009





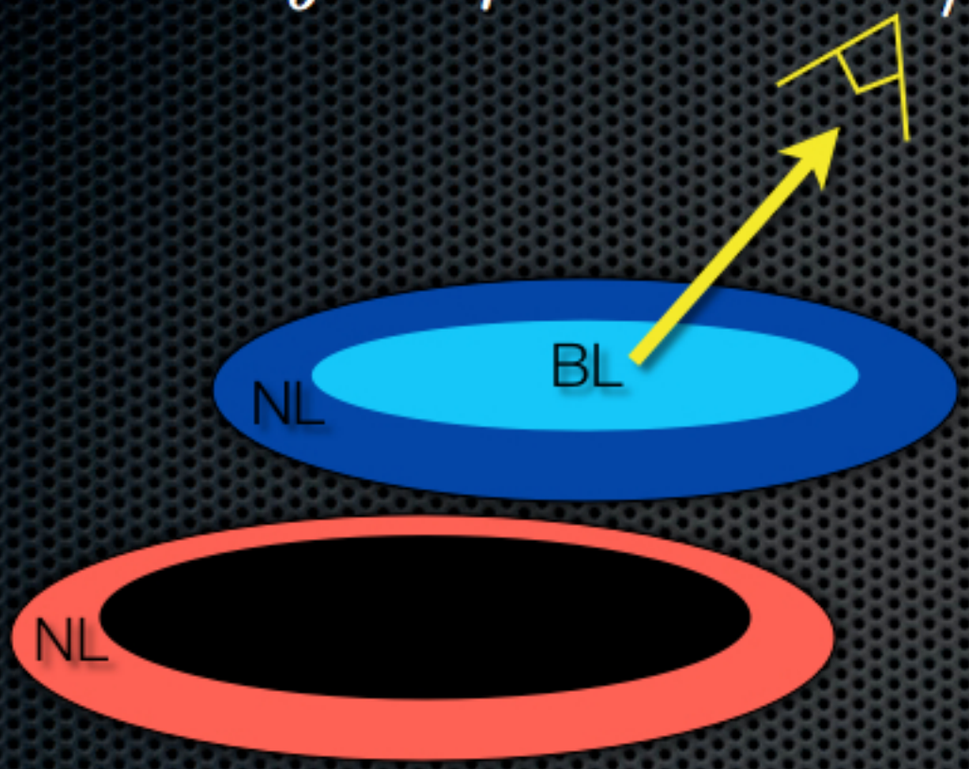
- Gap Refills:
- Milosavljevic & Phinney 2005, Shapiro 2010

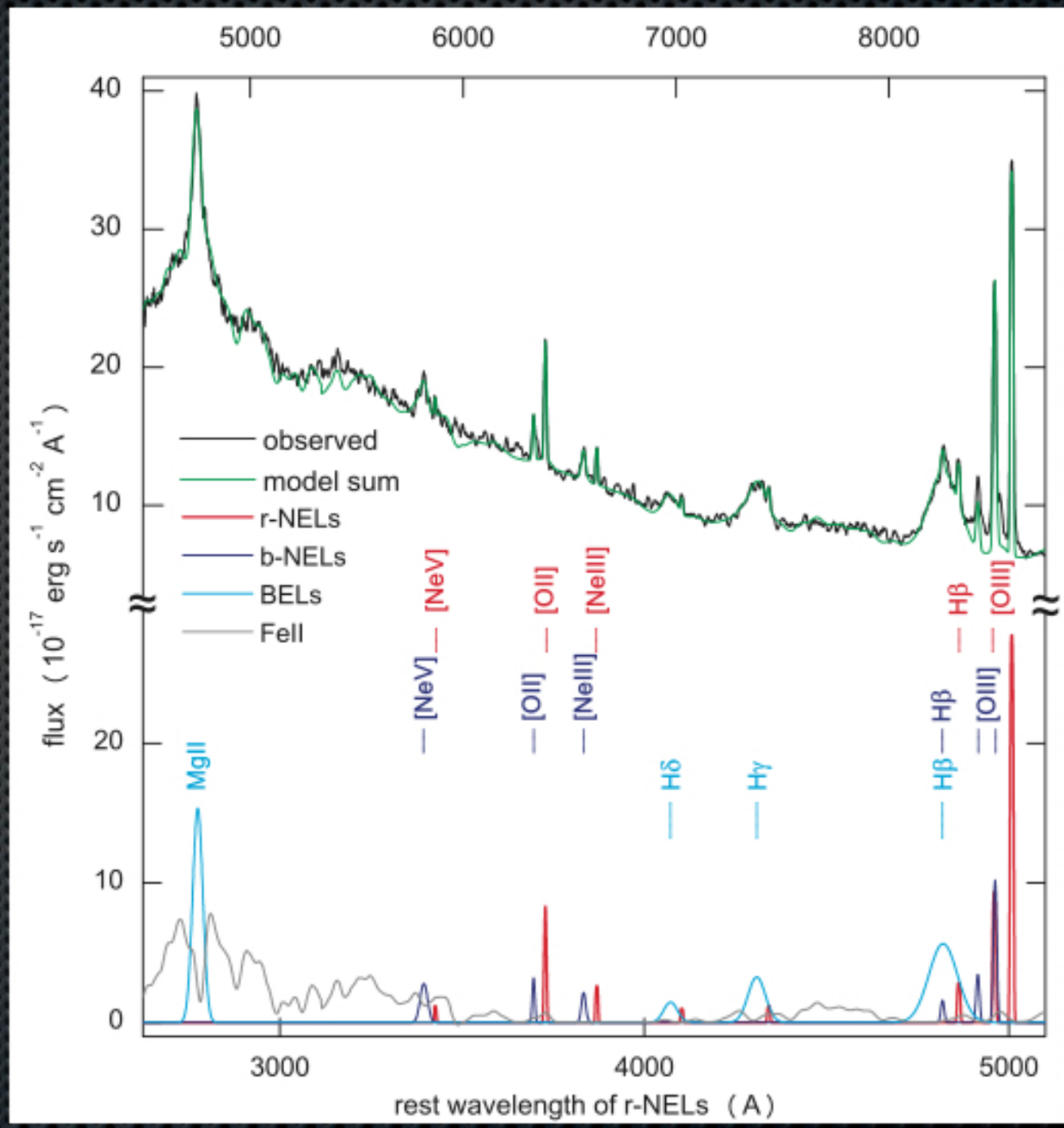
Recoiled SBH? SDSS J0927+2943

Komossa, Zhou, Lu (2008)

$z = 0.713 \qquad r_{\rm BL} \sim 0.1{\rm pc}$

$v_b - v_r = 2650{\rm km/s}$





Other Explanations:

Heckman et al 2009, Shields et al. 2009,
Bogdanovic et al. 2009, Dotti et al. 2009

Another Similar Candidate:

SDSS J105041.35+345631.3 (Shields et al. 2009)

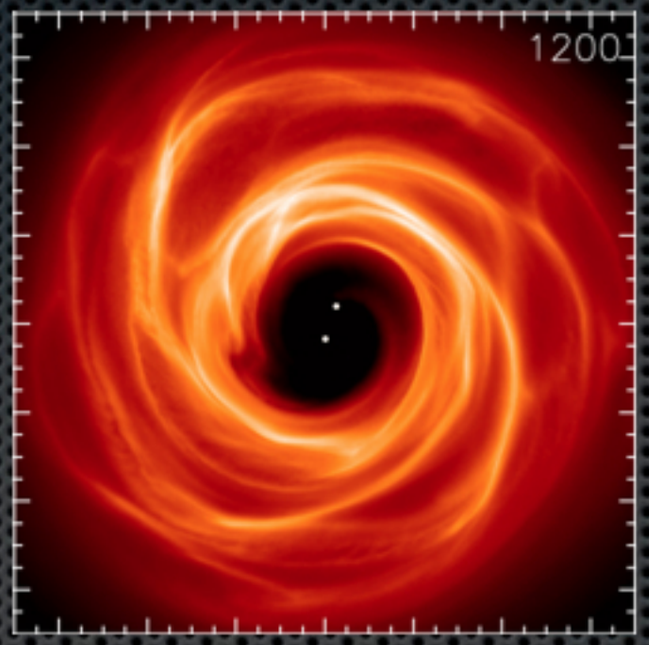
$v_{\rm BL} - v_{\rm NL} = 3500{\rm km/s}$

Circumbinary Disks

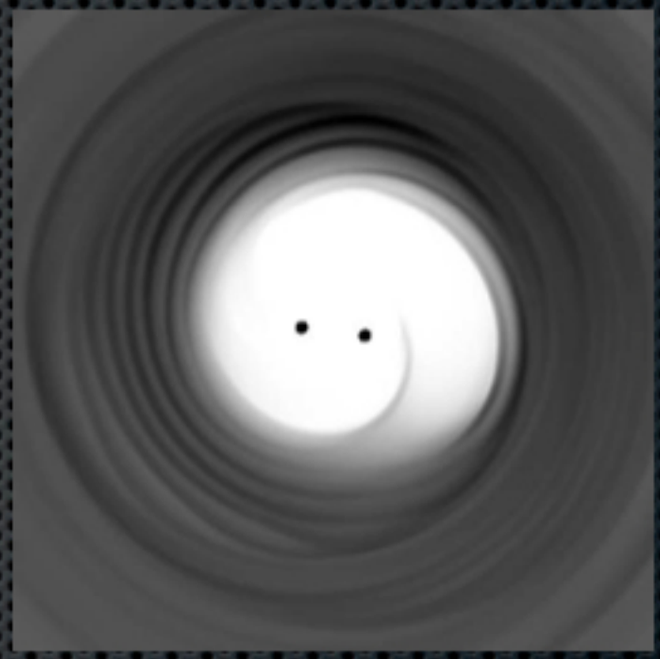
- a = BHB's radial separation

- Consider SMBHs with mass ratio $M_2/M_1 \sim 1$ ... though $M_2/M_1 << 1$ may be more common

- BHB exerts torque on gas via its time-varying mass quadrupole moment
- BHB torque restores some of the accreting gas' angular momentum , thereby diminishing the mass accretion rate through the gas
- Surface density tends to build-up at r ~ 2a as matter accretes faster beyond gap

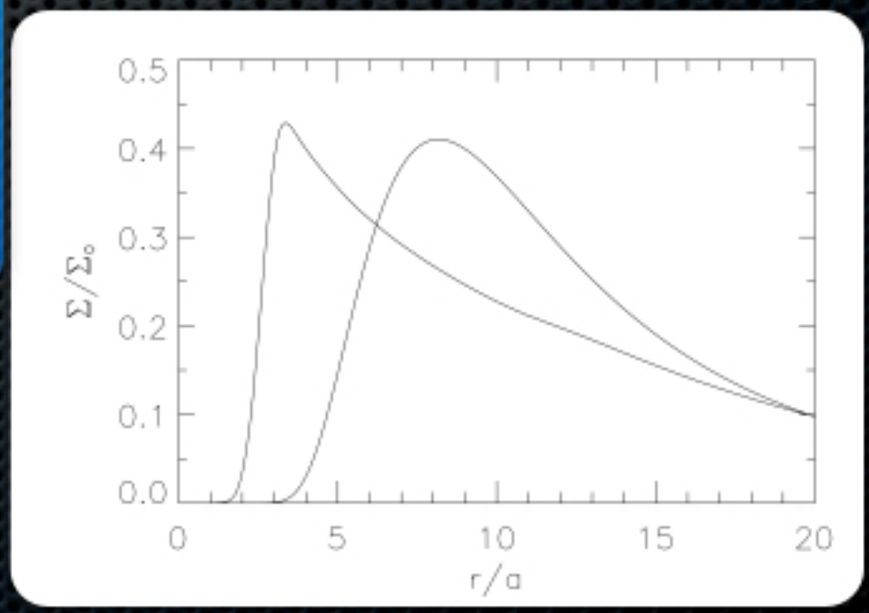


Cuadra et al 2009



MacFadyen & Milosavljevic 2008

Newtonian Hydrodynamic Simulations

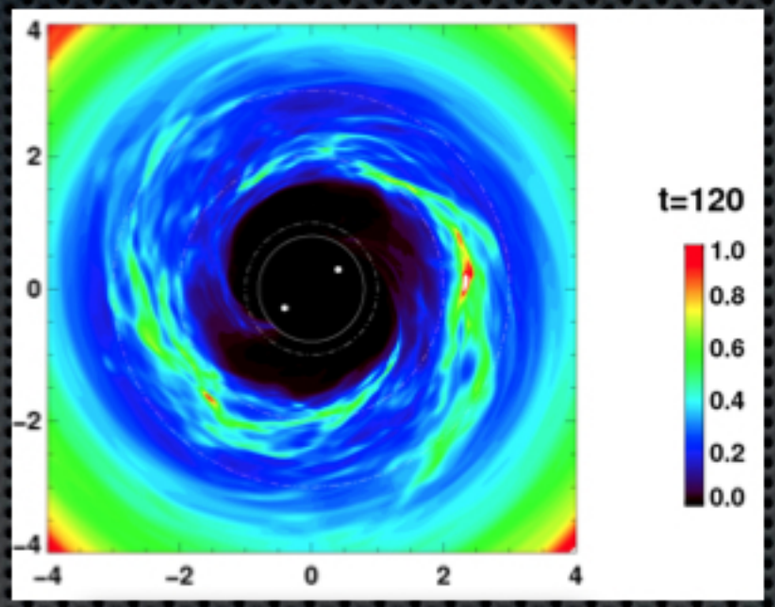


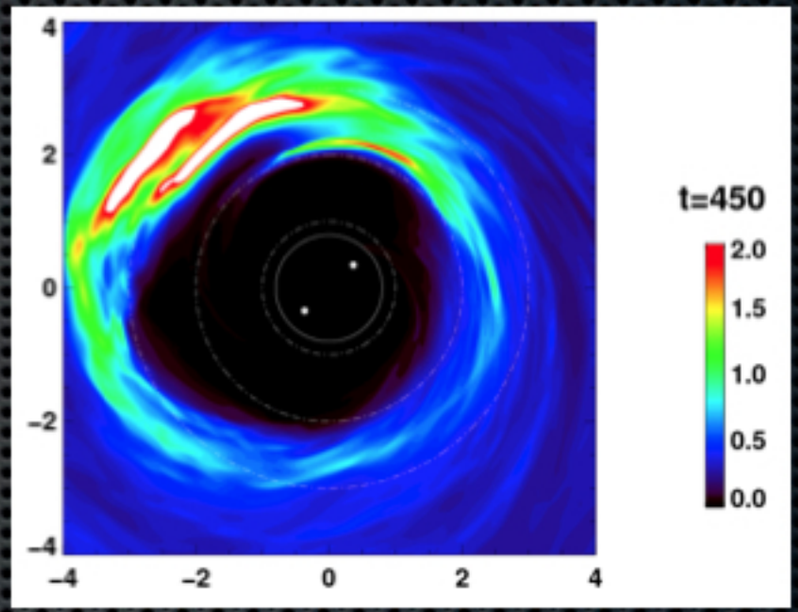
Circumbinary Disks

- a = BHB's radial separation

- Consider SMBHs with mass ratio $M_2/M_1 \sim 1$ ... though $M_2/M_1 << 1$ may be more common

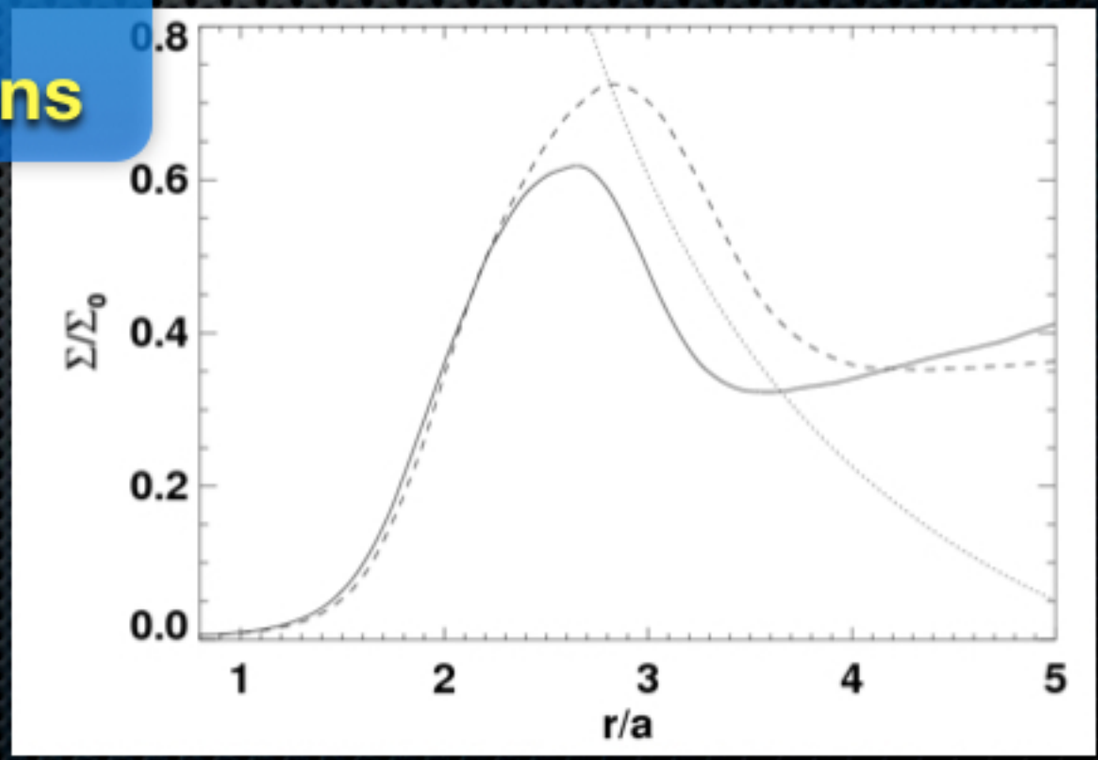
- BHB exerts torque on gas via its time-varying mass quadrupole moment
- BHB torque restores some of the accreting gas' angular momentum , thereby diminishing the mass accretion rate through the gas
- Surface density tends to build-up at r ~ 2a as matter accretes faster beyond gap
- MHD torques seem to be able to accrete more material through the gap, but gap still exists





Newtonian MHD Simulations

Shi++2011



Circumbinary Disks

- a = BHB's radial separation

- Consider SMBHs with mass ratio $M_2/M_1 \sim 1$ ... though $M_2/M_1 << 1$ may be more common

- BHB exerts torque on gas via its time-varying mass quadrupole moment

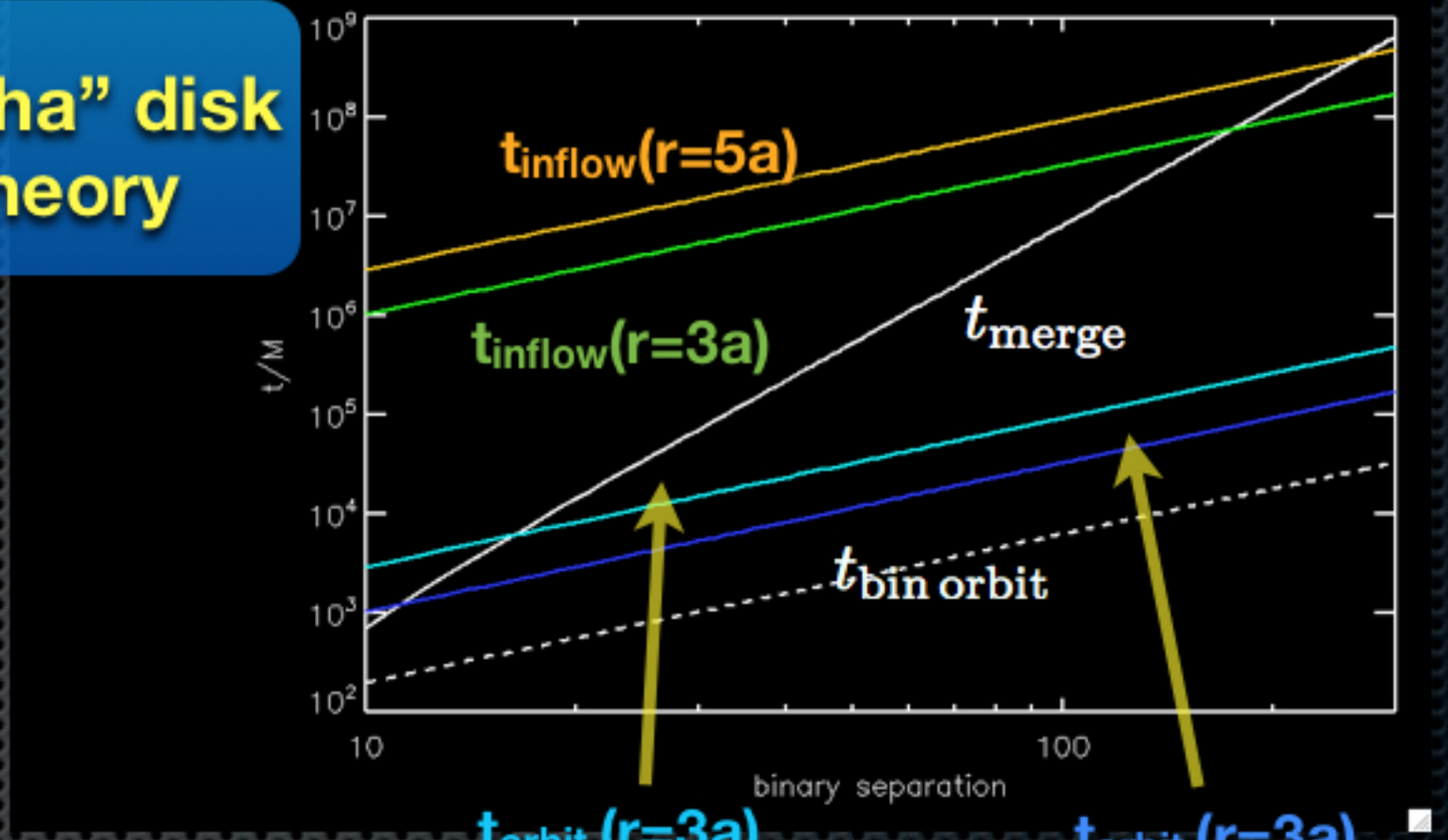
"alpha" disk theory

- BHB torque restores some of the accreting gas' angular momentum , thereby diminishing the mass accretion rate through the gas

- Surface density tends to build-up at r ~ 2a as matter accretes faster beyond gap

- MHD torques seem to be able to accrete more material through the gap, but gap still exists

- BHB torque on gas removes ang. mom. from BHB, meaning BHB may inspiral on inflow timescale until GW emission sheds ang. mom. even faster



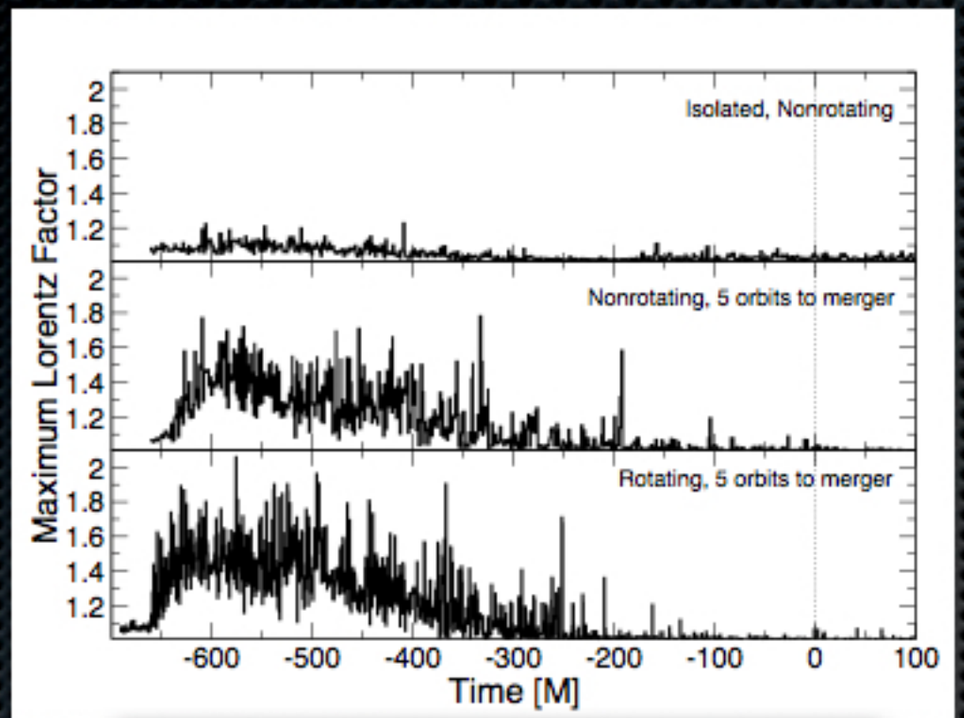
$\alpha = 0.1$

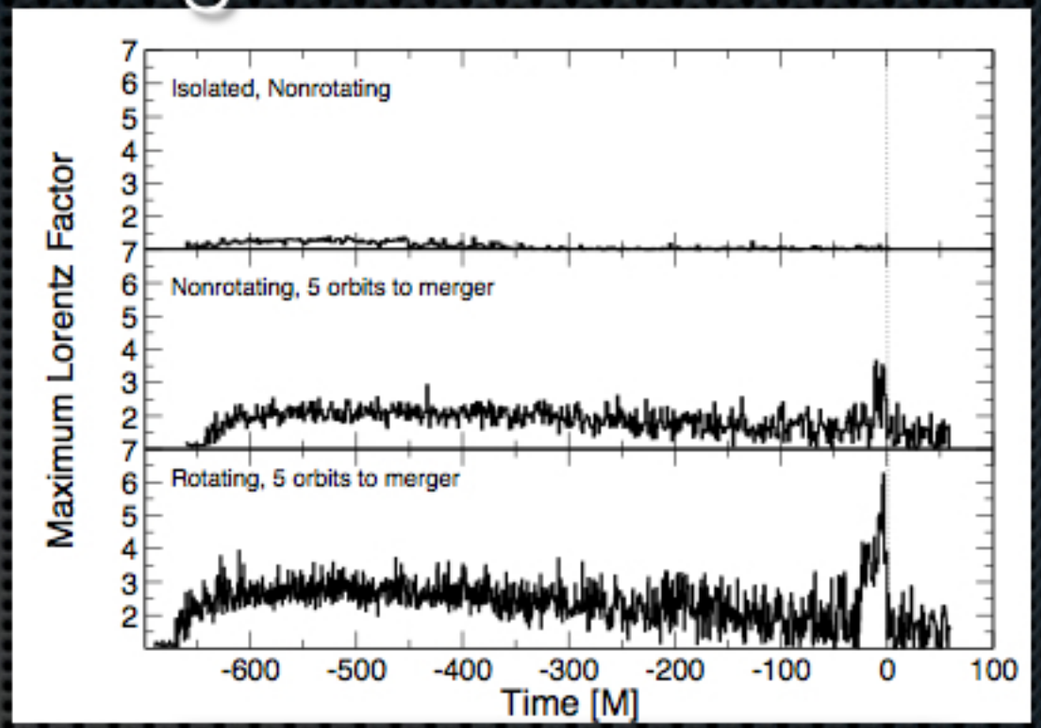
Can MHD torques keep the disk/gap near the binary?
or
At what radius is the MHD disk left behind?

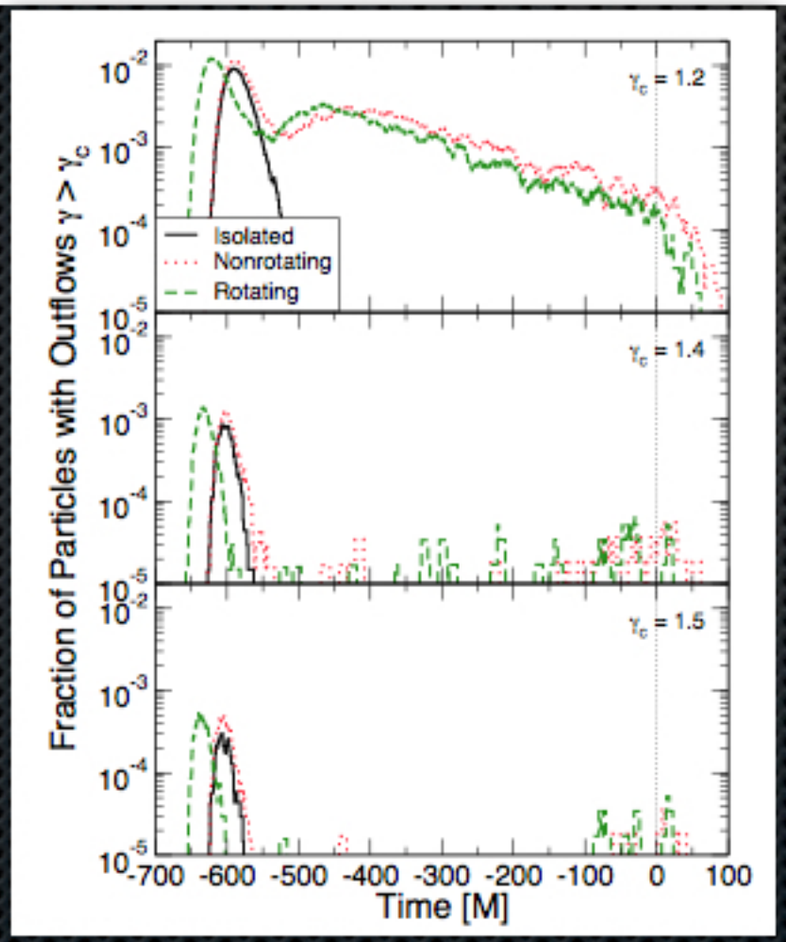
- Mass distribution is critical for accurate EM models as emissivity typically scales $\sim \rho^n$

van Meter et al 2009

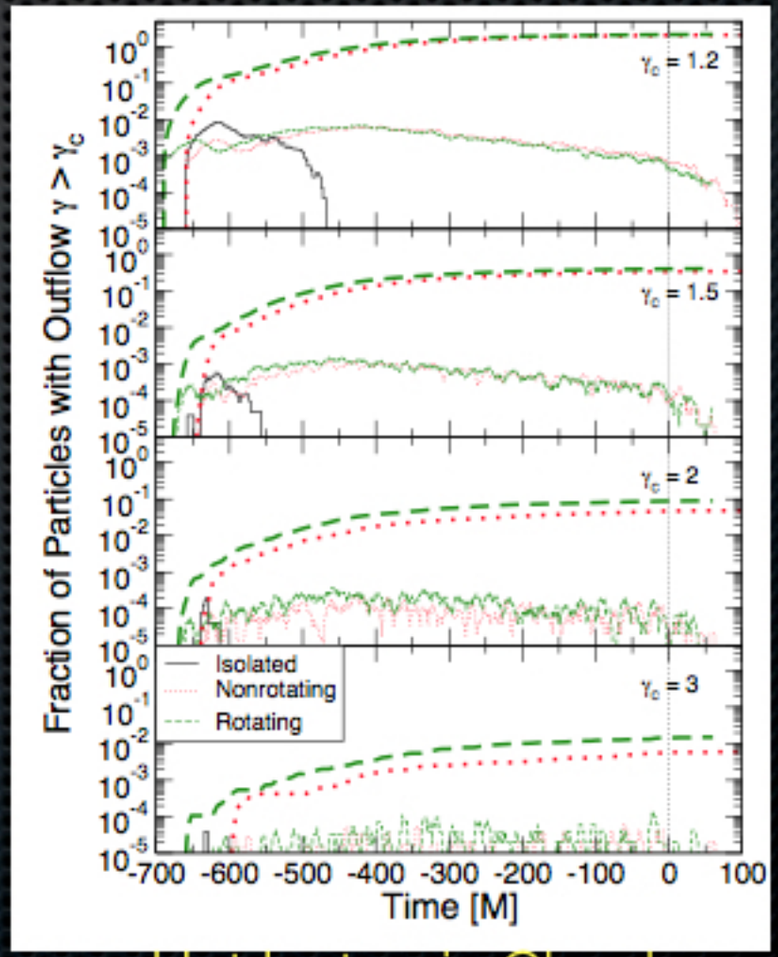
"Stirring Test Particles"





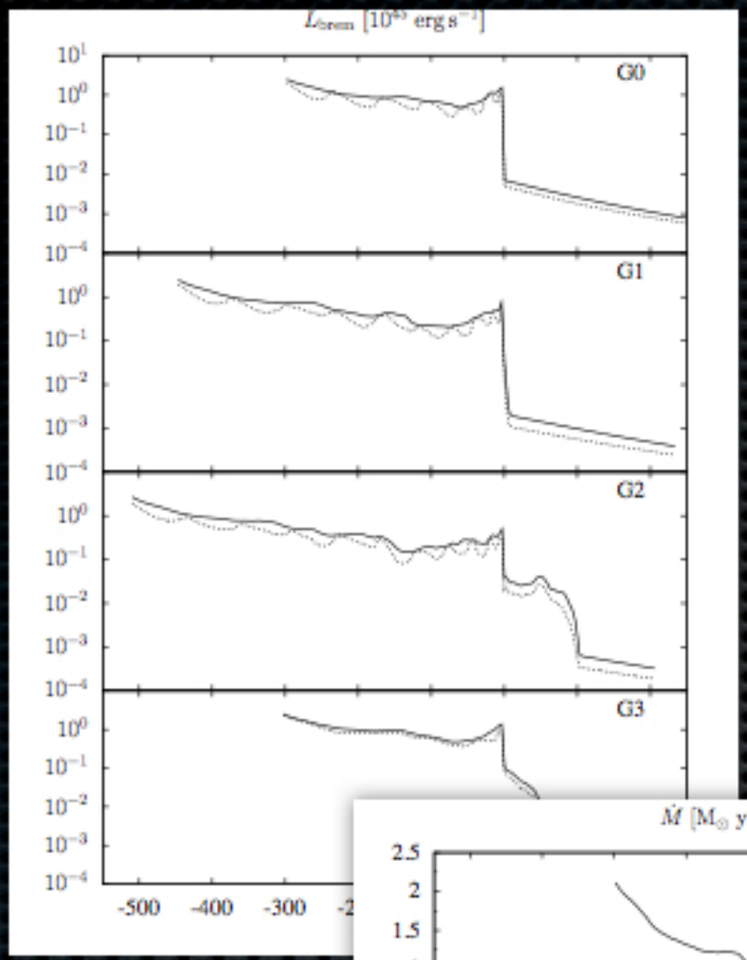


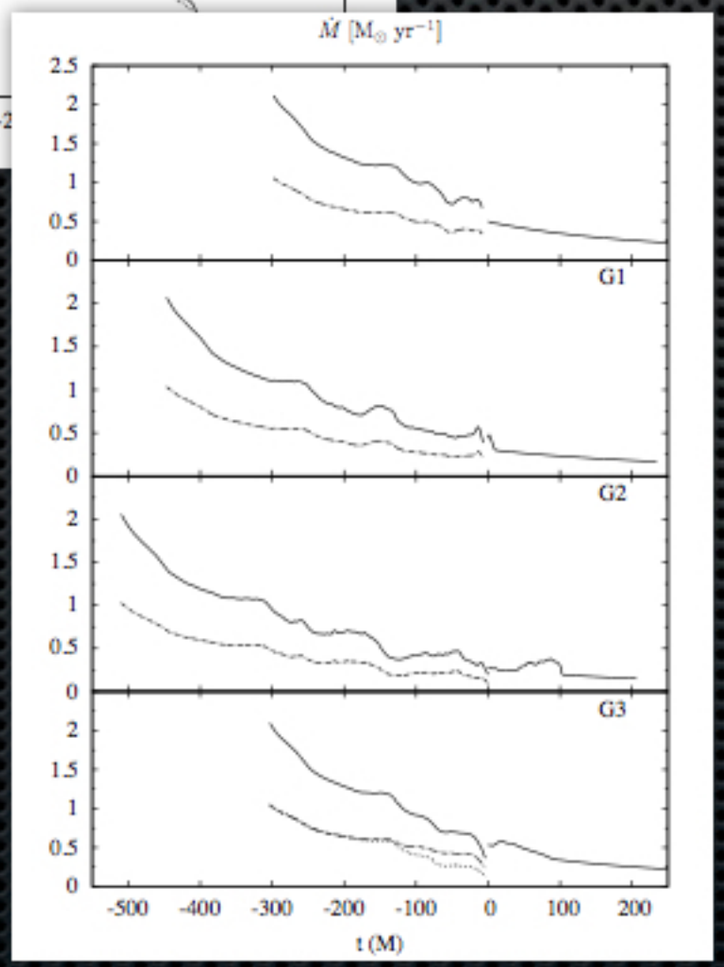
Warm Torus in Orbit



Hot Isotropic Cloud

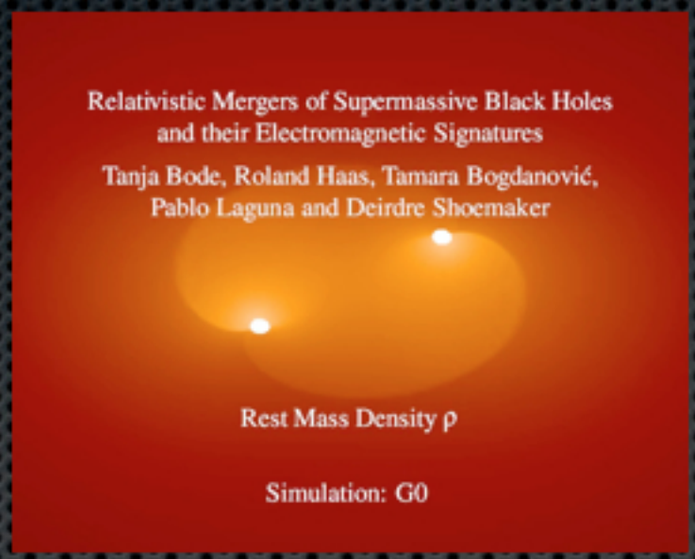
Bode et al. 2009



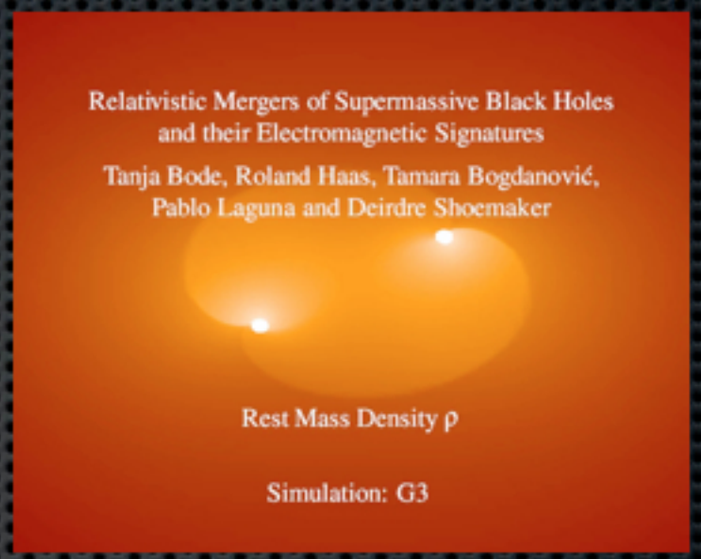


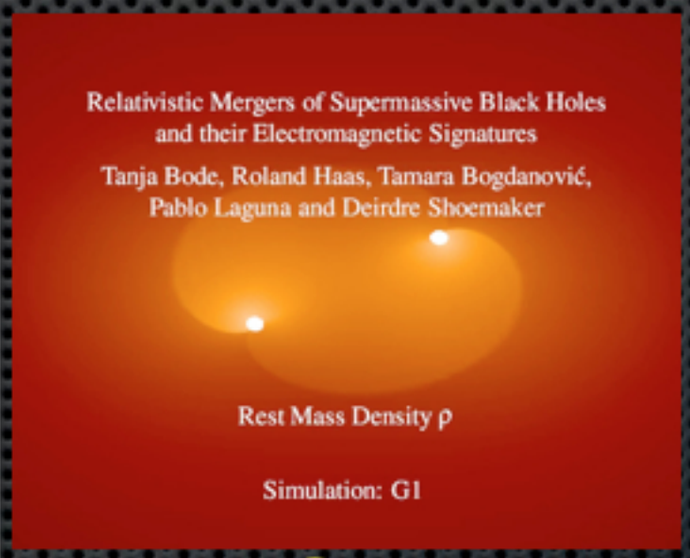
"Stirring Hot Gas"

G0

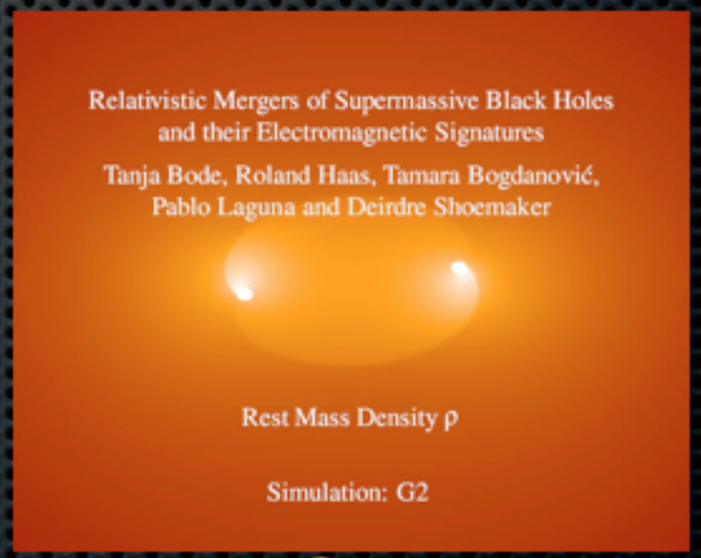


G3





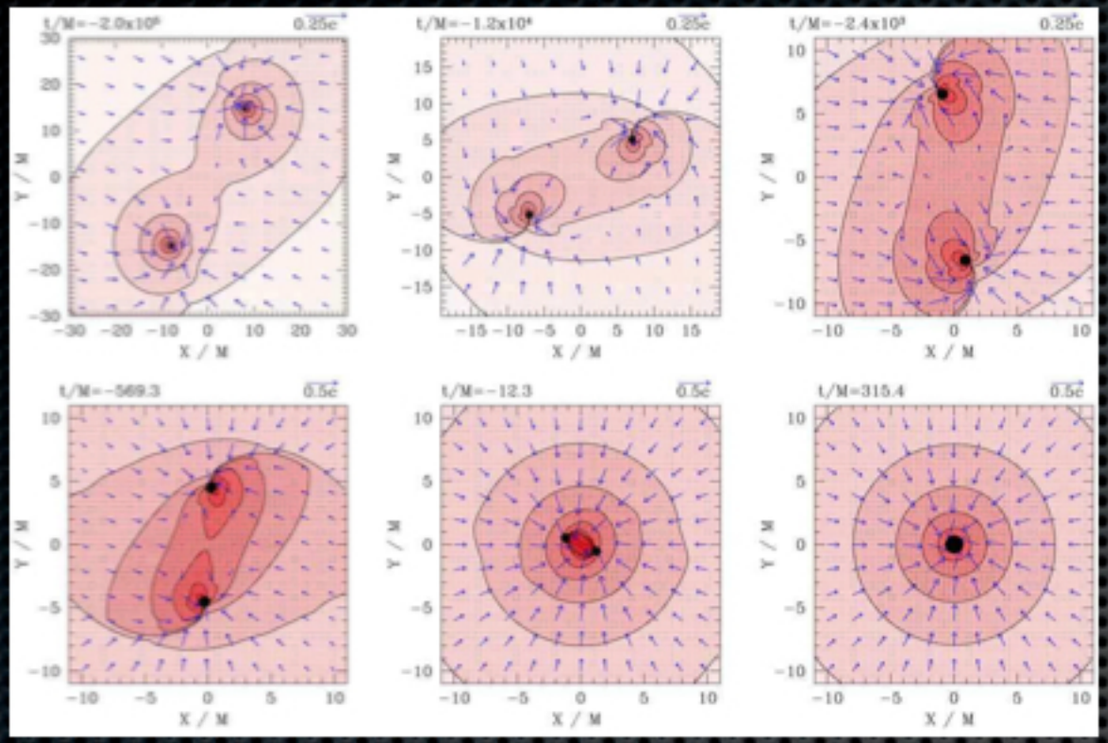
G1



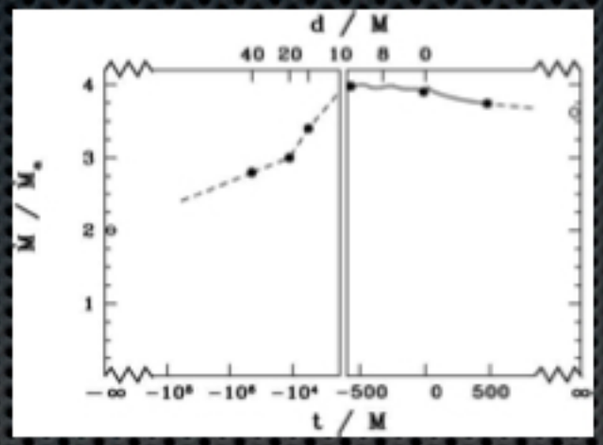
G2

Run	a_1/m	a_2/m	P^x/M	P^y/M	m_i/M	M_{ADM}/M
G0	0	0	-2.0902×10^{-3}	0.11237	0.5000	0.9878
G1	+0.4	+0.4	0	0.10862	0.4893	0.9875
G2	+0.6	+0.6	0	0.10677	0.4736	0.9874
G3	+0.4	-0.4	0	0.11237	0.4893	0.9878

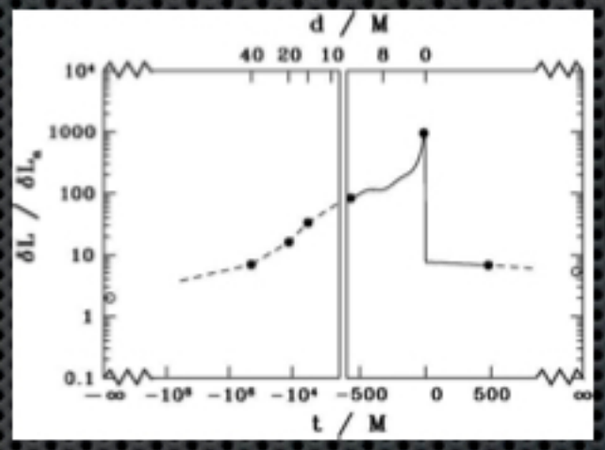
Farris, Liu, Shapiro 2009



Binary Bondi-Hoyle-Lyttleton Accretion



$$L_{ff}^{max} \approx 3 \times 10^{37}\, n_1^2\, T_6^{-3} M_6^3 \text{ erg s}^{-1} ,$$
$$L_{syn}^{max} \approx 3 \times 10^{43}\, n_1^2\, T_6^{-3} \beta_1^{-1} M_6^3 \text{ erg s}^{-1}$$

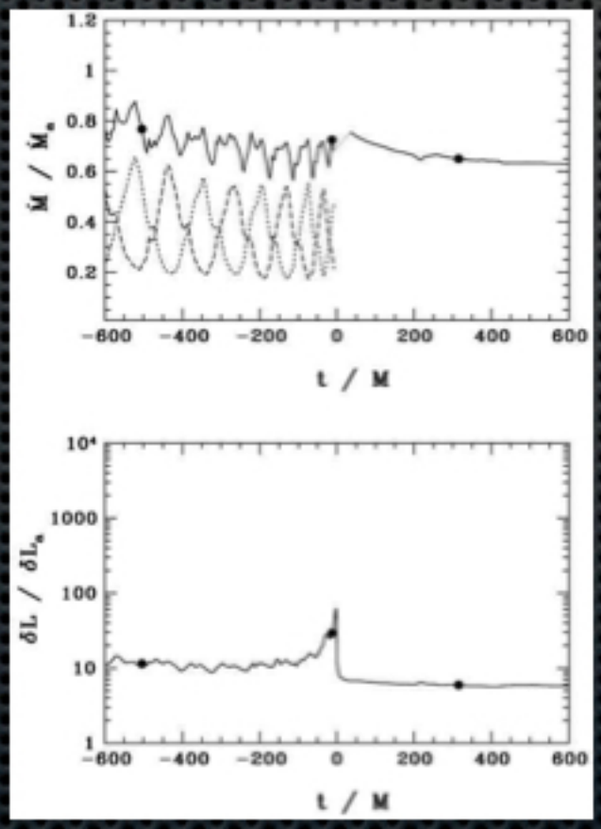


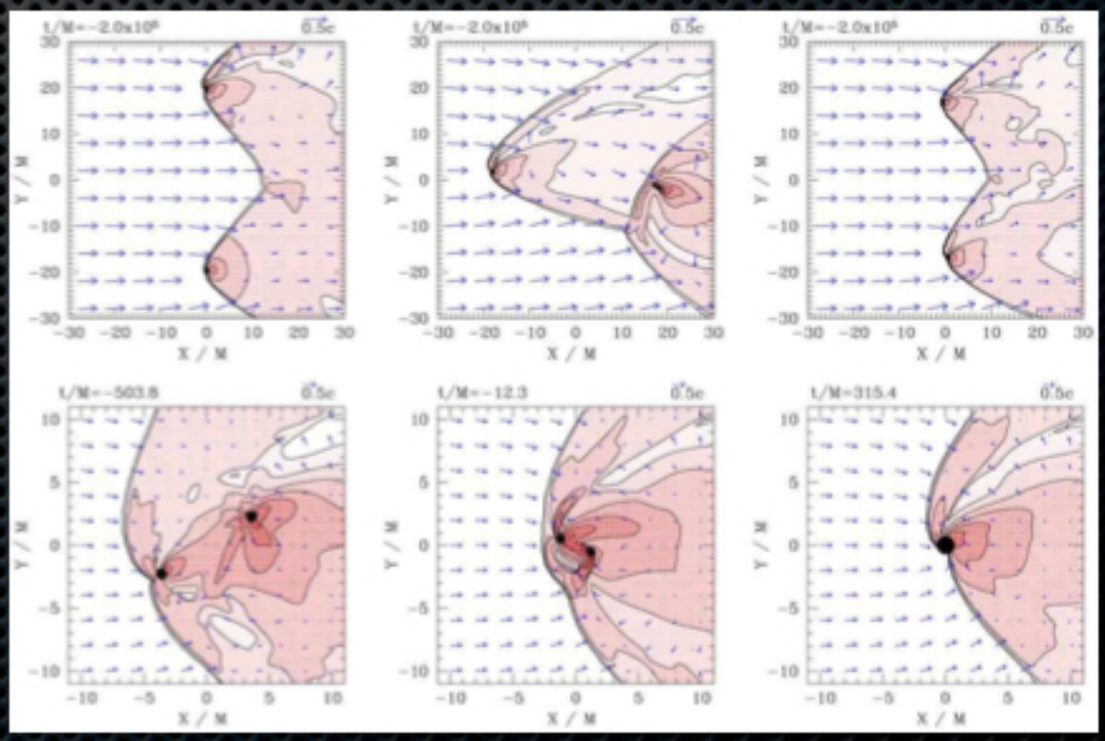
$$h\nu_{ff}^{max} \approx \frac{230 \text{ MeV}}{1+z} \text{ (RA2)}$$

$$h\nu_{syn}^{max} = \frac{100}{1+z}\, n_1^{1/2}\, T_6^{-3/4} \beta_1^{-1/2} \text{ eV (RA2)}$$

"Prototype"

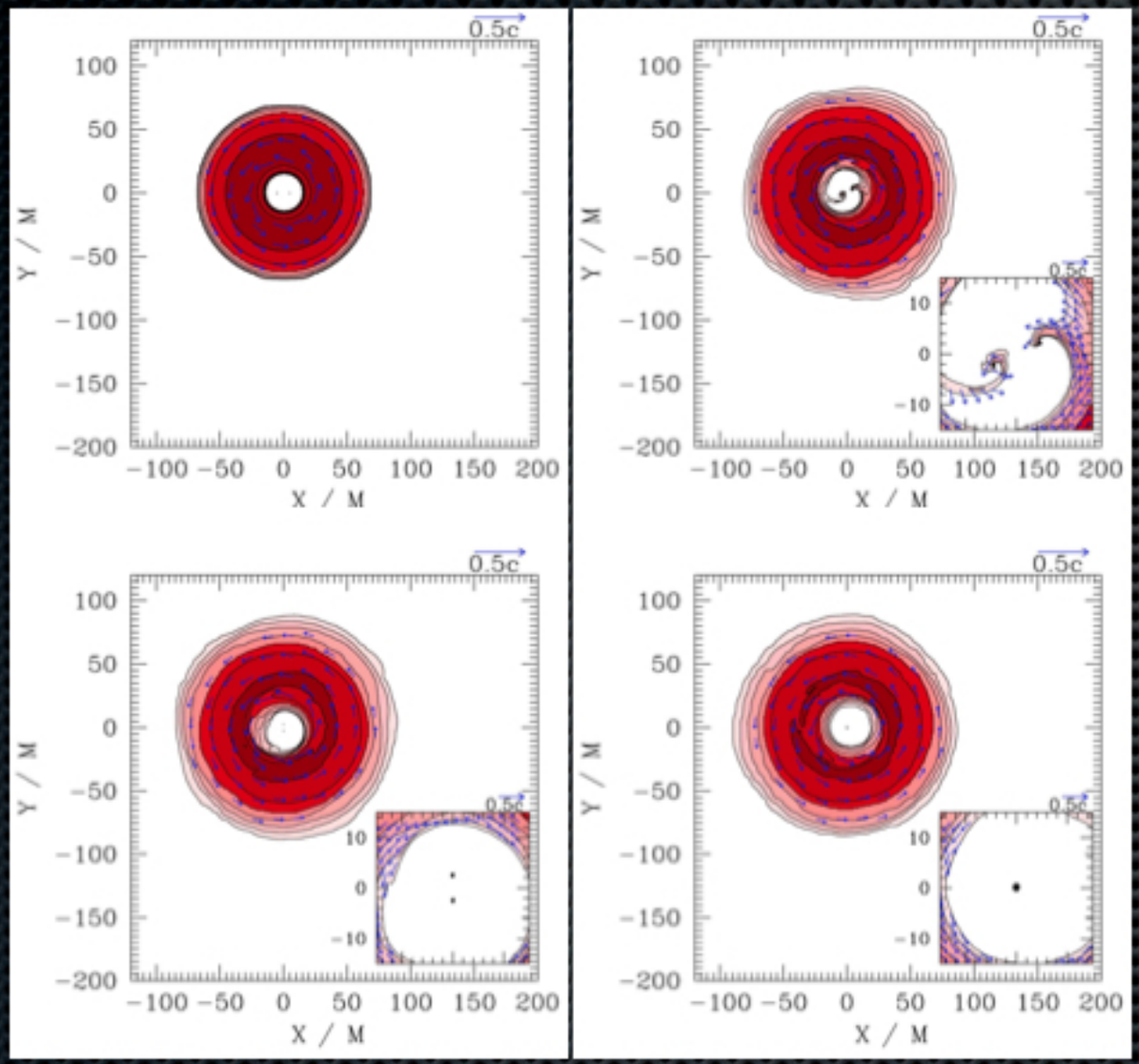
Boosted Temperature





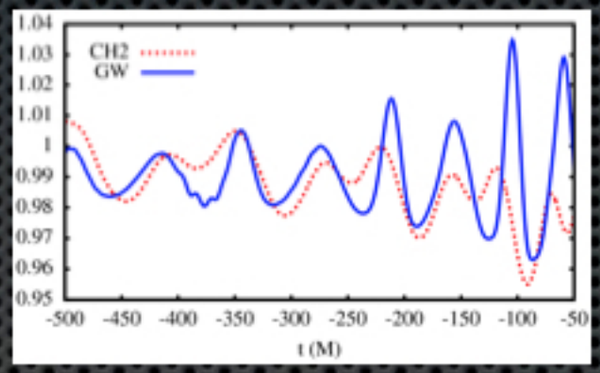
Farris, Liu & Shapiro 2011

- Full GR, circumbinary disk
- Hydro, no cooling, inviscid
- Thin disks

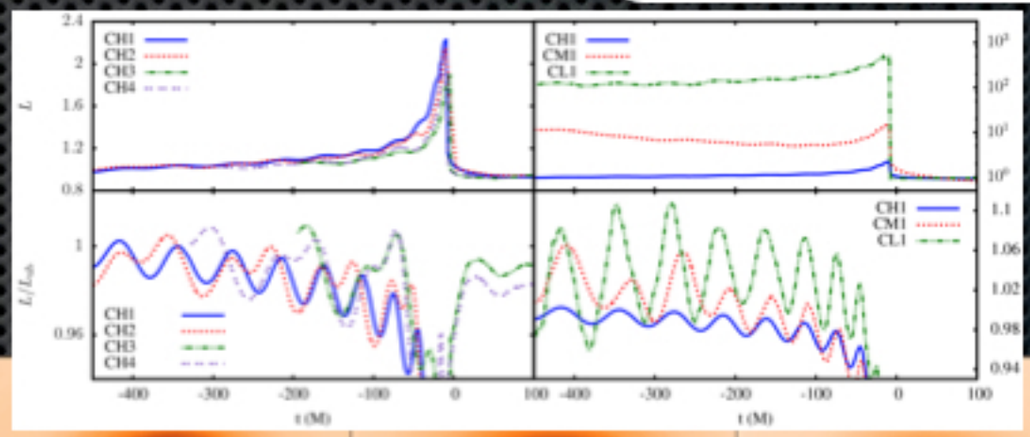


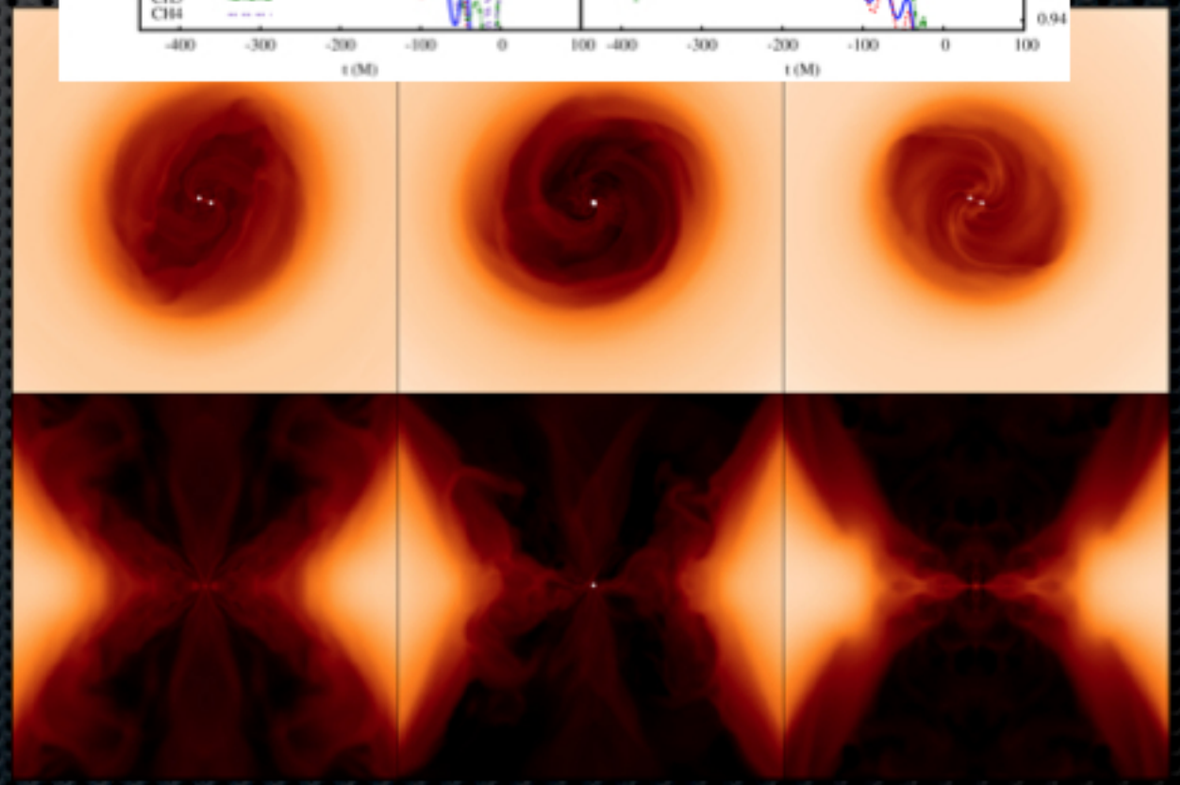
Bode++ 2011

- Full GR, circumbinary disk
- Hydro, no cooling, inviscid
- Thick disks



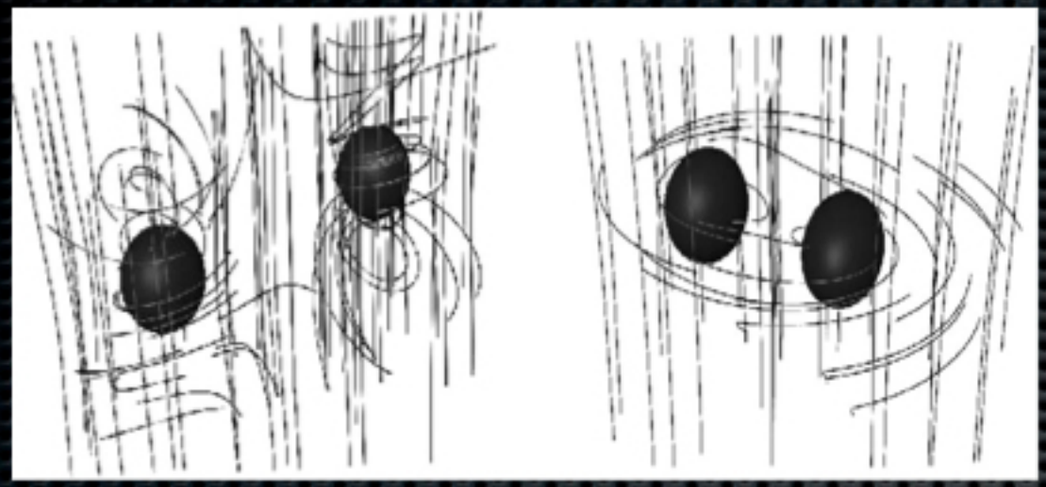
Case	q	$\vec{a}_1/m_1$	$\vec{a}_2/m_2$	T_0(K)
CH1	1	(0, 0, 0.6)	(0, 0, 0.6)	10^{12}
CM1	1	(0, 0, 0.6)	(0, 0, 0.6)	10^{11}
CL1	1	(0, 0, 0.6)	(0, 0, 0.6)	10^{10}
CH2	1/2	(0, 0, 0.6)	(0, 0, 0.6)	10^{12}
CH3	1/2	(−0.40, 0.44, −0.02)	(−0.16, 0.54, −0.21)	10^{12}
CH4	1/2	(−0.35, −0.47, 0.10)	(0.28, 0.44, 0.30)	10^{12}
				h/r
DA1	1	(0, 0, 0.6)	(0, 0, 0.6)	0.2
DB1	1	(0, 0, 0.6)	(0, 0, 0.6)	0.4
DC1 *	1	(0, 0, 0.6)	(0, 0, 0.6)	0.2
DA2	1/2	(0, 0, 0.6)	(0, 0, 0.6)	0.2
DA3	1/2	(−0.40, 0.44, −0.02)	(−0.16, 0.54, −0.21)	0.2
DA4	1/2	(−0.35, −0.47, 0.10)	(0.28, 0.44, 0.30)	0.2

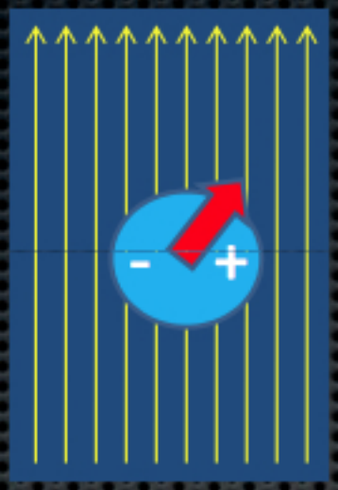




BBH Merger in Magnetic Field

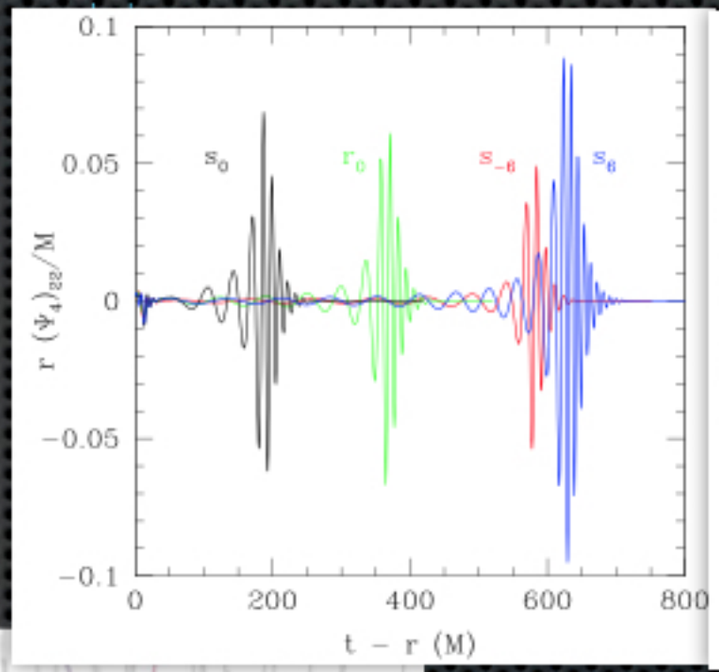
Palenzuela et al 2009

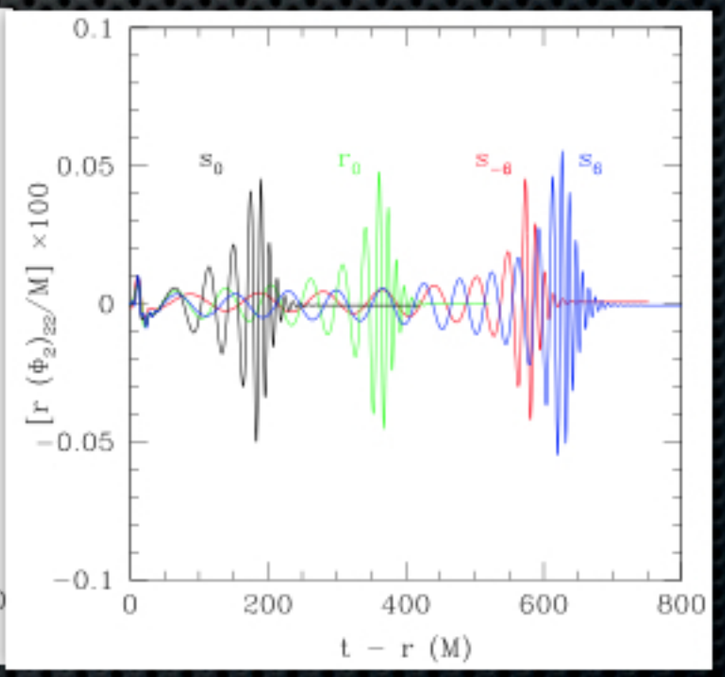




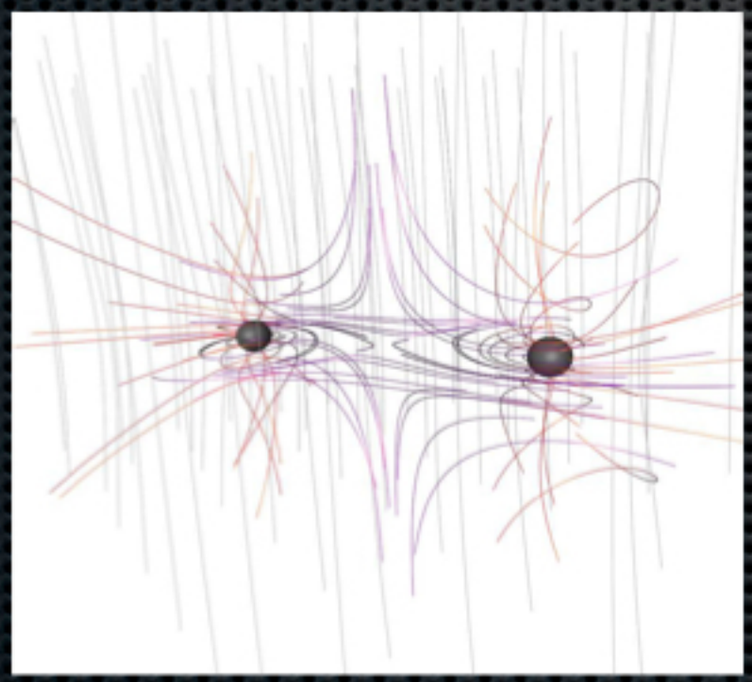
GW

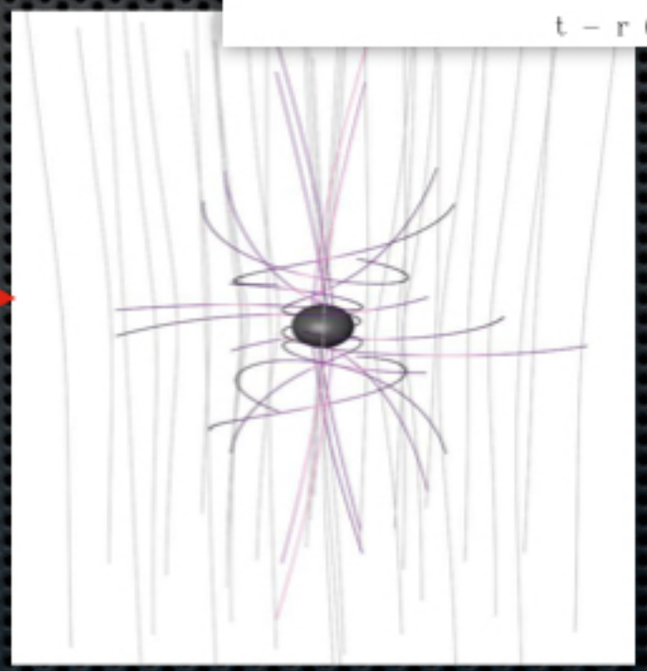
EM





Mosta et al 2009



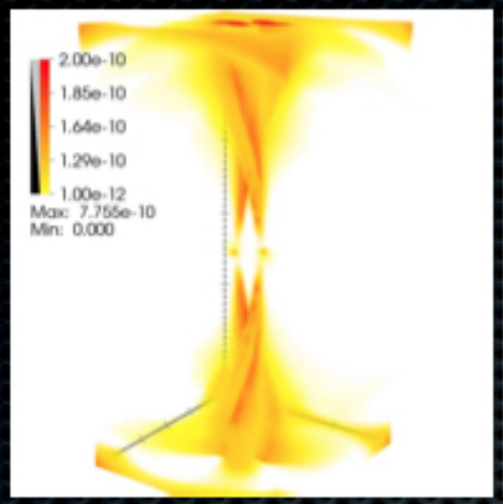


$$\frac{E_{\rm EM}^{\rm rad}}{M} \simeq 10^{-15}\left(\frac{M}{10^8\,M_\odot}\right)^2\left(\frac{B}{10^4\,{\rm G}}\right)^2$$

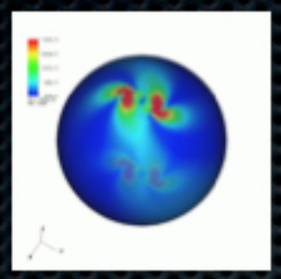
$$L_{\rm EM} \equiv \frac{E_{\rm EM}^{\rm rad}}{\tau} \simeq 10^{-4}\left(\frac{B}{10^4\,{\rm G}}\right)^2 L_{\rm Edd}$$

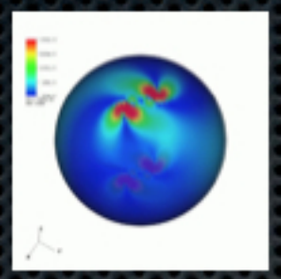
$$\nu = 10^{-4}(10^8 M_\odot/M){\rm Hz}$$

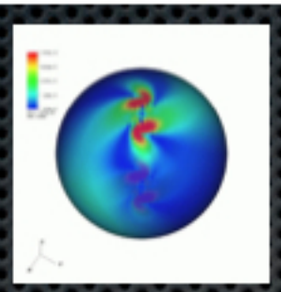
Palenzuela++ 2010, 2010

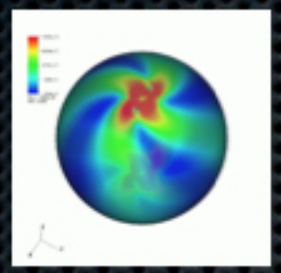


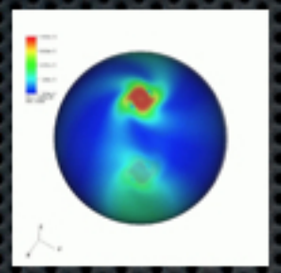


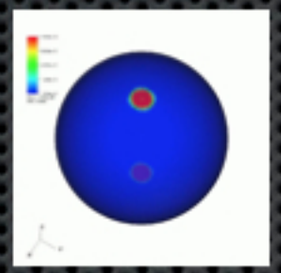




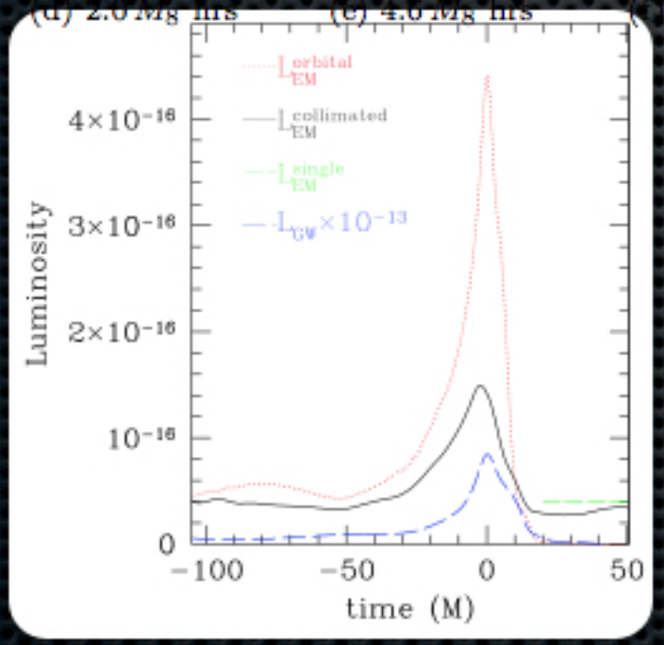




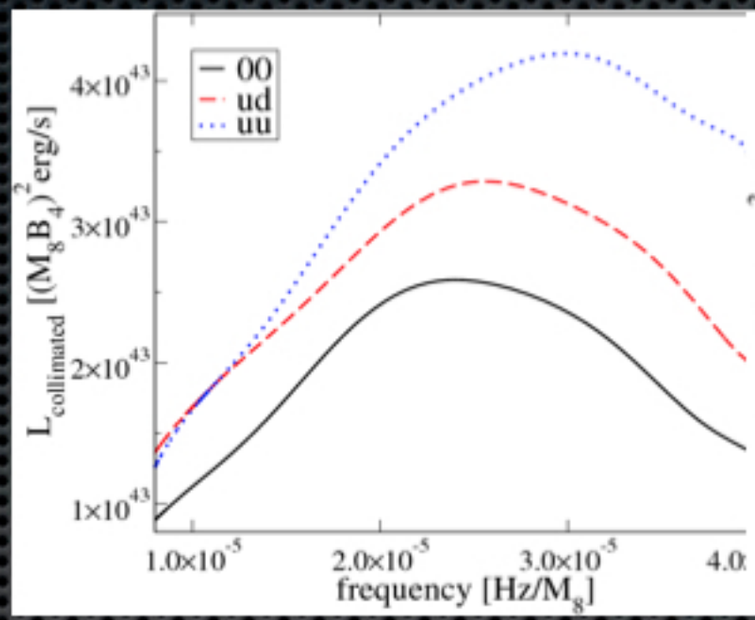


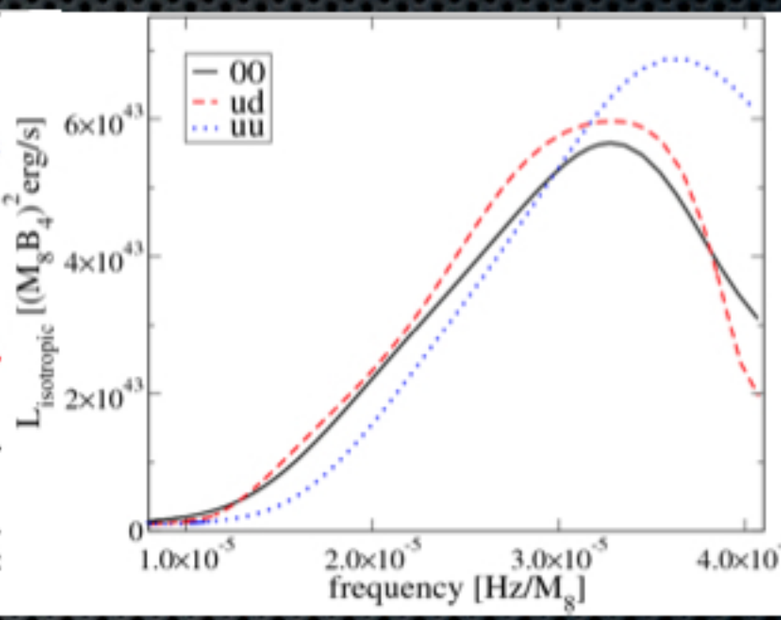


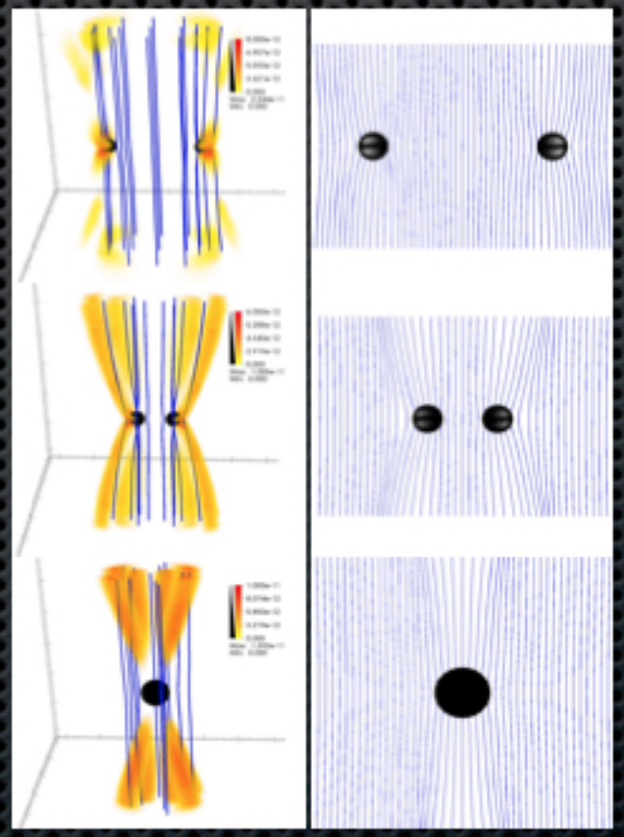
(d) 2.0 M_8 hrs (e) 4.0 M_8 hrs (f) 6.8 M_8 hrs



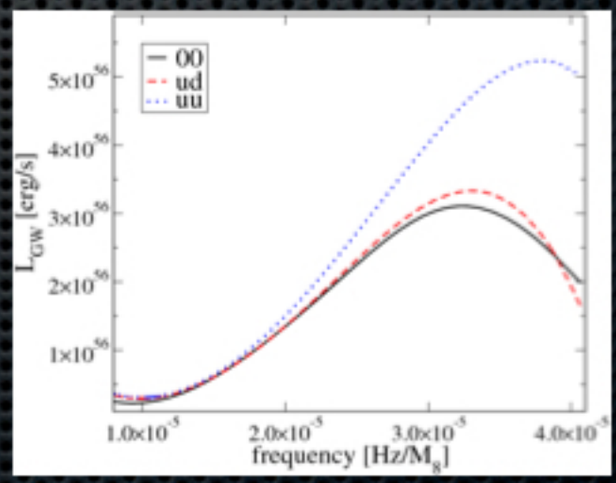
BBH Merger in FF Plasma: Jets







Head-on BBH



Neilsen++ 2011

Prior Work:

Refs.	Gravity Model	Matter Model	Code	Algorithm	Resolved Horizons	Timescales
Farris++	GR	Hydro Cloud (cold) Hydro Disk	UI's Cactus	Eulerian, HRSC	YES	$t_{disk} > t_{shrink}$
Bode++	GR	Hydro Cloud (hot) Hydro Disk	ET/Cactus	Eulerian, HRSC	YES	$t_{disk} > t_{shrink}$
Palenzuela++	GR	EM & Force-free plasma	HAD & Whisky (w/ Mosta)	Eulerian, FD	YES	N/A
MacFadyen & Milosavljevic	Newtonian	(cold) Hydro Disk	FLASH	Eulerian, HRSC	NO	$t_{disk} < t_{shrink}$
Cuadra++	Newtonian, self-gravity	(cold) Hydro Disk	Gadget-2	SPH	NO	$t_{disk} < t_{shrink}$
Shi++	Newtonian	(cold) Hydro Disk	Zeus	Eulerian, FD	NO	$t_{disk} < t_{shrink}$

Acronyms: **UI** = Univ. of Illinois, **ET** = Einstein Toolkit, **HAD** = Hydro. ADaptive mesh refinement, **HRSC** = High-Resolution Shock-Capturing, **FD** = finite difference

Prior Work:

Refs.	Gravity Model	Matter Model	Code	Algorithm	Resolved Horizons	Timescales
Farris++	GR	Hydro Cloud (cold) Hydro Disk	UI's Cactus	Eulerian, HRSC	YES	$t_{disk} > t_{shrink}$
Bode++	GR	Hydro Cloud (hot) Hydro Disk	ET/Cactus	Eulerian, HRSC	YES	$t_{disk} > t_{shrink}$
Palenzuela++	GR	EM & Force-free plasma	HAD & Whisky (w/ Mosta)	Eulerian, FD	YES	N/A
Ours	2.5PN	(cool) MHD Disk	HARM3d	Eulerian, HRSC	NO (not yet)	$t_{disk} <=> t_{shrink}$
MacFadyen & Milosavljevic	Newtonian	(cold) Hydro Disk	FLASH	Eulerian, HRSC	NO	$t_{disk} < t_{shrink}$
Cuadra++	Newtonian, self-gravity	(cold) Hydro Disk	Gadget-2	SPH	NO	$t_{disk} < t_{shrink}$
Shi++	Newtonian	(cold) MHD Disk	Zeus	Eulerian, FD	NO	$t_{disk} < t_{shrink}$

Acronyms: **UI** = Univ. of Illinois, **ET** = Einstein Toolkit, **HAD** = Hydro. ADaptive mesh refinement, **HRSC** = High-Resolution Shock-Capturing, **FD** = finite difference

Why do we need GR?

- Most important near the black holes, though relativistic winds/jets may span large radial extent;
- Newtonian dynamics cannot easily fake relativistic dynamics;
  - lack of proper critical surfaces, horizons;
  - MHD Riemann solution is different;
  - lack of coupling between velocity components;
    - particularly important for relativistic shear flow;
  - Magnetic energy density should carry inertia;
- Relativistic effects critical for accurate radiation predictions

Why do we need Dynamic GR?

- Orbital dynamics hard to predict (e.g. spin-orbit effects);
  - PN can help at large binary separation;
  - Gravitational wave influence on gas/disk?
    - Maybe right before plunge?
  - Dynamical horizons affect innermost flow
  - time-dependent spacetime results in complicated redshifting environment (need Dyn GR ray-tracing)

ET/GRHydro Pros and Cons:

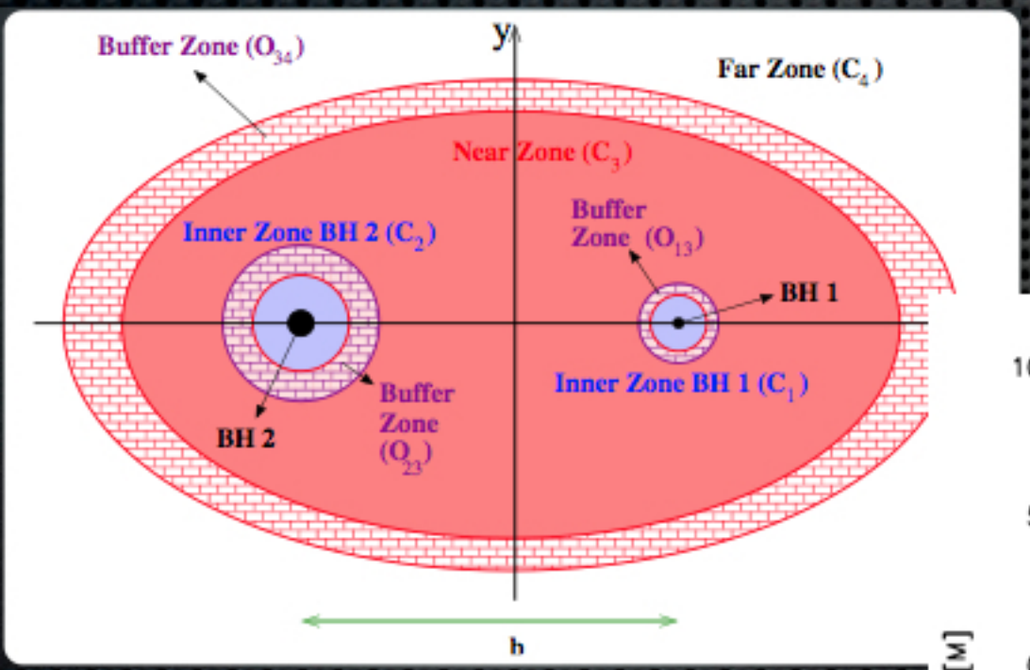
- Full dynamical GR
- In principle, able to do entire problem range;
- Block Structured AMR
- Mature code, used by many to study BBH mergers
- Hydro ready, MHD is almost there (with divergence cleaning)
- Can be slow:
  - Soln of Einstein's equations adds work;
  - Analysis modules (aka thorns) hinder good scaling to >256 cores
- Cartesian coordinates not well-suited to problem
- Excessive dissipation at AMR boundaries
- Hydro/MHD data analysis is not as mature

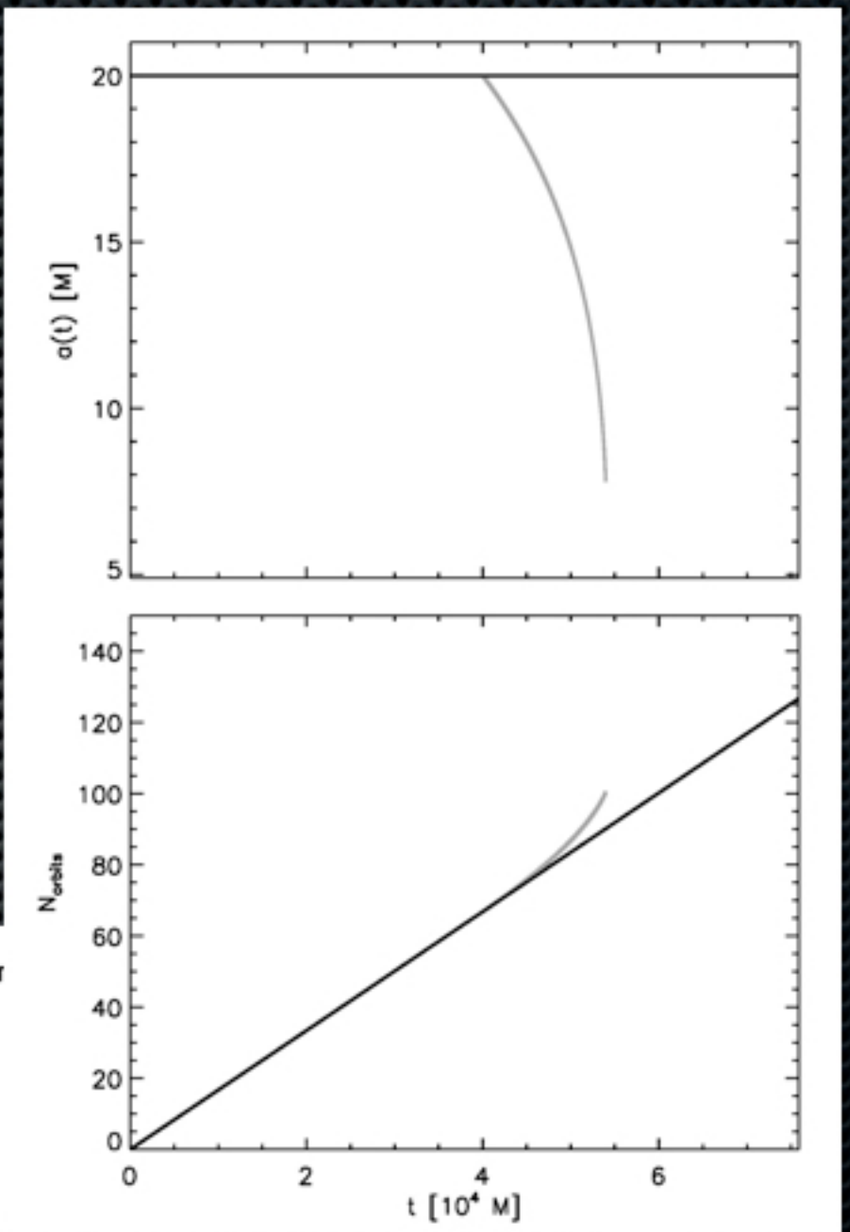
Harm3d Pros and Cons:

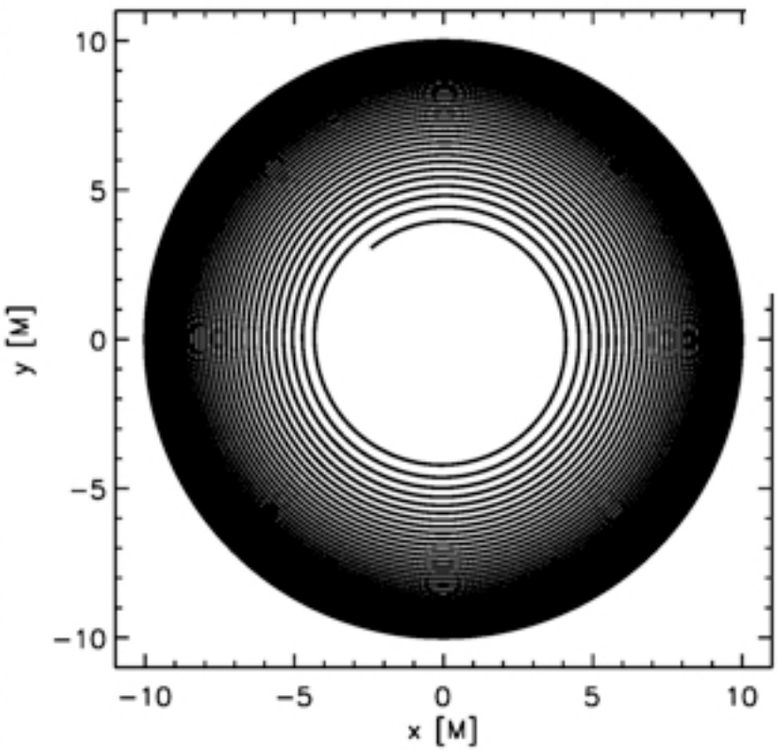
- 3d grMHD code, used primarily for single black hole disks
- Spherical coordinates available
  - well-suited to geometry of the disk (minimal numerical viscosity/ resistivity)
  - minimizes number of grid points
- Code scales well to >3000 cores
- Code is simple (to me) and more readily developable
- MHD ready to go
- 2 ray-tracing codes are “compatible” with its data format
- No dynamical GR (problem scope is limited)
- No AMR (though FMR, and potentially dynamical FMR, pseudo-Langrangian)

“2.5PN Spacetime” ~ BBH Spacetime Yunes++2006

- 2.5PN error --> O($(M/r)^3$) error
- 2.5PN metric used as initial data for Num. Rel.
- Rotation modeled as rigid rotations at PN’s $\Omega_{\rm bin}$
- a_0 = 20M
- Domain: r = [0.75 a_0, 13a_0] = [15M, 260M]
- Keep binary at fixed separation until t = 40,000M
- For t > 40,000M, let BHB inspiral according to PN







Harm3d

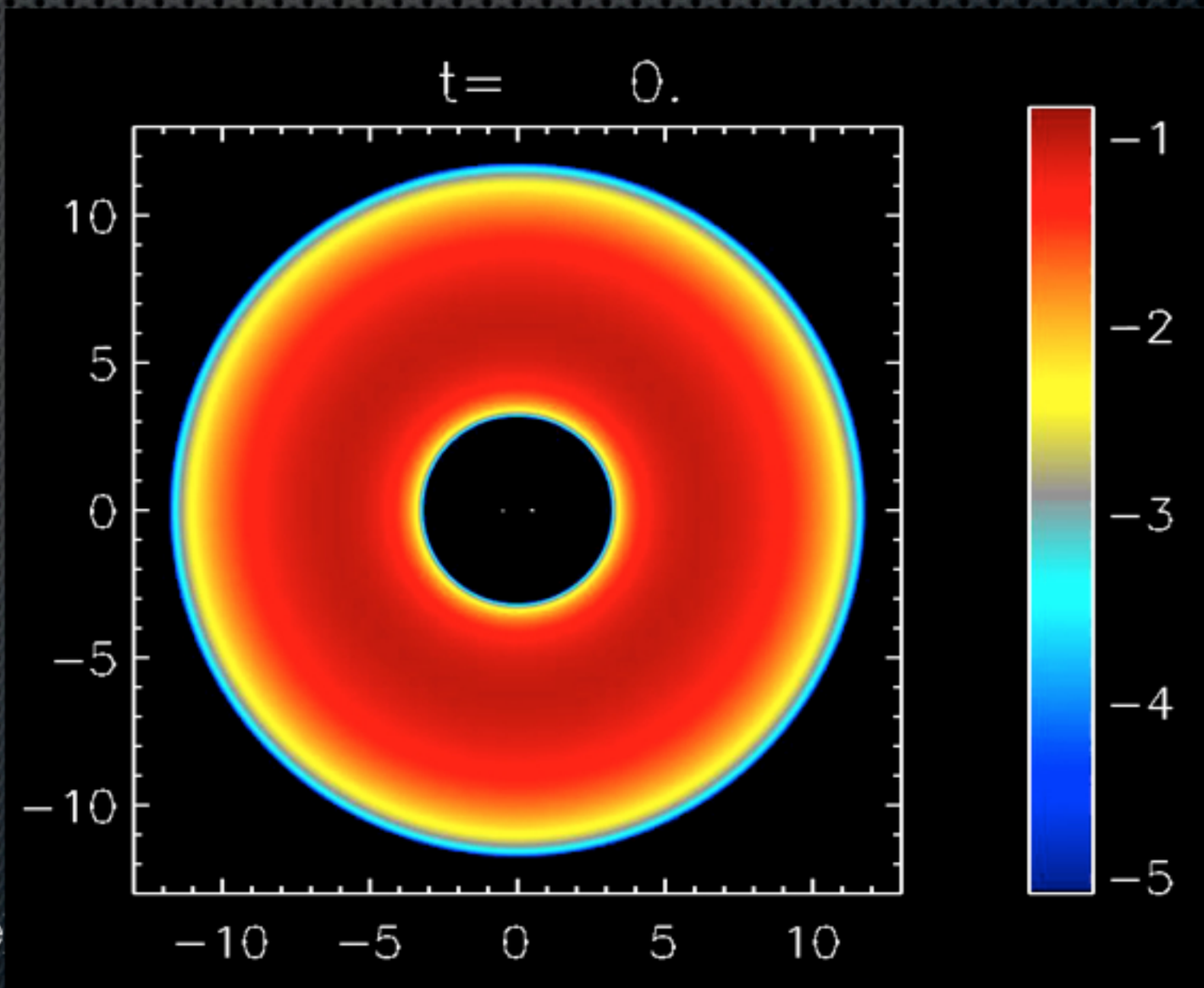
Harm3d written largely independent of chosen coordinate system (covariance)

- GRMHD code
- Assumes metric is static;
- Efficiency through static Fixed Mesh Refinement (FMR);

- Added support for non-analytic arbitrary metrics
  - Modified arbitrary connection calculation to handle time-dependent spacetimes;
  - Modified update schedule to update metric functions;
- Added entropy cooling function (Penna++2010)
- Spherical coordinates reduce numerical viscosity;
- Using FMR means that we lack extra dissipation at AMR boundaries (see EXTRA SLIDES)

Initial Data

- Disk extended over $r = [3\ a_0, 10\ a_0]$
- Pressure maximum $rp = 5\ a_0$
- Poloidal Magnet field following density contours
- Near "Equilibrium" Disk solution using time-averaged spacetime
- Cool to constant entropy s.t. H/r=0.1
- Cooling function can be used as an emissivity in a GR ray-tracing code....



Grid:

- Resolves the MRI
- Resolves the spiral density waves
- $\Delta r \sim r\Delta\theta \sim r\Delta\phi$
- Full azimuthal extent (resolve dominant m=1 mode
- Nr x Nth x Nph = [300,160,400]

Better Initial Data

Chakrabarti (1985)
De Villiers, Hawley, Krolik (2003)

How do we minimize the initial transient behavior of a circumbinary disk?

Our typical UBH data assumes metric with the same zero components as Boyer-Lindquist, and is typically only approximately "solved"

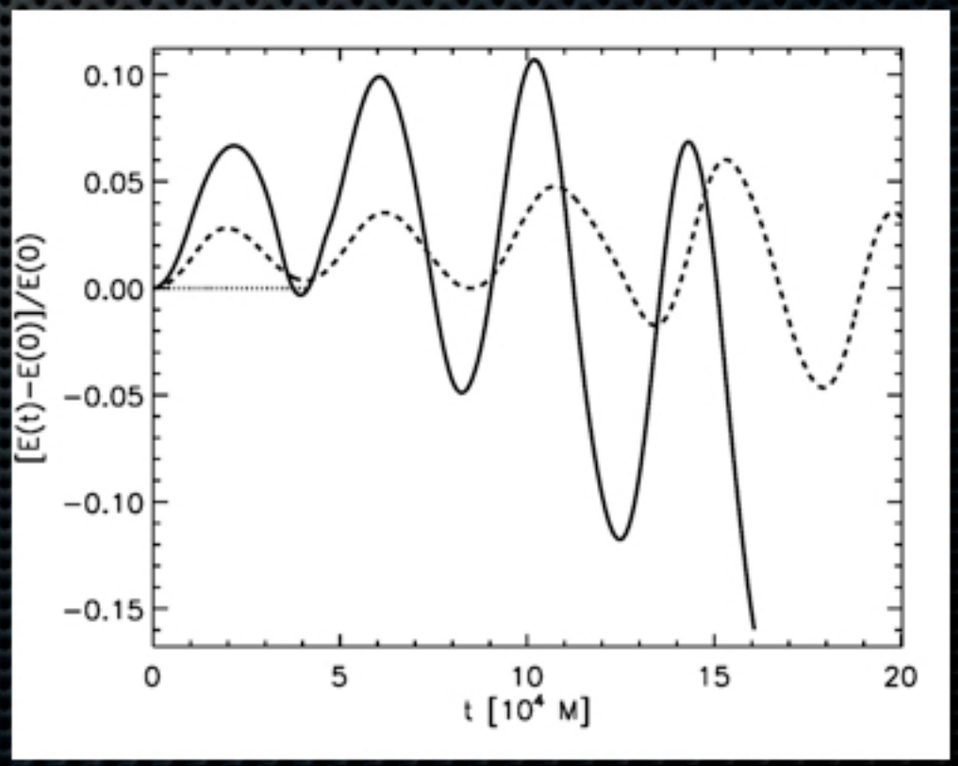
$$g_{\mu\nu} = \begin{bmatrix} g_{tt} & 0 & 0 & g_{t\phi} \\ 0 & g_{rr} & 0 & 0 \\ 0 & 0 & g_{\theta\theta} & 0 \\ g_{t\phi} & a & 0 & g_{\phi\phi} \end{bmatrix}$$

$$\bar{g}_{\mu\nu} = \frac{\int g_{\mu\nu}\sqrt{g_{\phi\phi}}d\phi}{\int \sqrt{g_{\phi\phi}}d\phi}$$

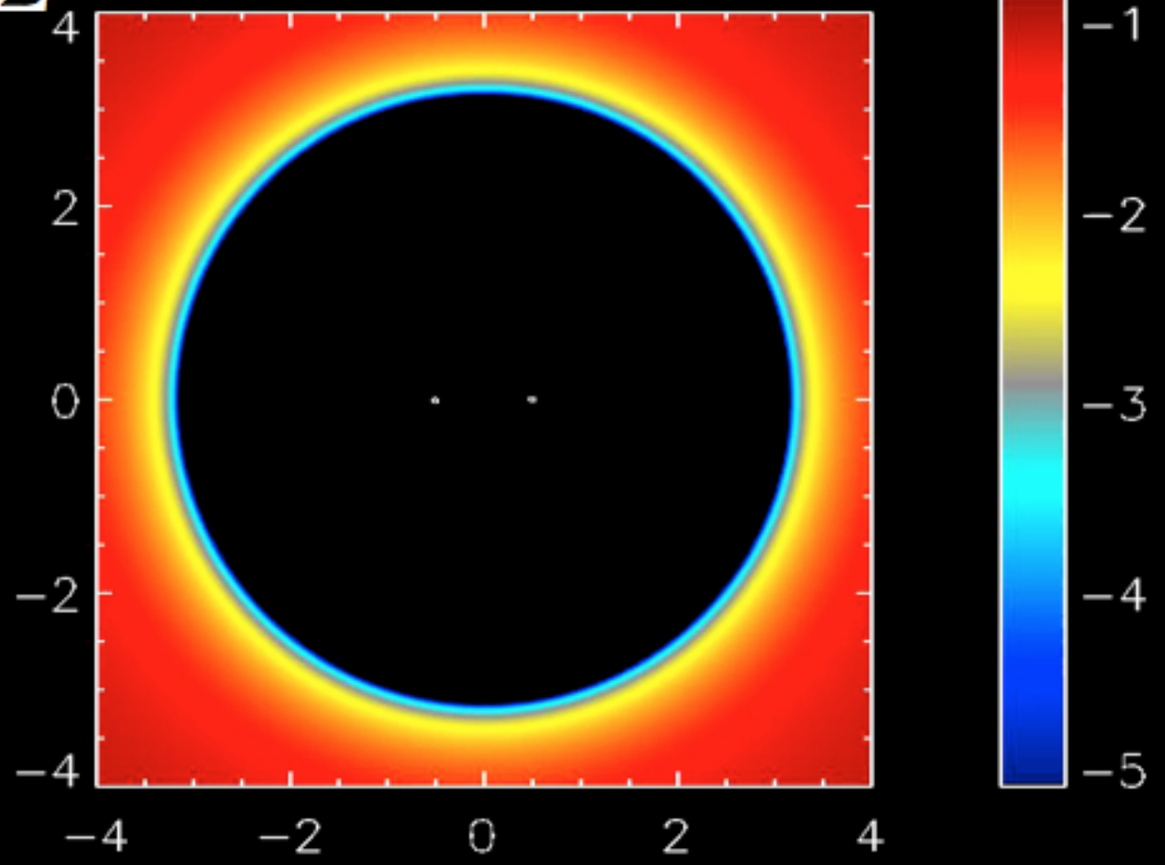
Need to "generalize" to arbitrary axi-sym spacetimes? Can we find "time-independent", "phi-symmetric" spacetime that approximates "time-averaged" BBH spacetime?

$$\bar{g}_{\mu\nu} = \begin{pmatrix} \bar{g}_{tt} & \bar{g}_{tr} & \bar{g}_{t\theta} & \bar{g}_{t\phi} \\ \bar{g}_{tr} & \bar{g}_{rr} & \bar{g}_{r\theta} & \bar{g}_{r\phi} \\ \bar{g}_{t\theta} & \bar{g}_{r\theta} & \bar{g}_{\theta\theta} & \bar{g}_{\theta\phi} \\ \bar{g}_{t\phi} & \bar{g}_{r\phi} & \bar{g}_{\theta\phi} & \bar{g}_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \bar{g}_{tt} & 0 & 0 & \bar{g}_{t\phi} \\ 0 & \bar{g}_{rr} & 0 & 0 \\ 0 & 0 & \bar{g}_{\theta\theta} & 0 \\ \bar{g}_{t\phi} & 0 & 0 & \bar{g}_{\phi\phi} \end{pmatrix}$$

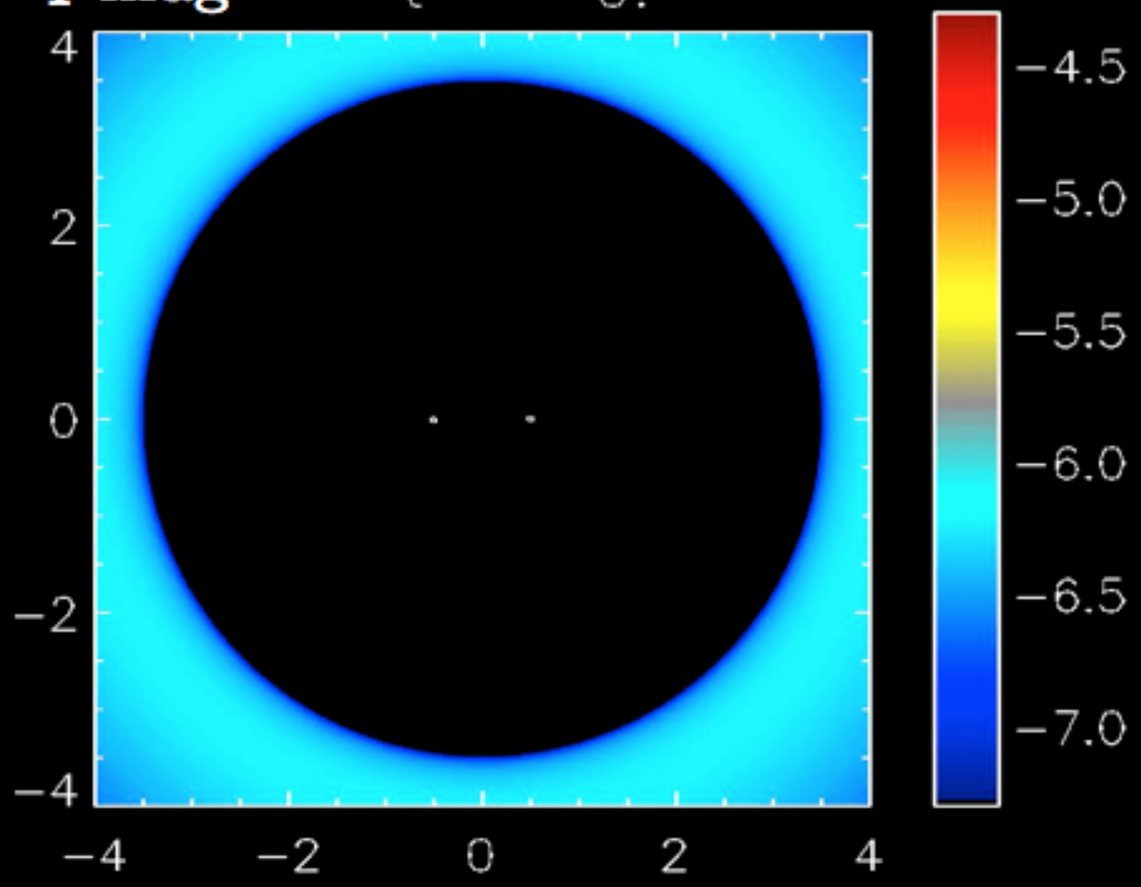
$$\Omega_{K\pm} = -\frac{1}{\partial_r g_{\phi\phi}}\left[\partial_r g_{t\phi} \pm \sqrt{(\partial_r g_{t\phi})^2 - (\partial_r g_{\phi\phi})(\partial_r g_{tt})}\right]$$



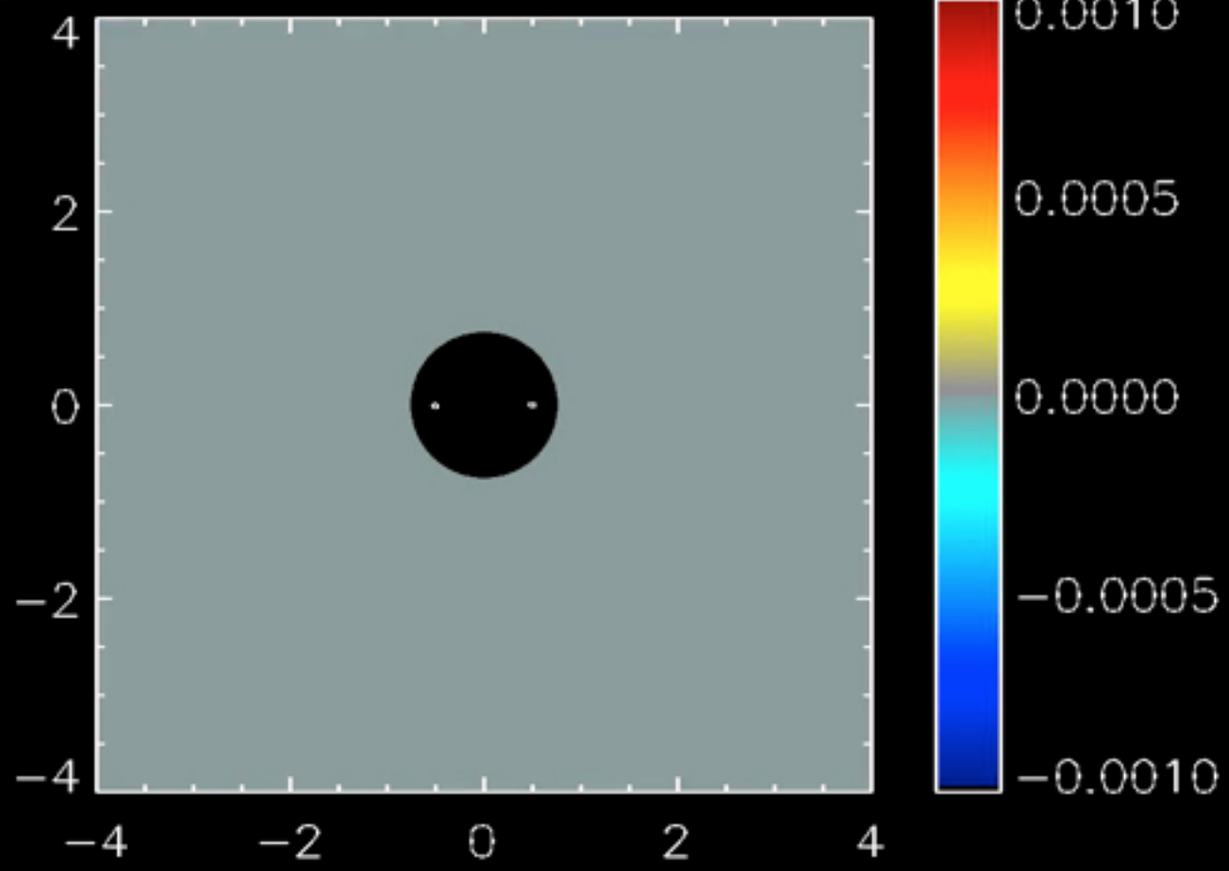
Σ t= 0.



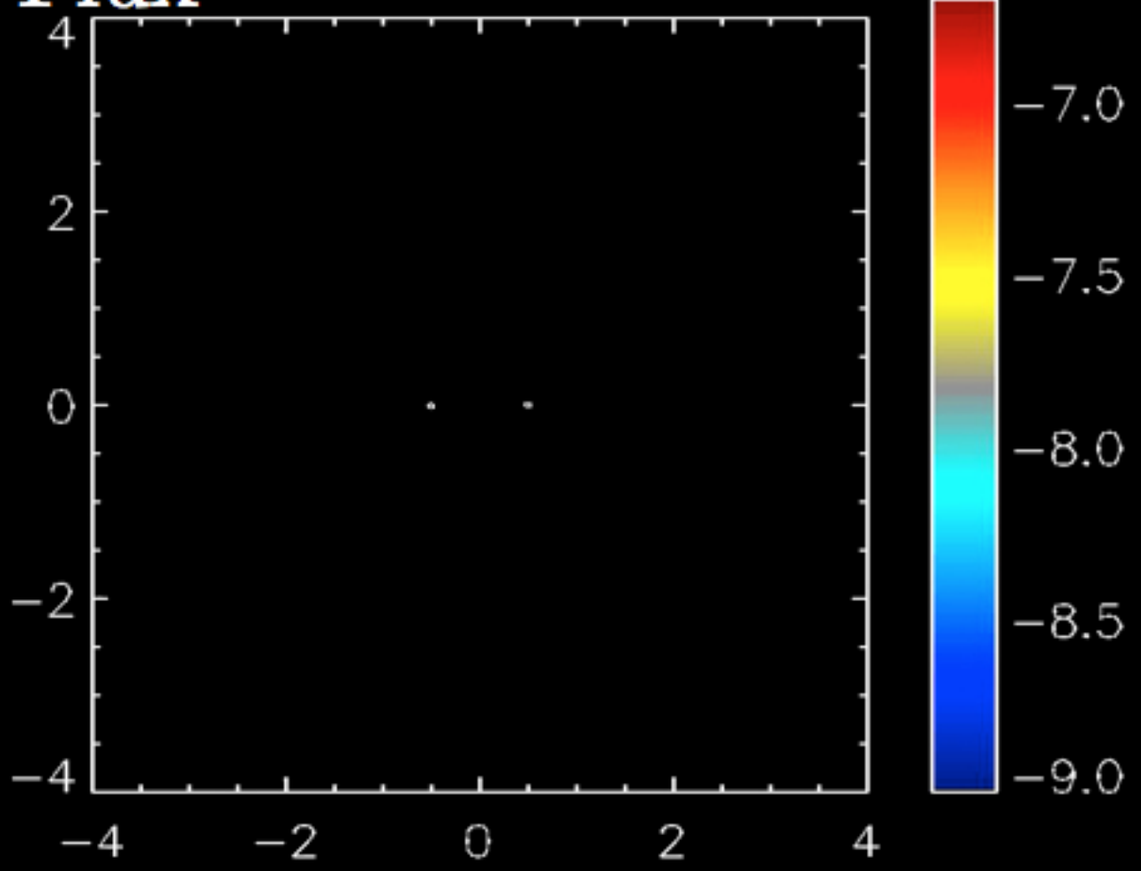
p_{mag} t= 0.



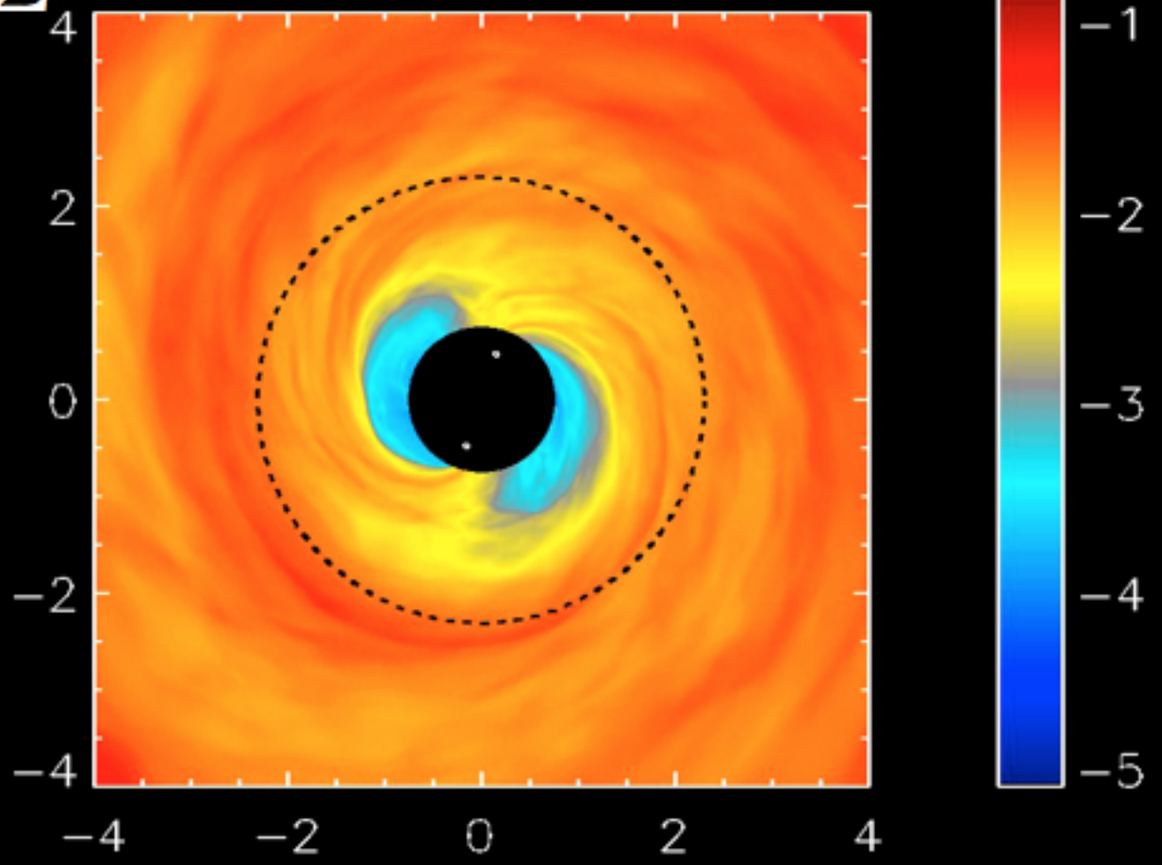
ρu^r t= 0.



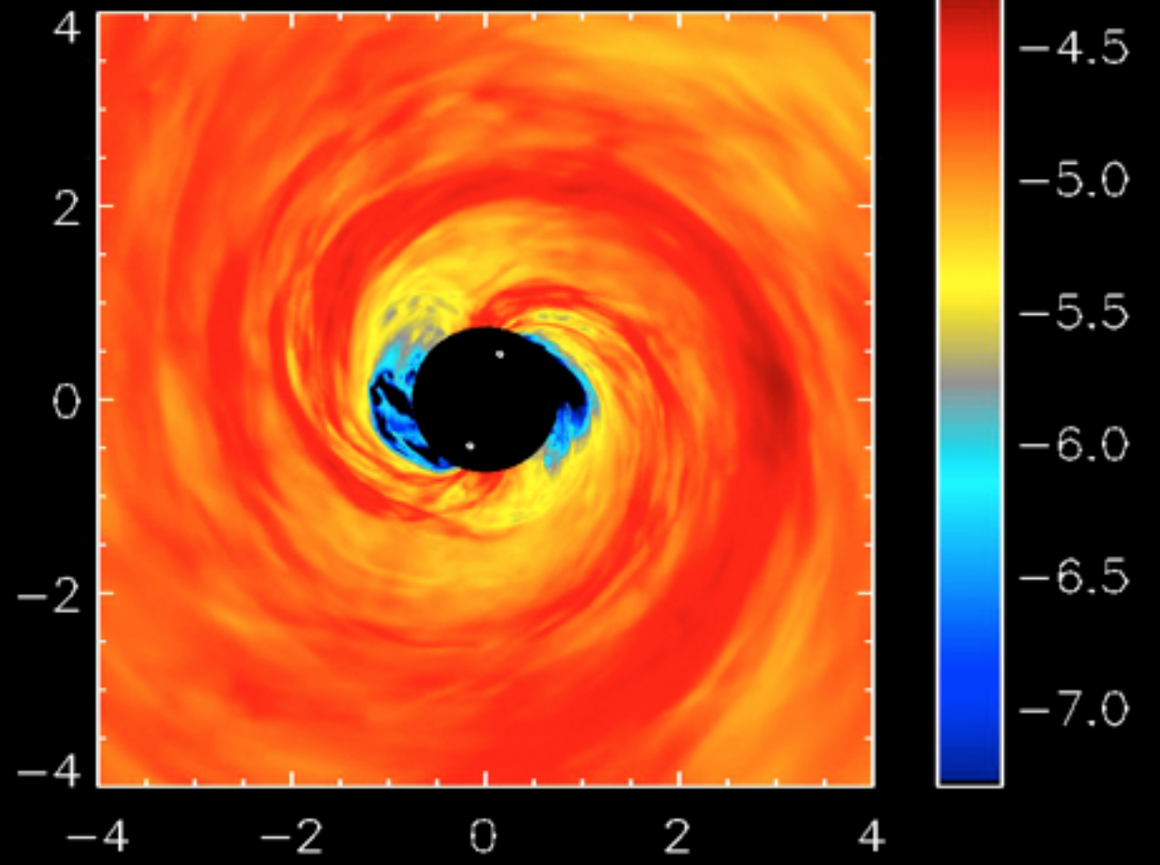
Flux t= 0.



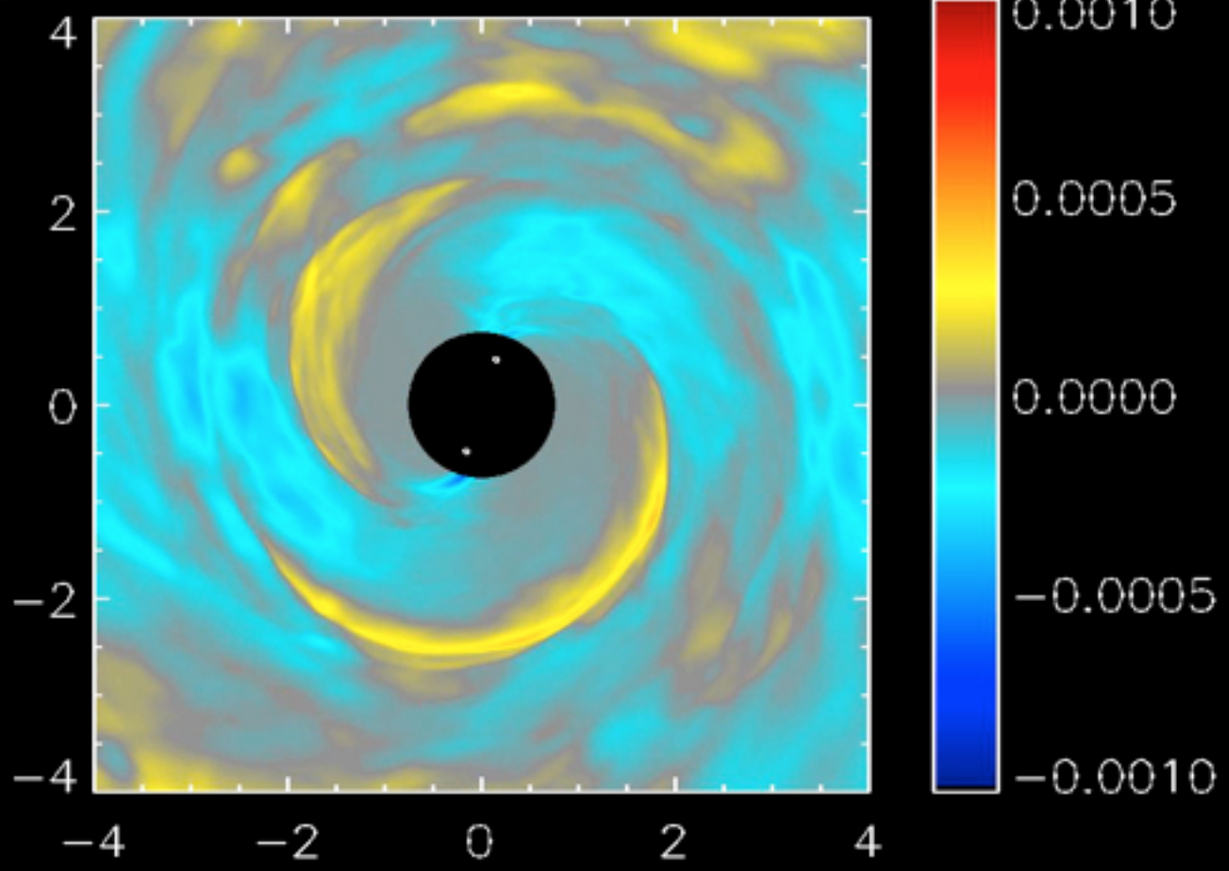
Σ t=30960.



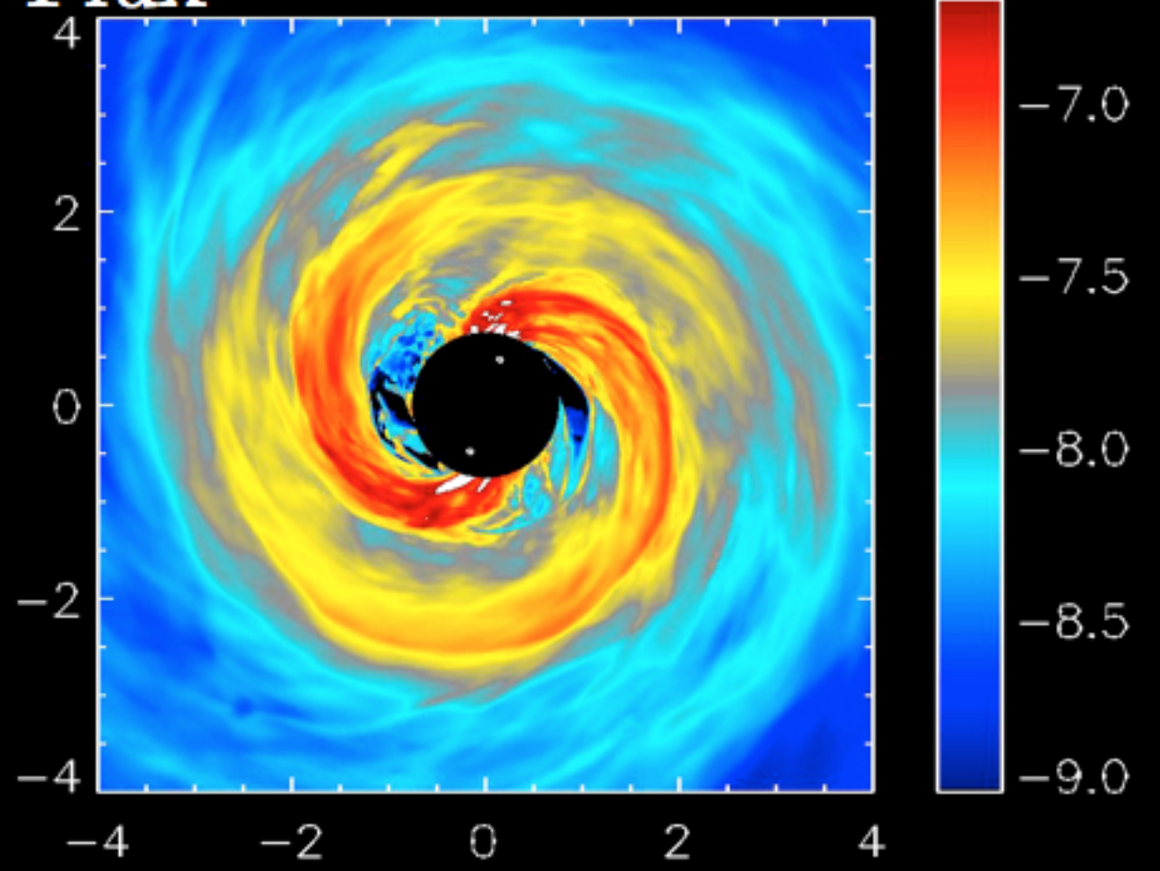
p_{mag} t=30960.



ρu^r t=30960.

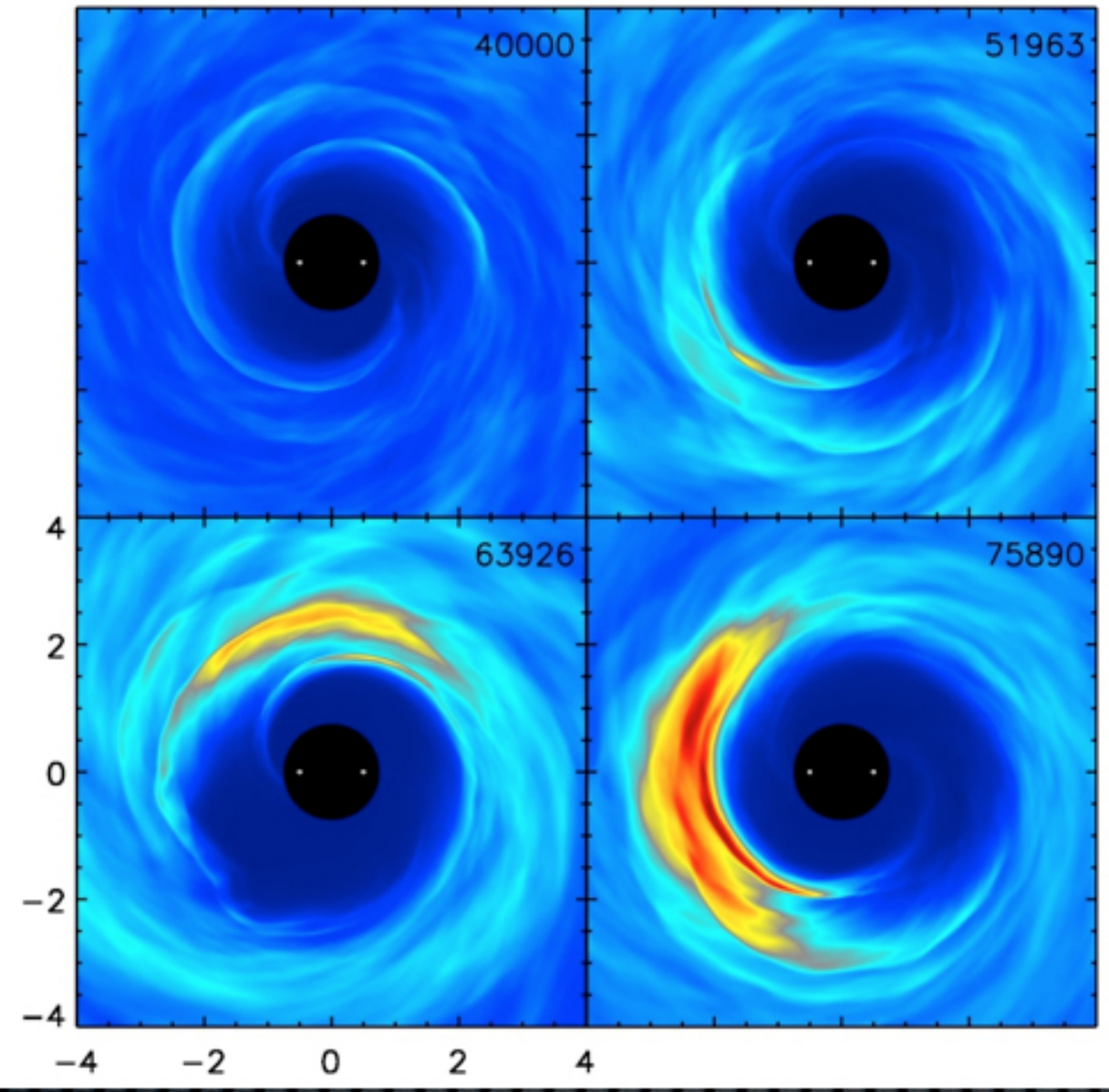


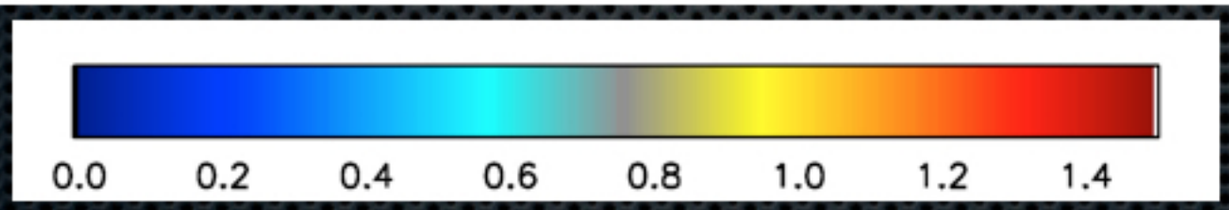
Flux t=30960.



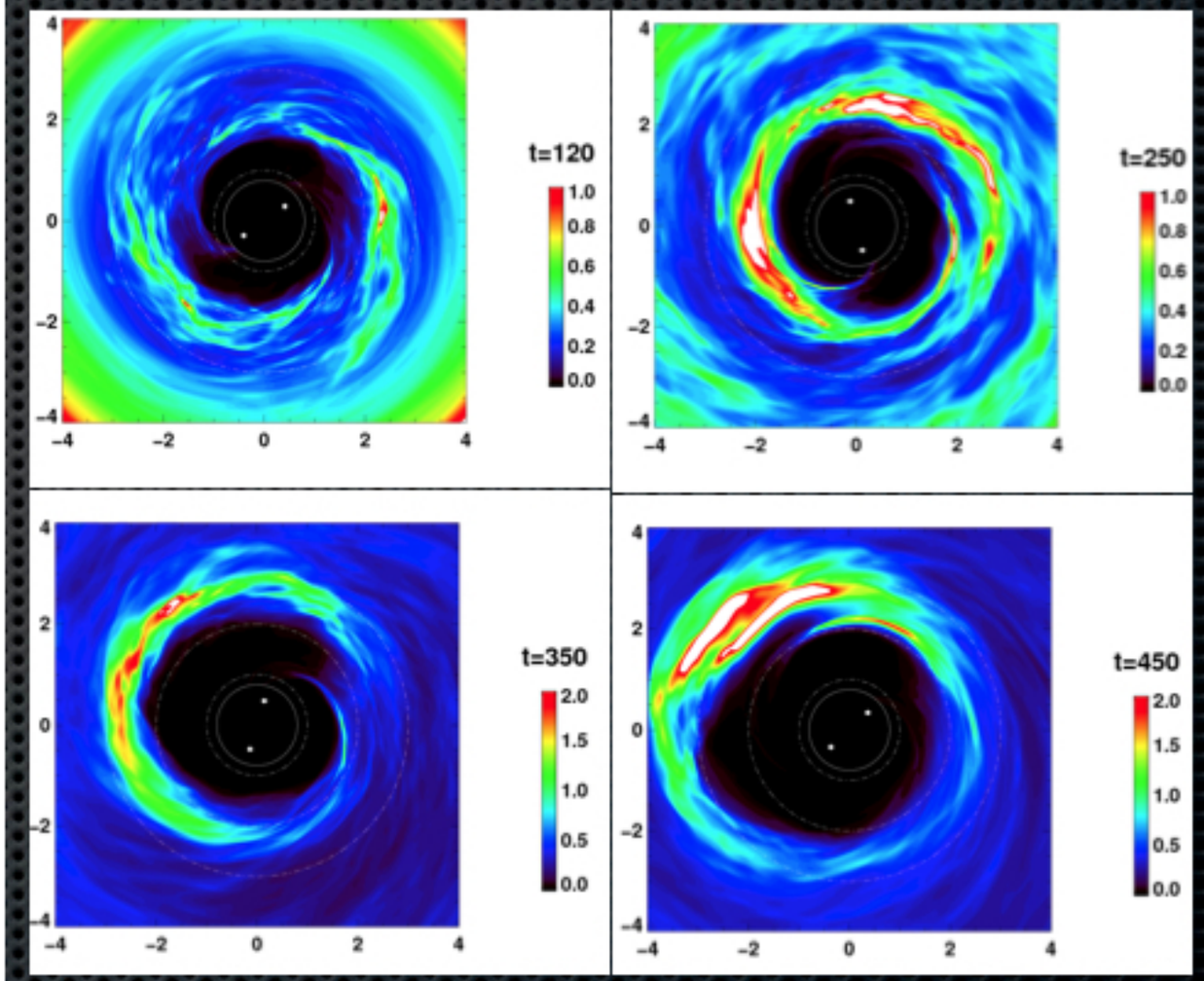
The "Lump"

$$\Sigma(r,\phi) \equiv \int d\theta \sqrt{-g}\rho / \sqrt{g_{\phi\phi}}$$





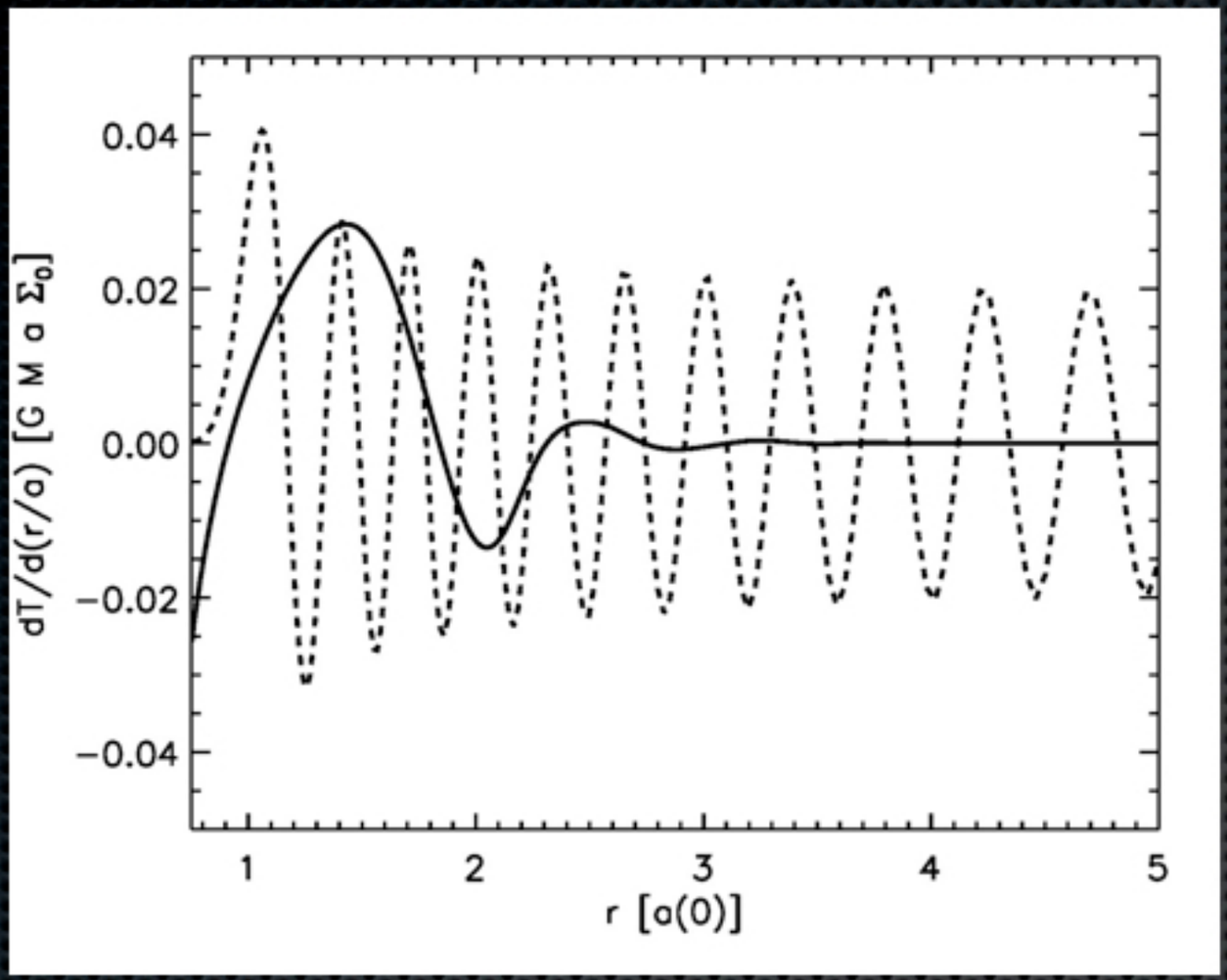
Ours

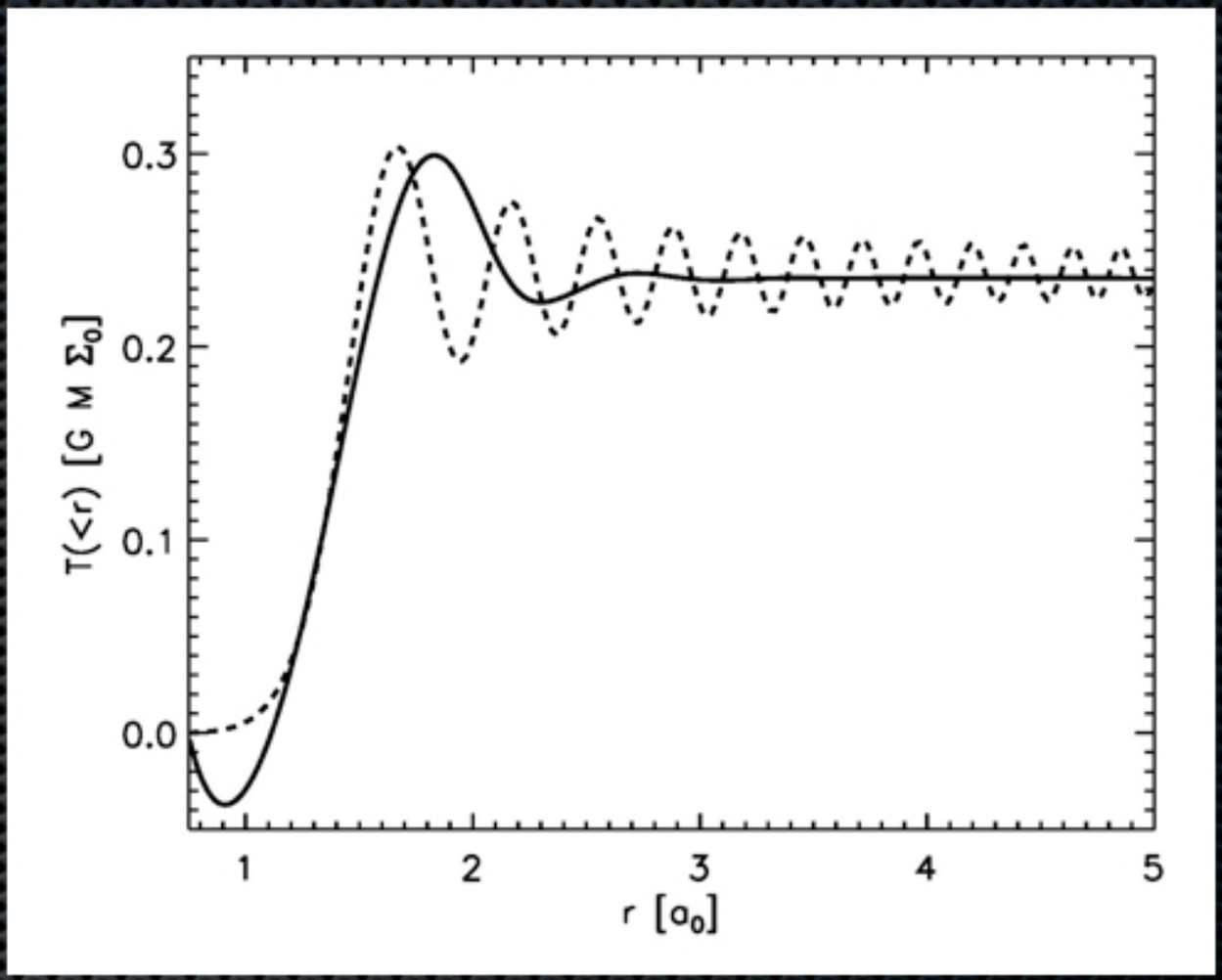


Newtonian MHD:
Shi++2012

Binary Torque Density

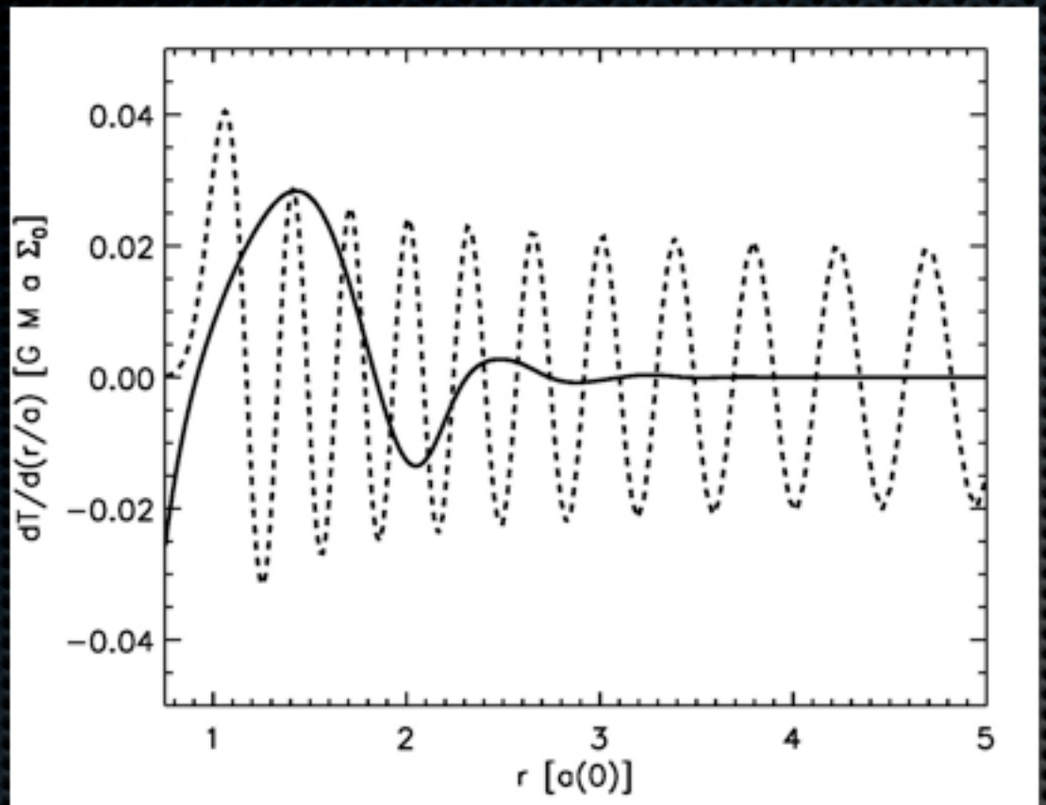
$$\frac{dT}{dr} = \int \sqrt{-g}\, T^{\mu}{}_{\nu} \Gamma^{\nu}{}_{\mu\phi}\, d\theta d\phi$$

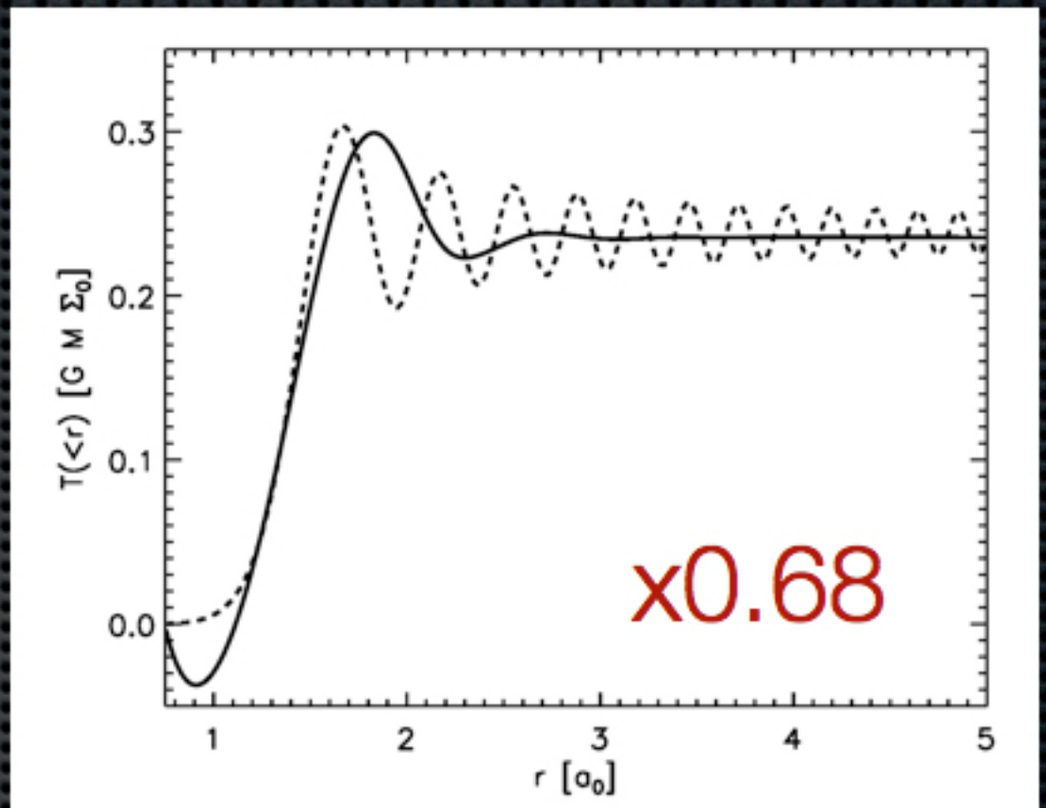


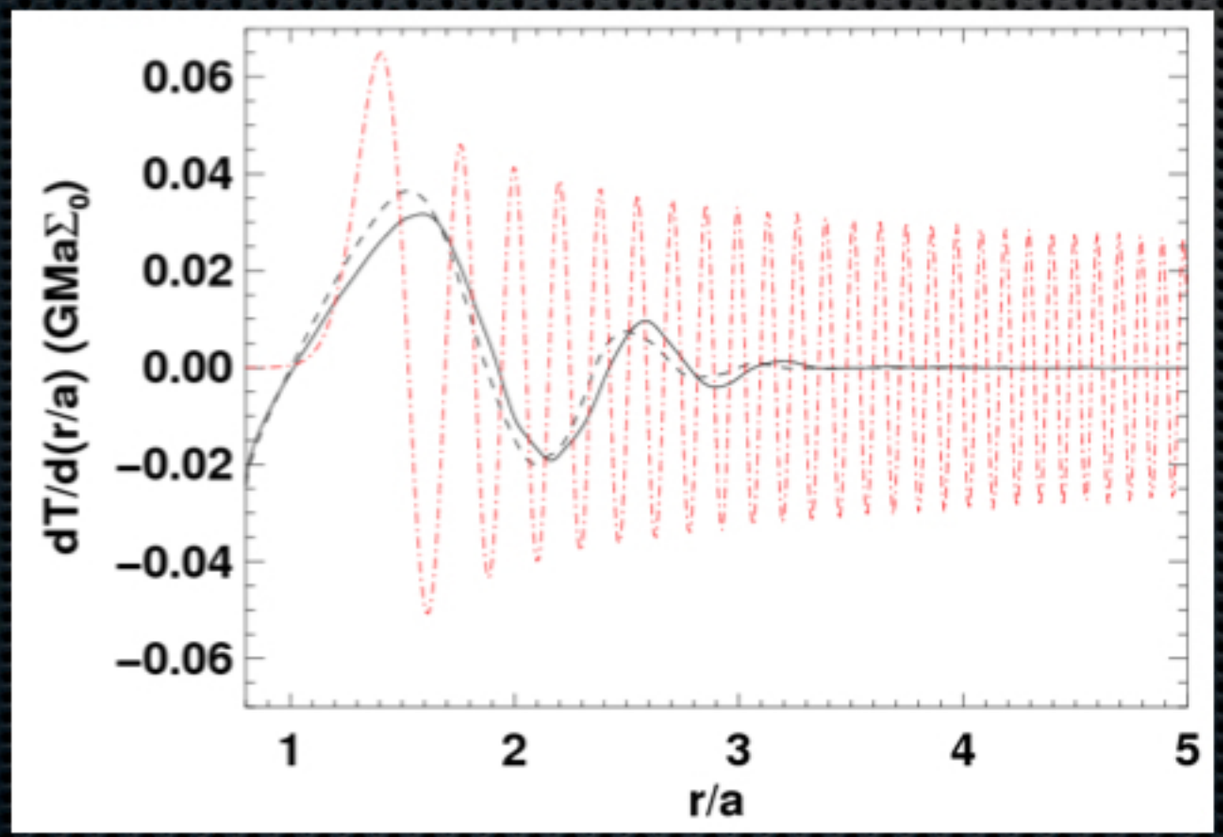


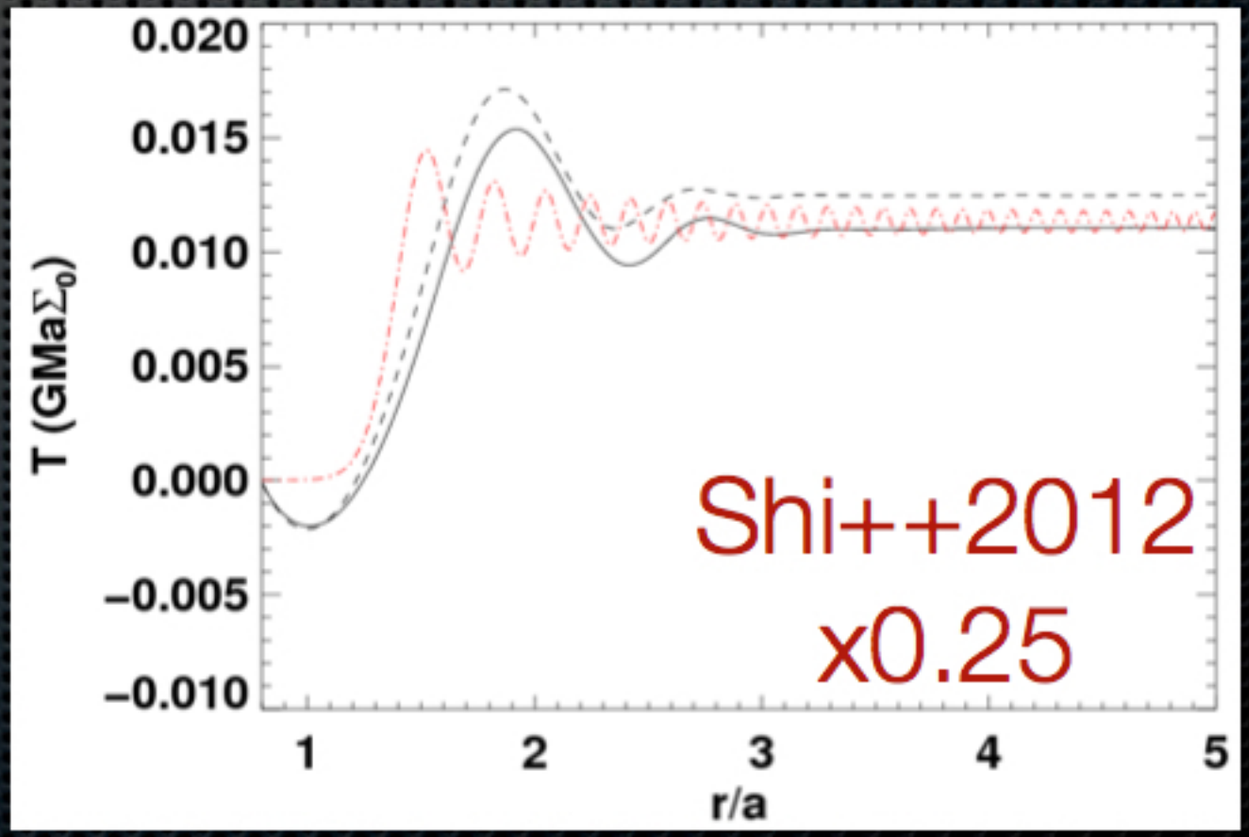
Binary Torque Density

$$\frac{dT}{dr} = \int \sqrt{-g}\, T^{\mu}{}_{\nu} \Gamma^{\nu}{}_{\mu\phi}\, d\theta d\phi$$



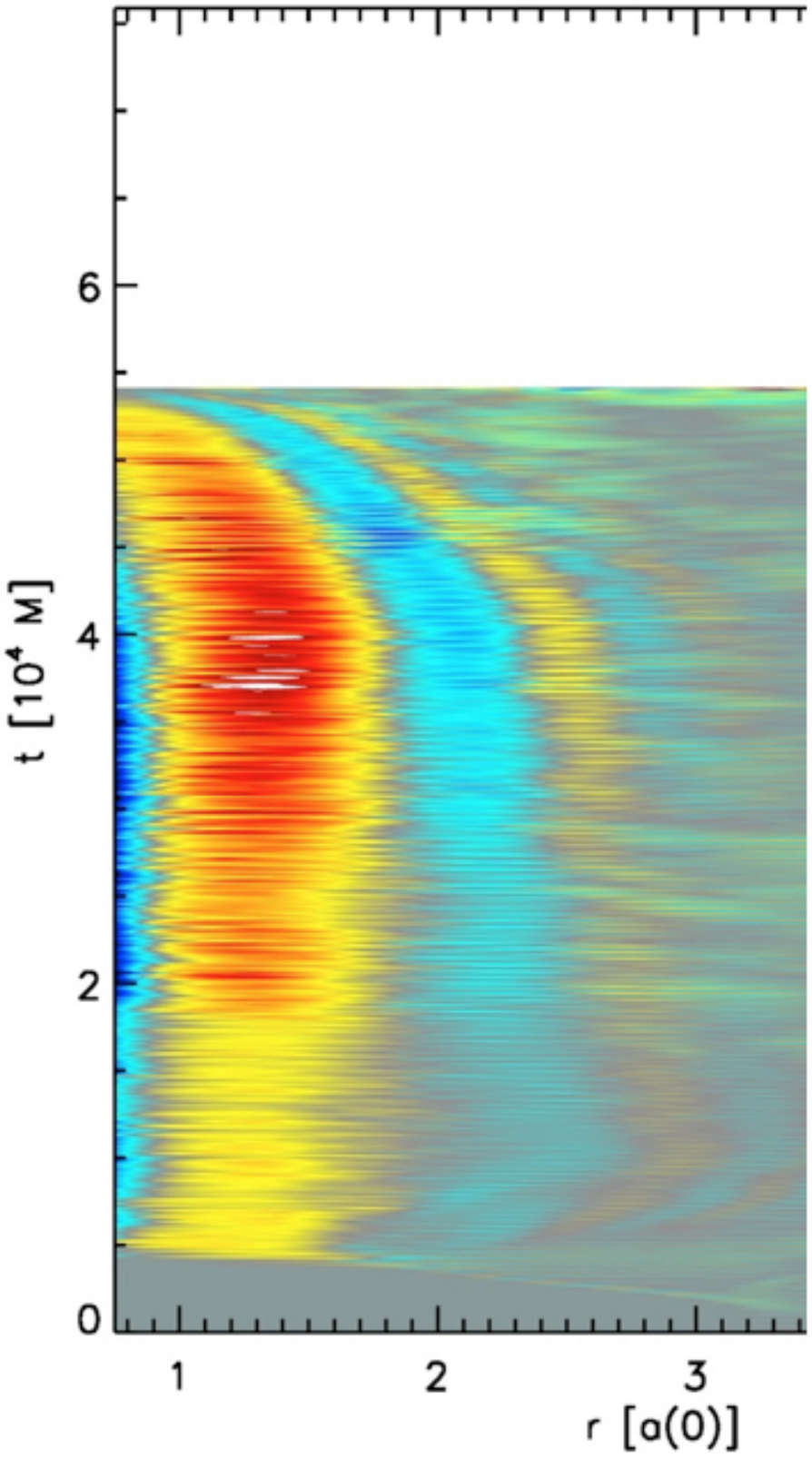


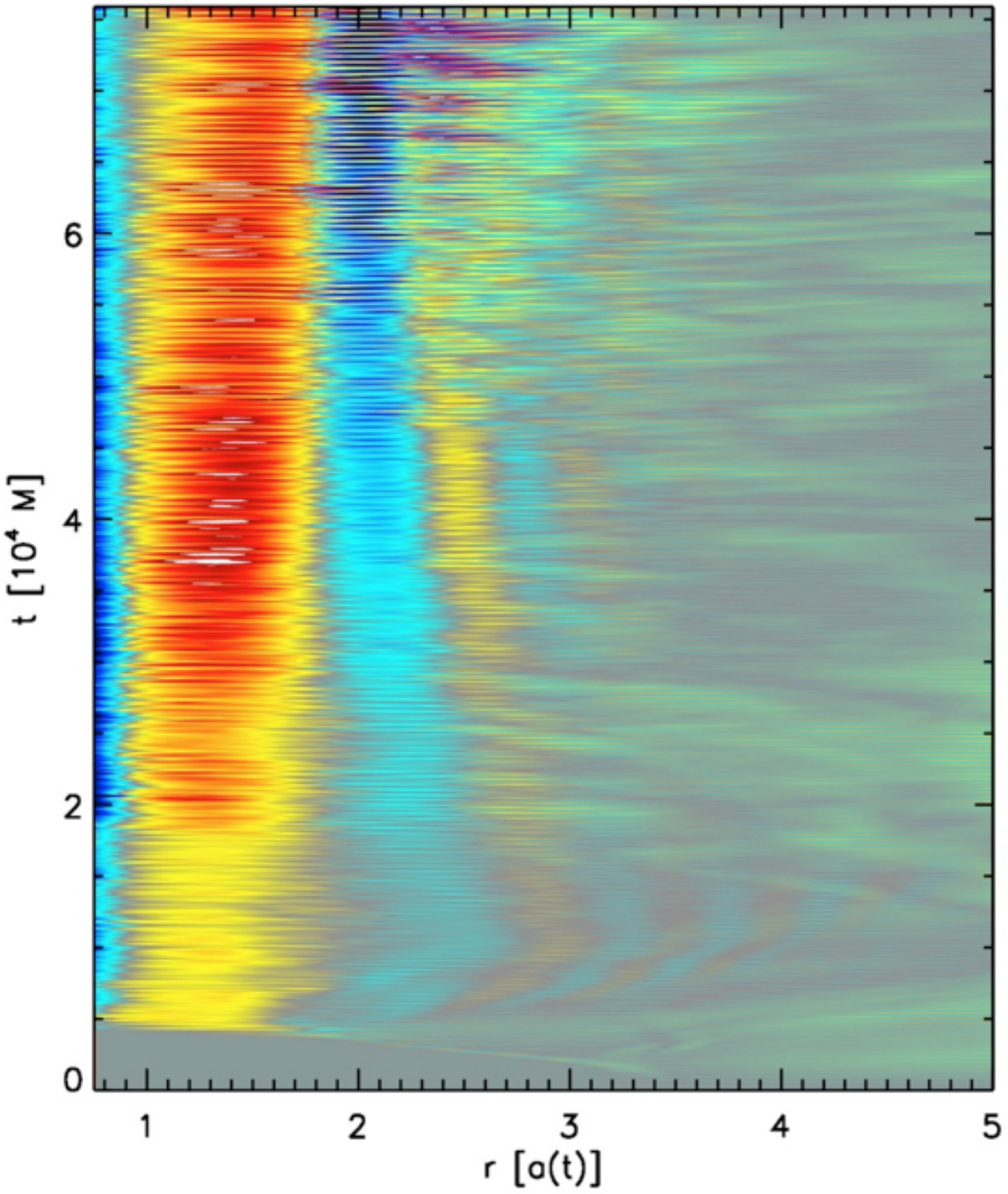


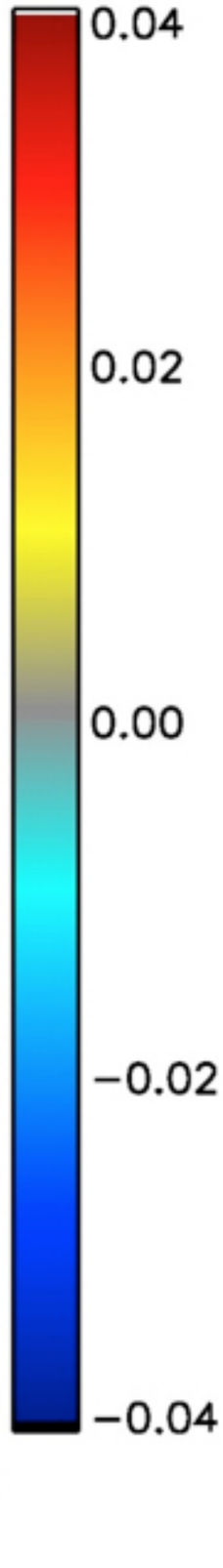


Binary Torque Density

$$\frac{dT}{dr} = \int \sqrt{-g}\, T^{\mu}{}_{\nu} \Gamma^{\nu}{}_{\mu\phi}\, d\theta d\phi$$

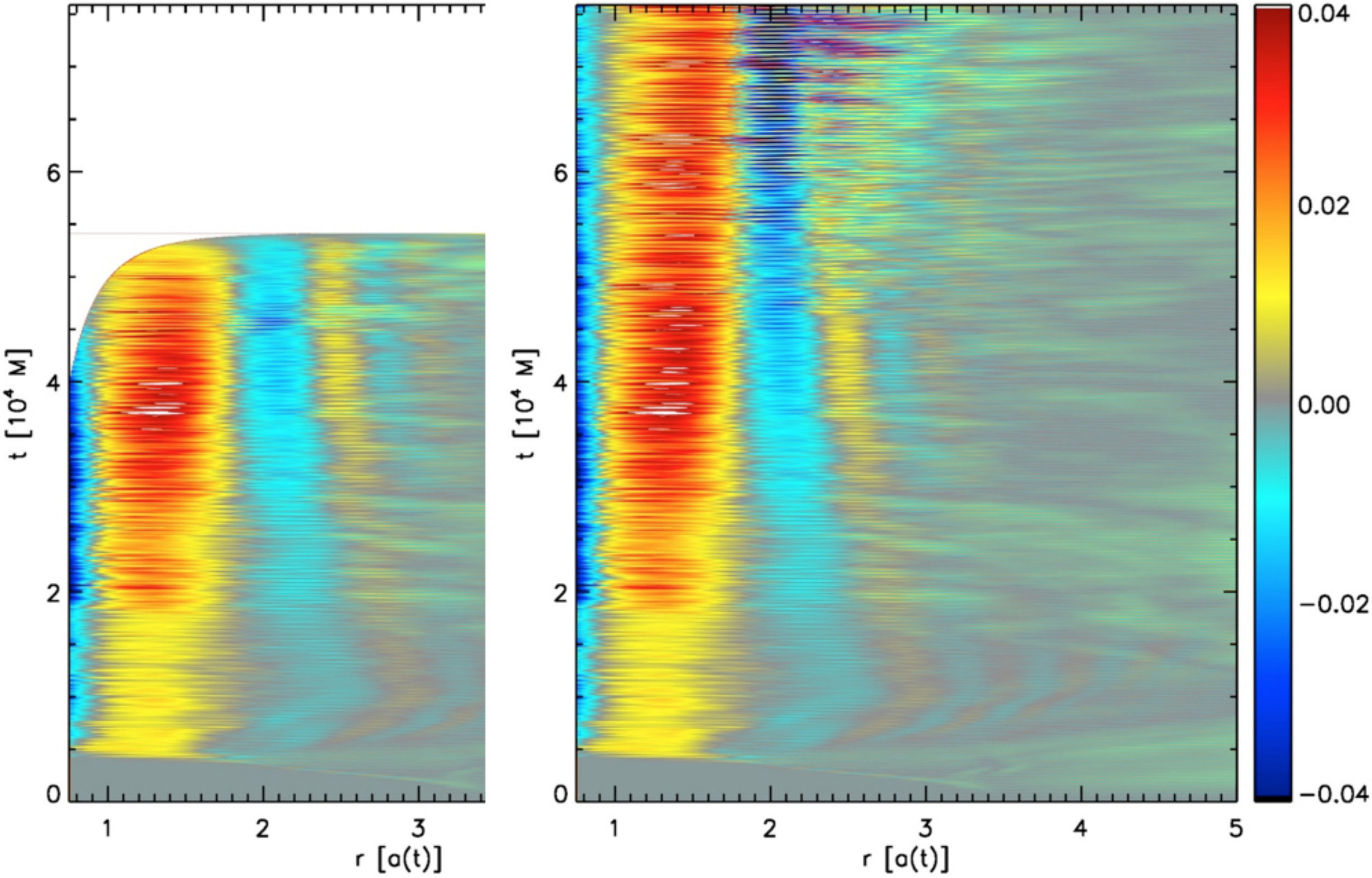




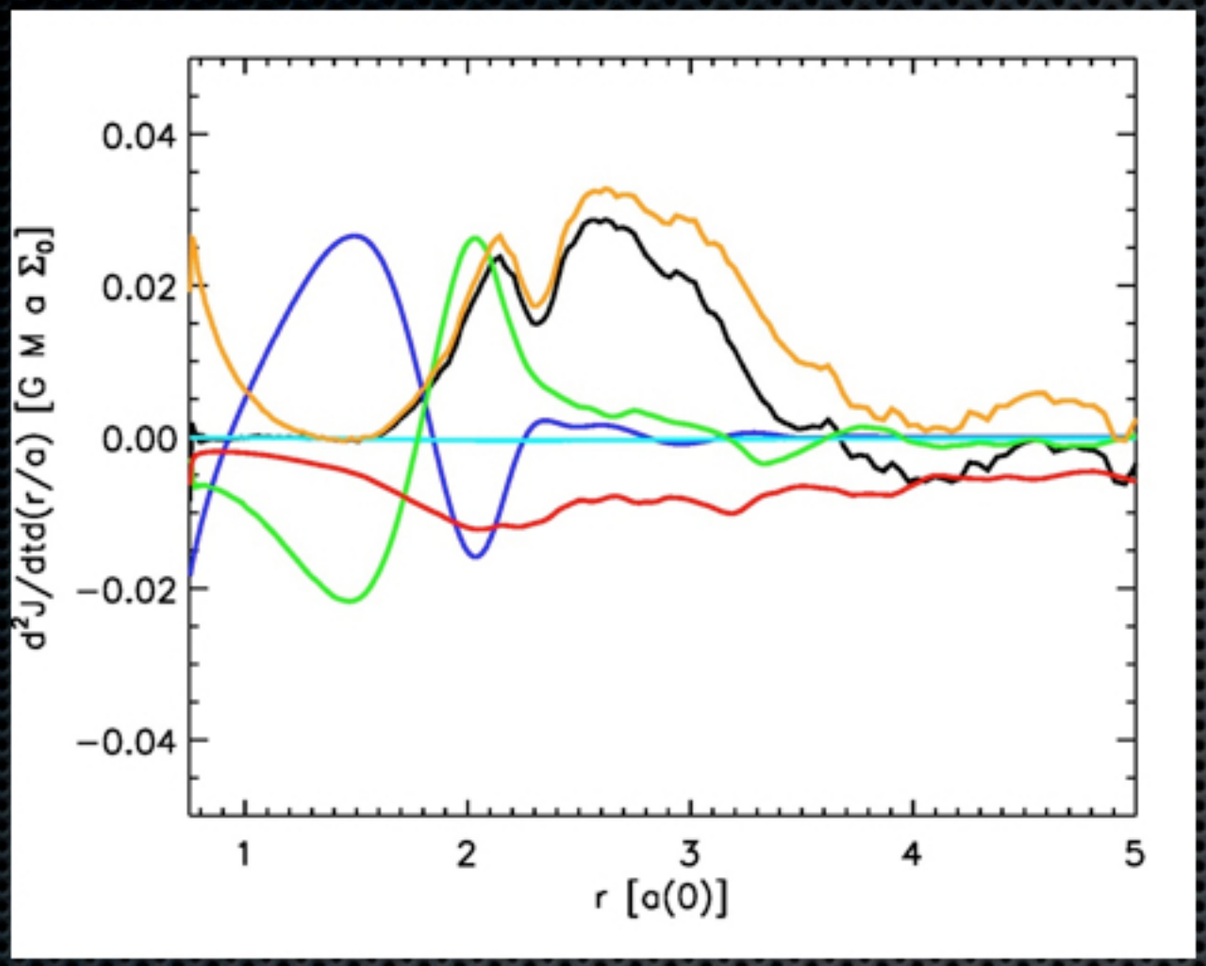


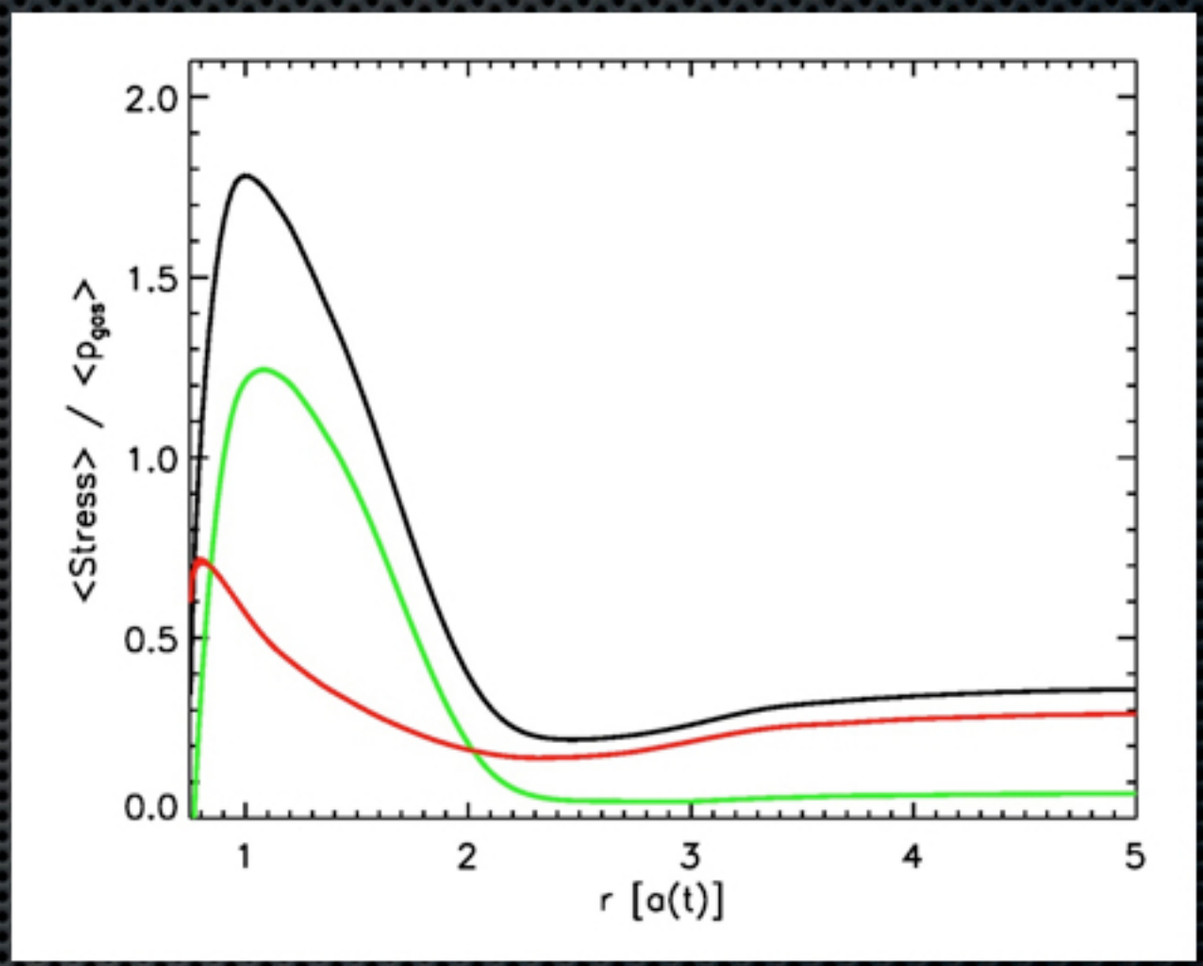
Binary Torque Density

$$\frac{dT}{dr} = \int \sqrt{-g}\, T^{\mu}{}_{\nu} \Gamma^{\nu}{}_{\mu\phi}\, d\theta d\phi$$



Angular Momentum Transport



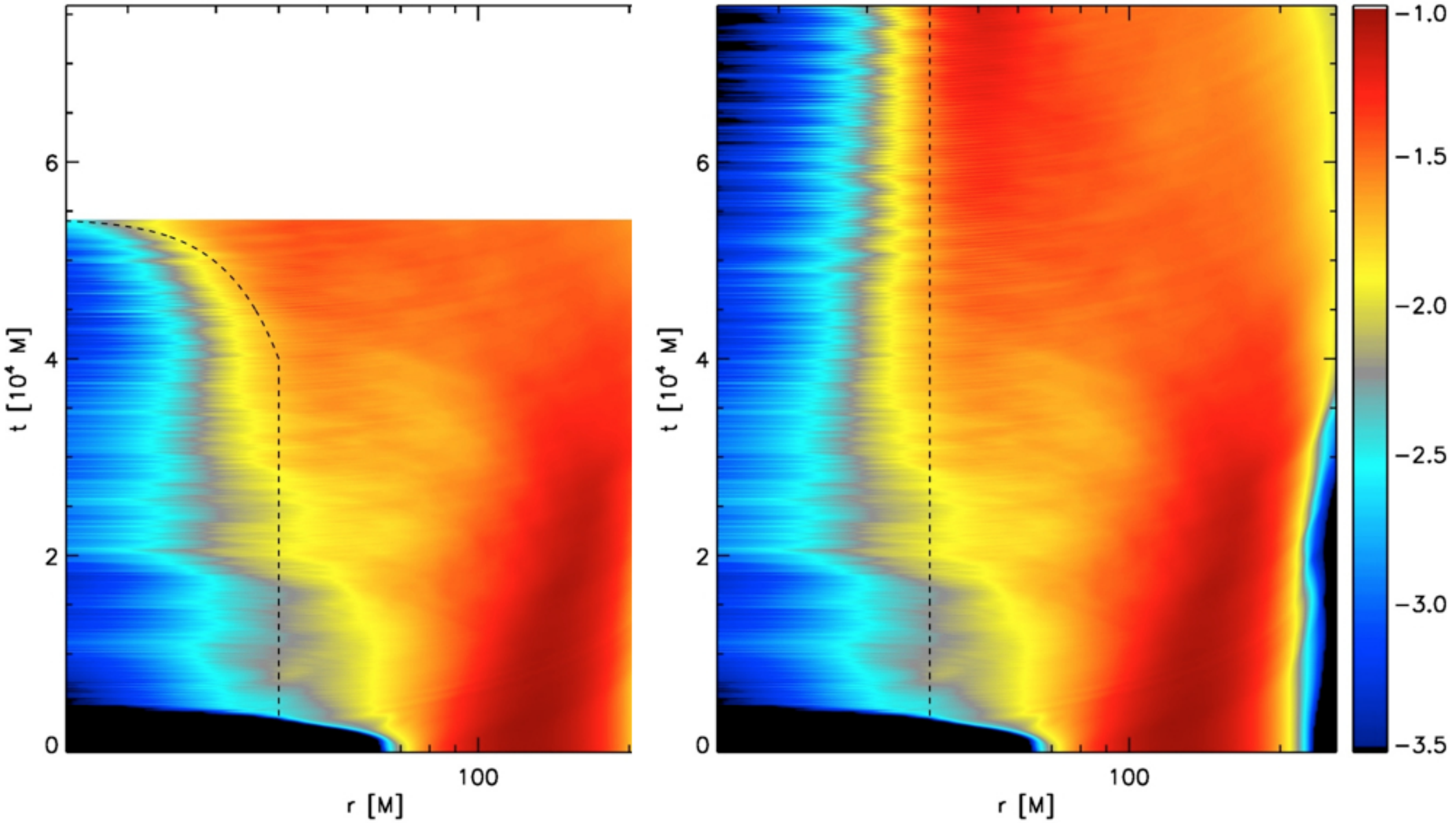


$$\partial_r \partial_t J = \frac{dT}{dr} - \{\mathcal{L} u_\phi\} - \partial_r \{M^r{}_\phi\} - \partial_r \{R^r{}_\phi\} - \partial_r \{A^r{}_\phi\}$$

$$\left(= [\text{Bin.}] - [\text{Rad.}] - [\nabla\text{Maxwell}] - [\nabla\text{Reynolds}] - [\nabla\text{Advected}]\right)$$

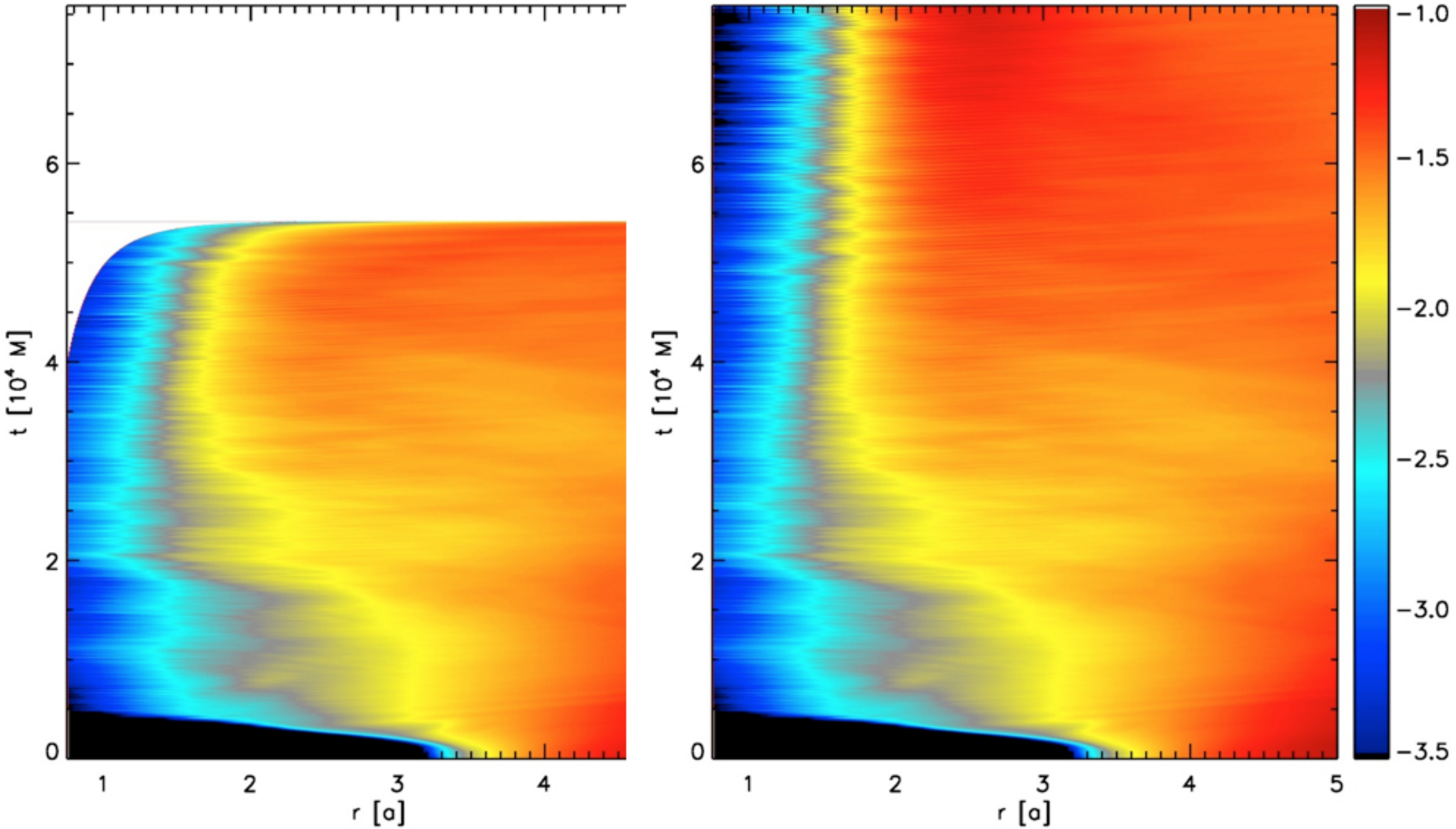
Surface Density

$$\Sigma(r,\phi) \equiv \int d\theta \sqrt{-g}\rho / \sqrt{g_{\phi\phi}}$$

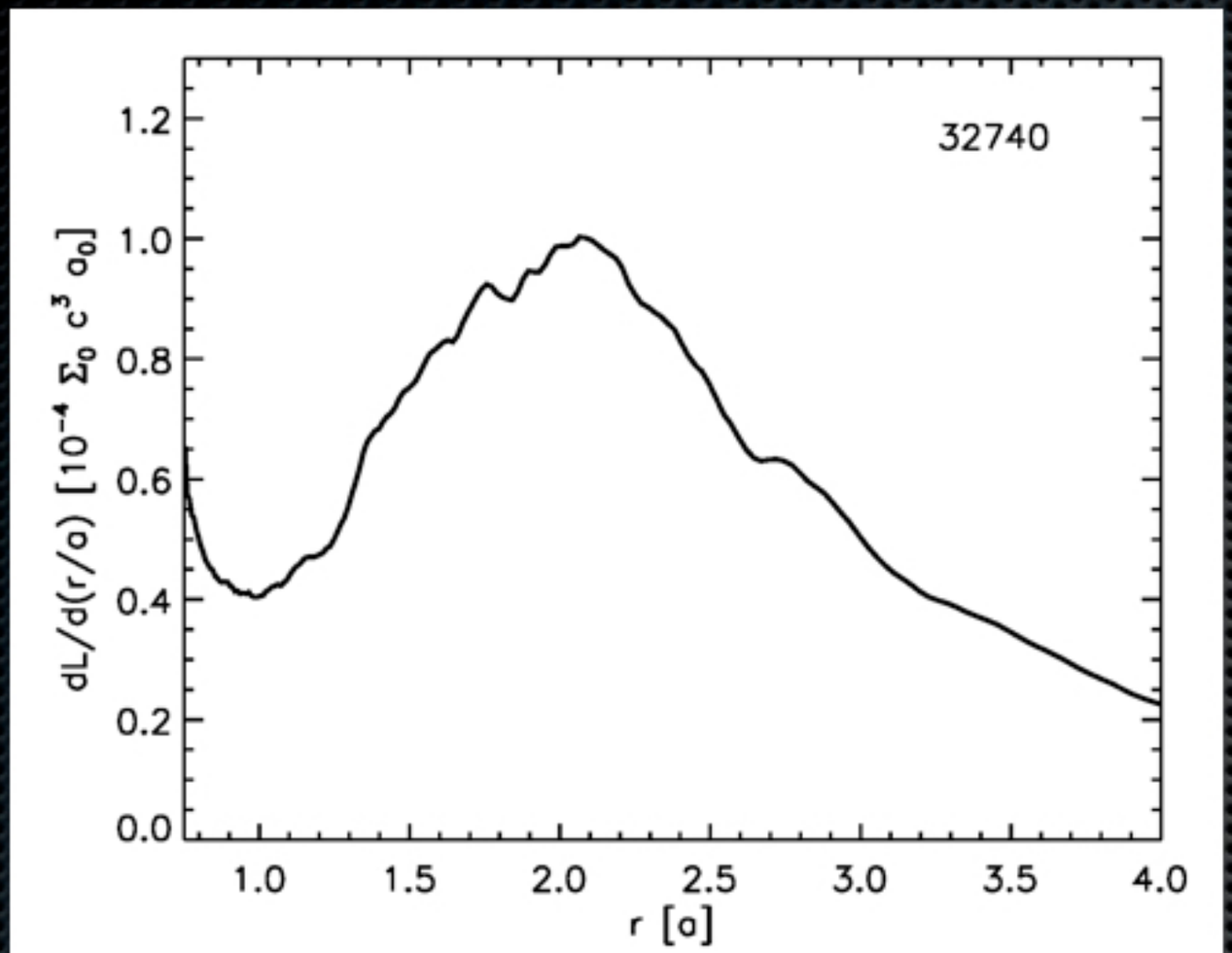


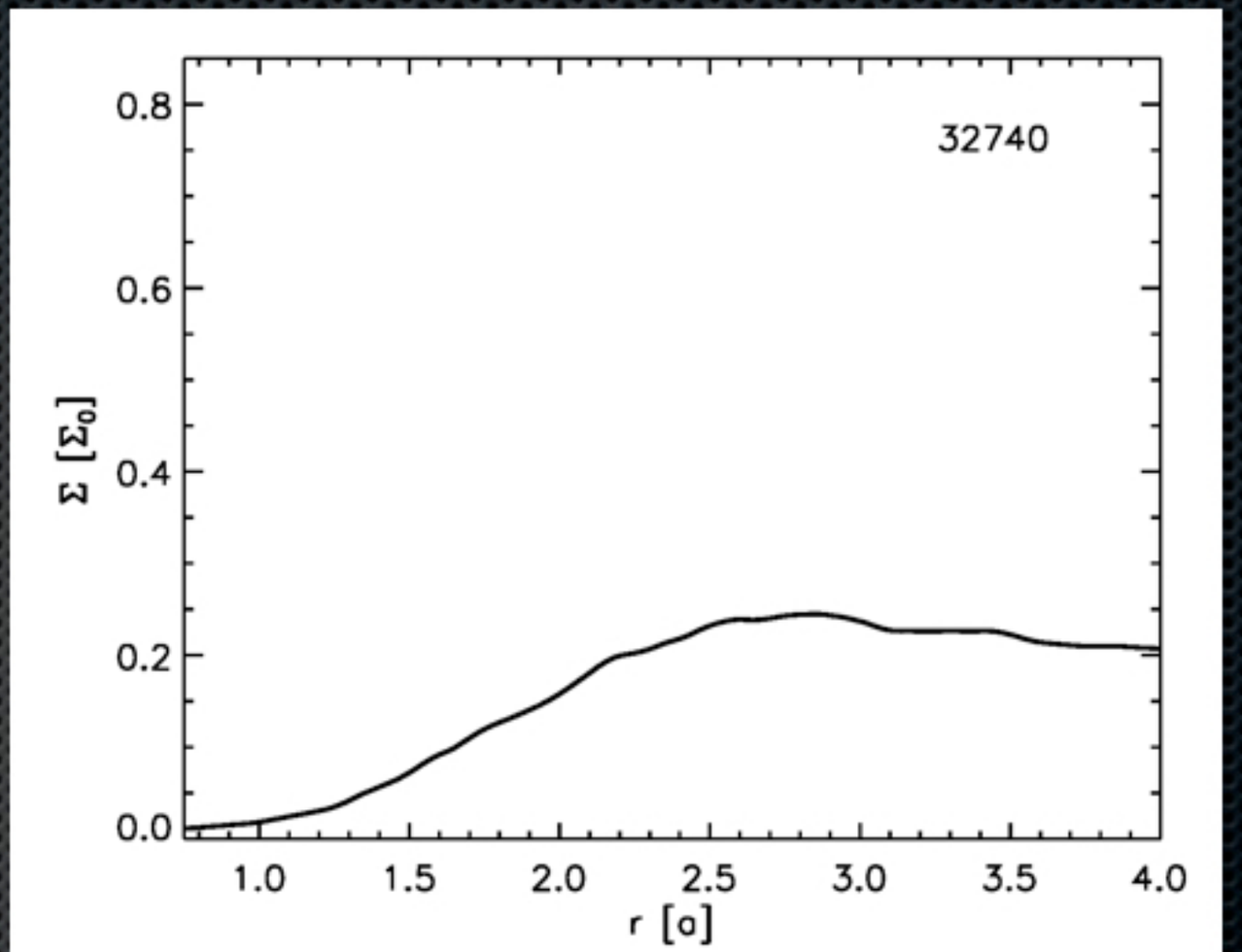
Surface Density

$$\Sigma(r,\phi) \equiv \int d\theta \sqrt{-g}\rho / \sqrt{g_{\phi\phi}}$$



Disk-Binary Decoupling





Binary-disk separation when:

$$t_{\rm gr} = \frac{5}{64}\left(\frac{a}{M}\right)^4 \frac{(1+q)^2}{q} M \ll t_{\rm in} = \alpha^{-1}(H/r)^{-2}(d\ln\Sigma/d\ln r)^{-1}\Omega^{-1} = \alpha^{-1}(H/r)^{-2}(d\ln\Sigma/d\ln r)^{-1}(r/r_g)^{3/2}M.$$

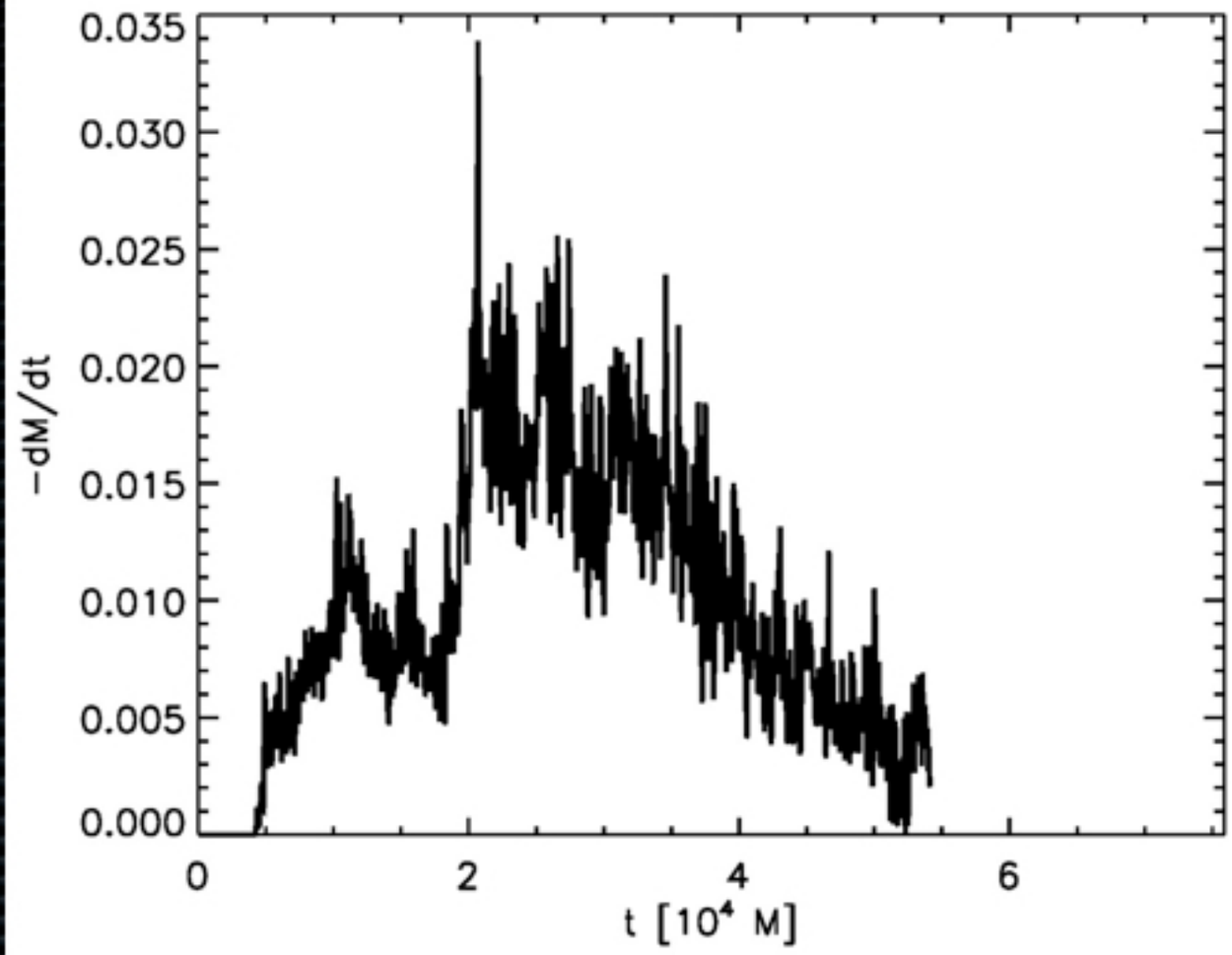
Commonly Imagined:

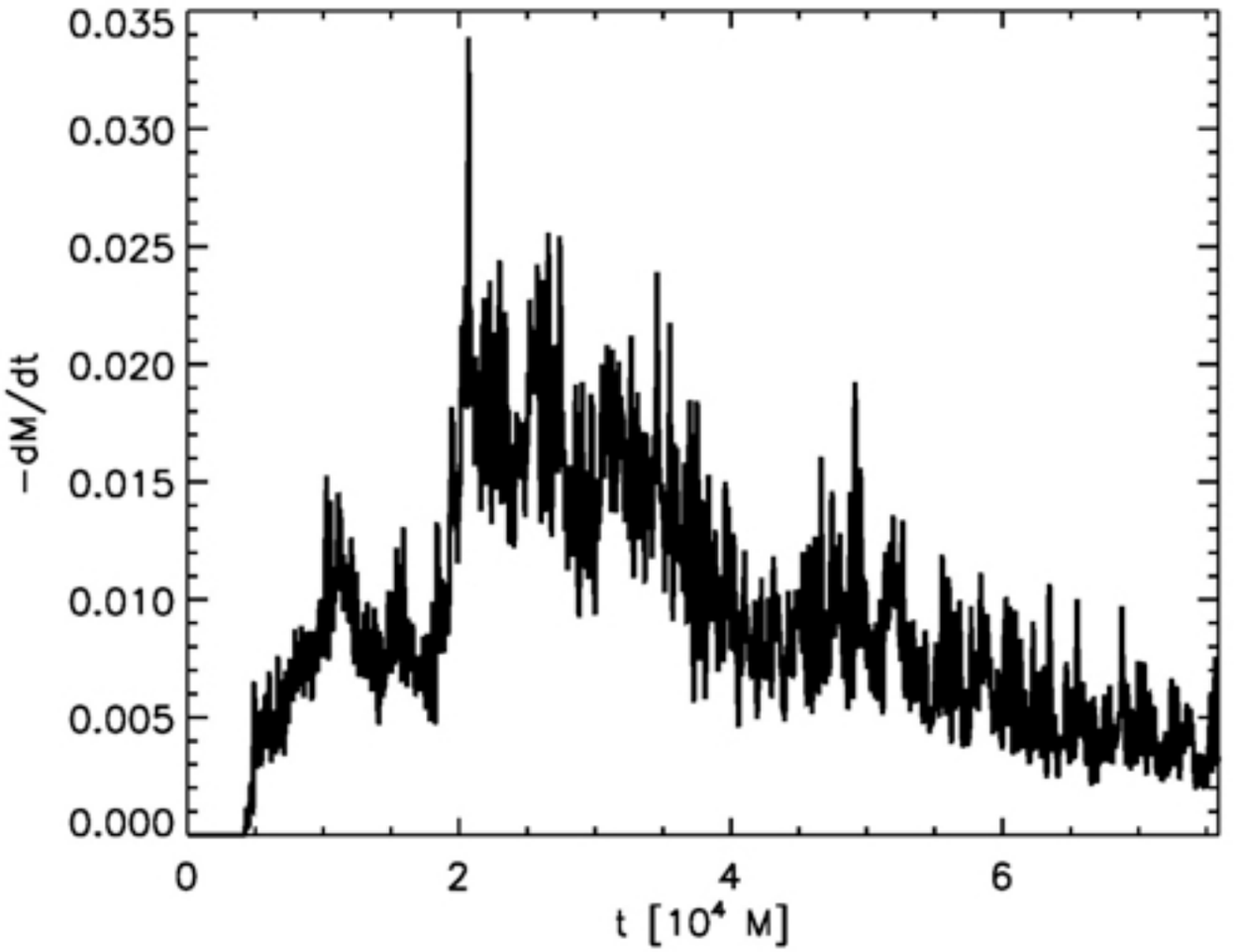
$$a_{\rm dec} = 70\,(d\ln\Sigma/d\ln r)^{-2/5}\left(\frac{H/r}{0.15}\right)^{-4/5}\left(\frac{\alpha}{0.01}\right)^{-1}$$

Ours:

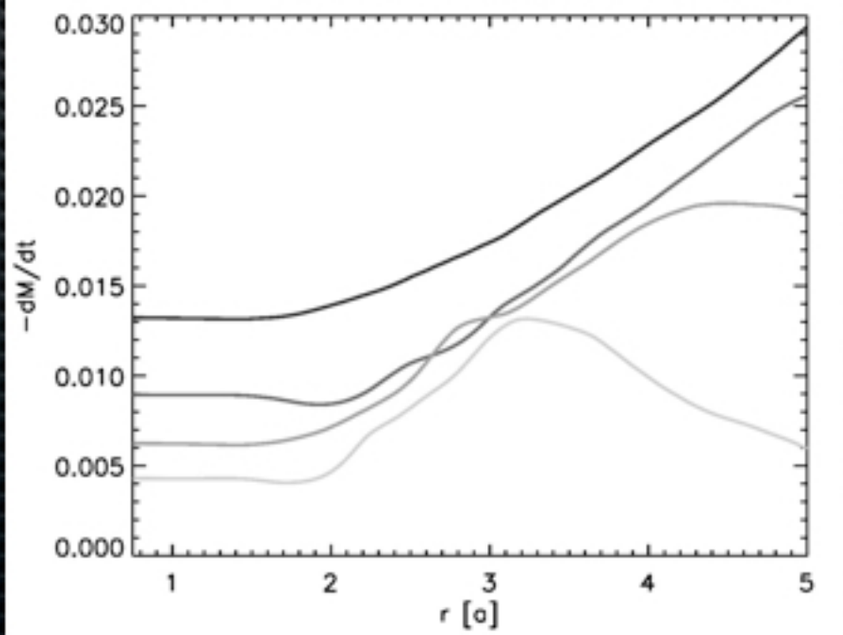
$$a_{\rm dec} \simeq 11\,(d\ln\Sigma/d\ln r/6)^{-2/5}\left(\frac{H/r}{0.15}\right)^{-4/5}\left(\frac{\alpha}{0.7}\right)^{-1}$$

Accretion Rate:

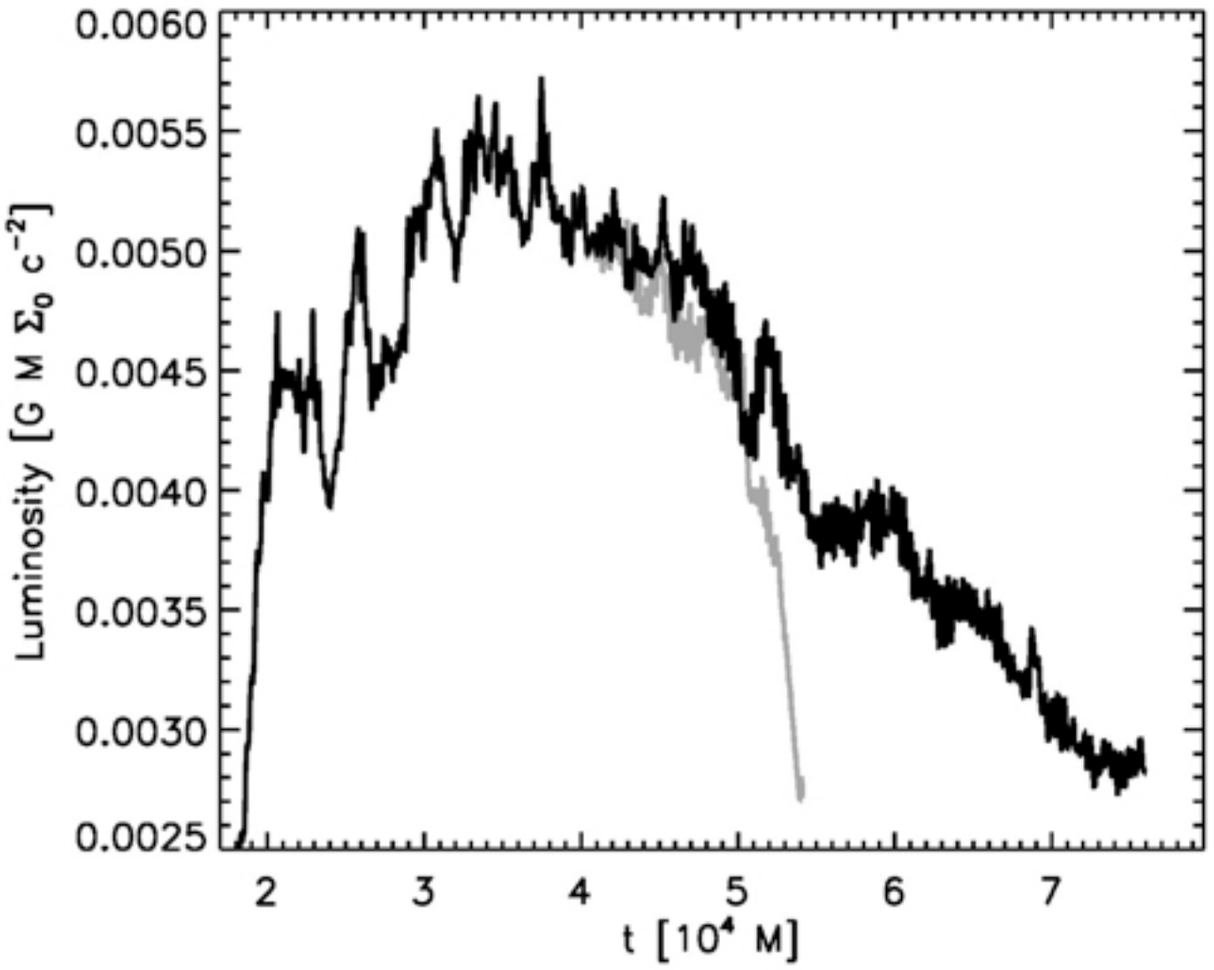


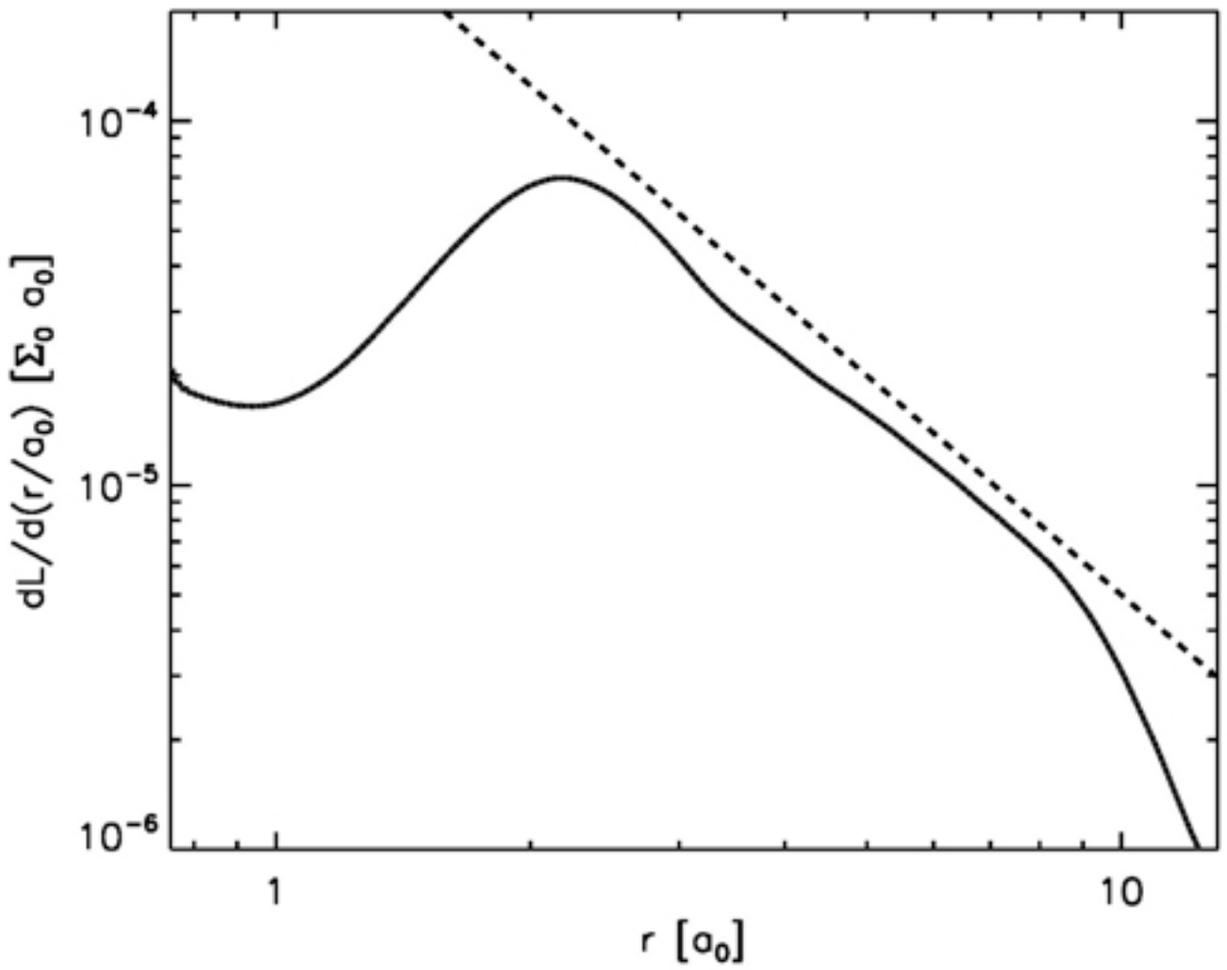


- Both decrease over time
- $Mdot_{Run1} > 0.3\ Mdot_{Run2}$
- Decrease due largely to torques
- Decrease of Run1 also due to decoupling



Luminosity





$$\frac{dL}{dr/a_0} = 4 \times 10^{-4} (\dot{M}/0.01)(r/a_0)^{-2} \Sigma_0 a_0.$$

$$L_{\rm disk} \simeq 2.4 \times 10^{40} (\hat{L}/10^{-3}) M_6 \tau_0 \ {\rm erg/s}.$$

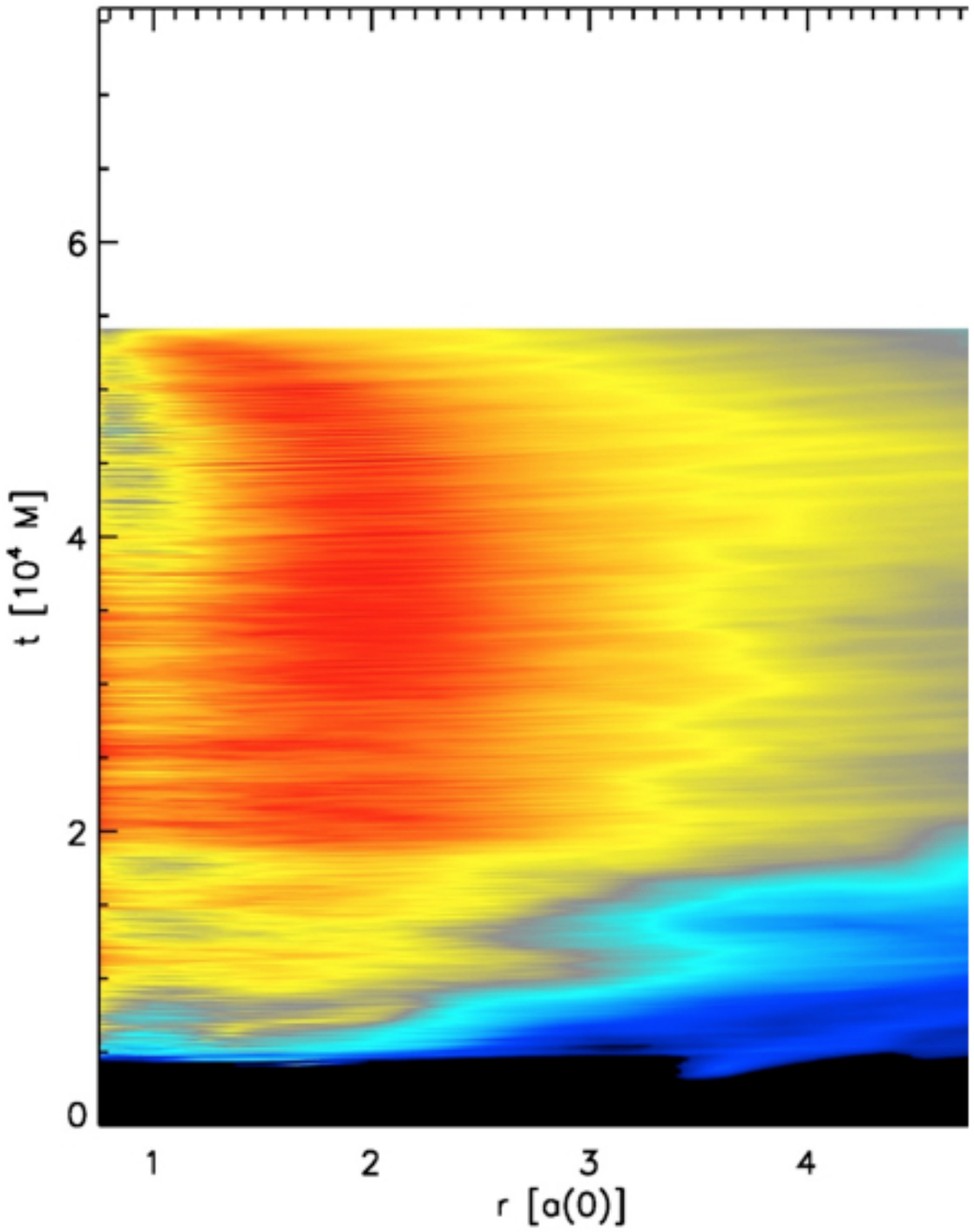
$$T_{\rm eff} \simeq 4 \times 10^4 (\hat{L}/10^{-3})^{1/4} M_6^{-1/4} \tau_0^{1/4} \ {\rm K},$$

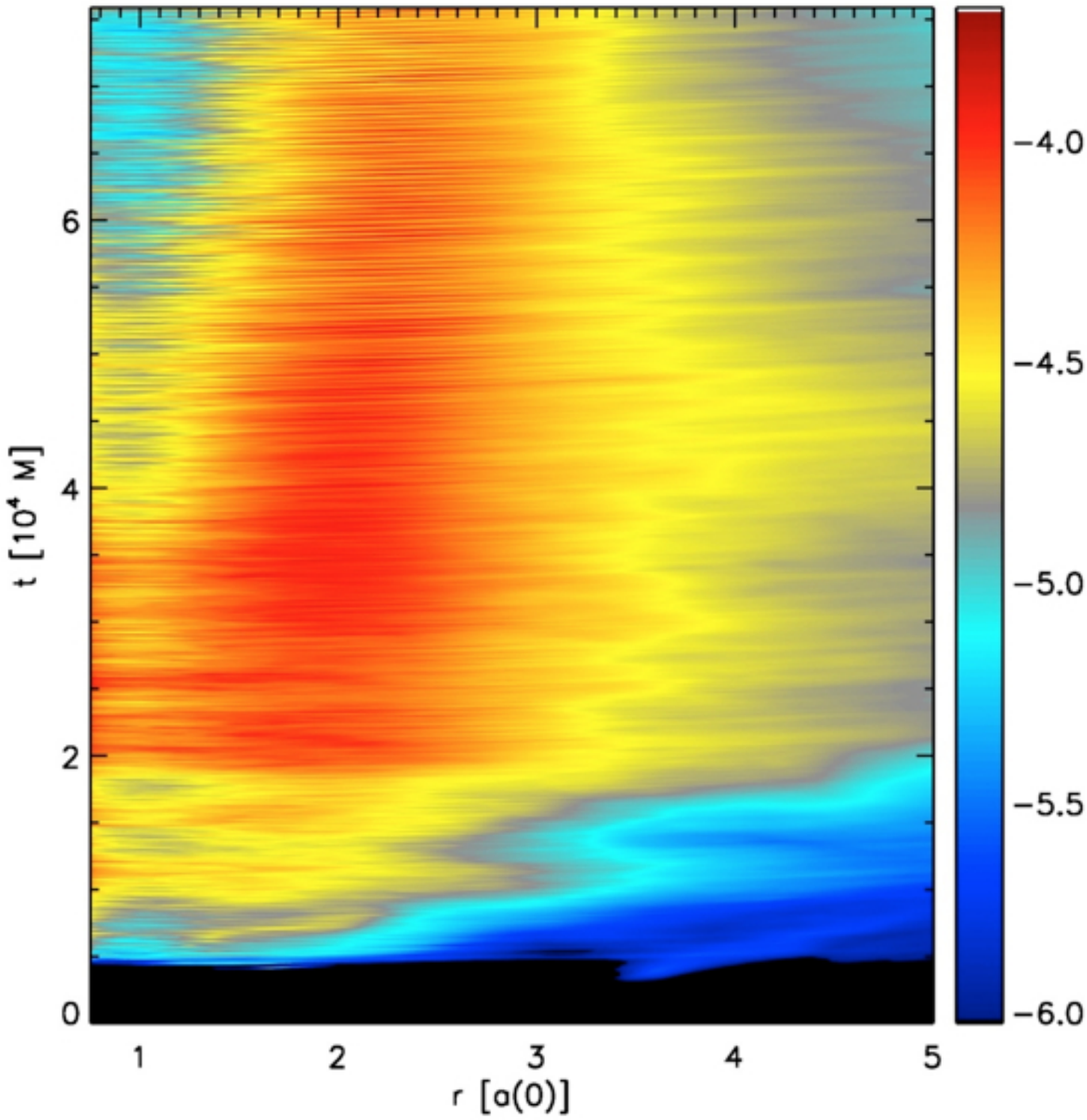
$$\tau_0(r = 20M) \sim \ 2 \times 10^3 (\alpha/0.1)^{-1} (\eta/\dot{m})$$

Typical for a AGN

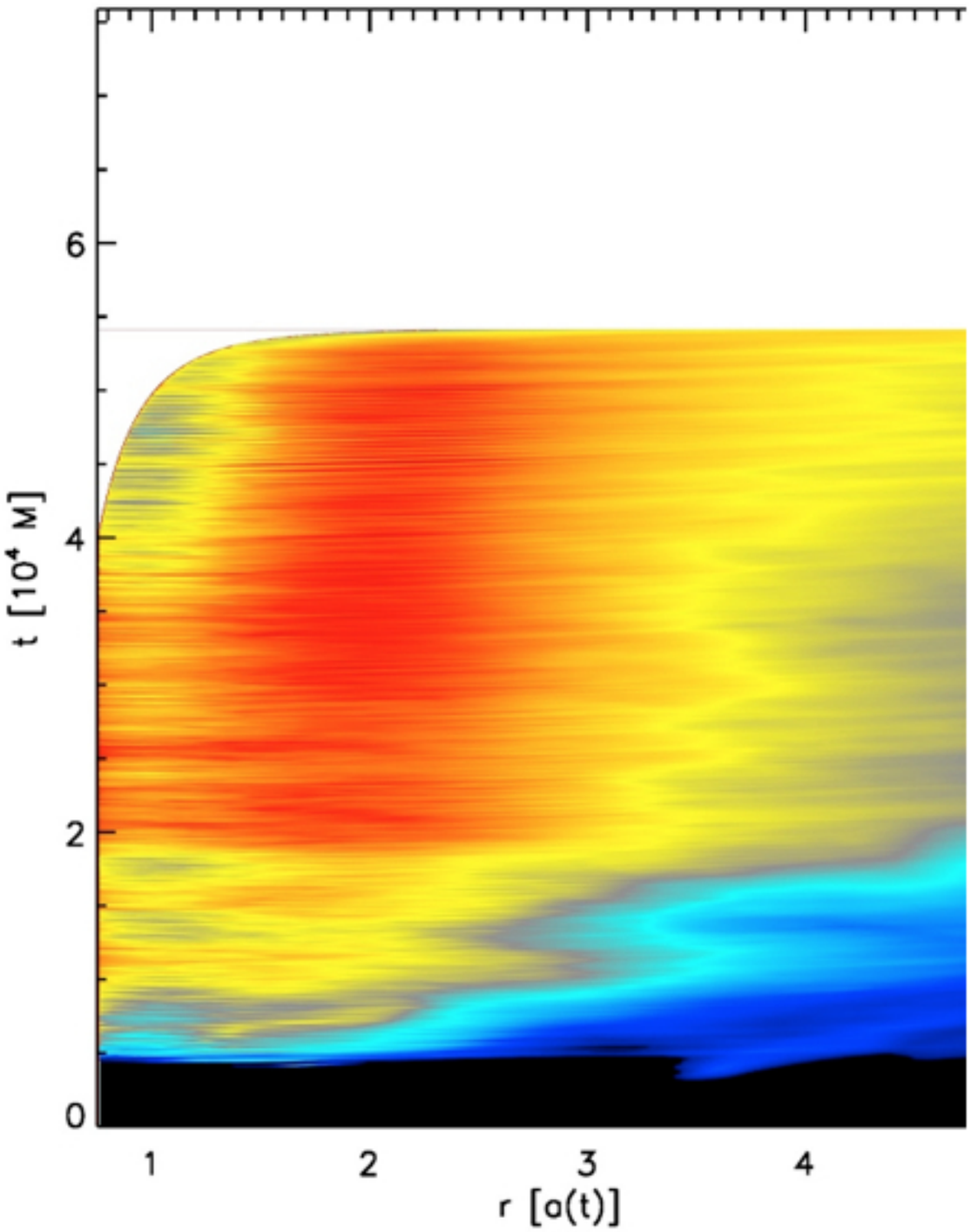
--> peak in UV assuming thermal emission

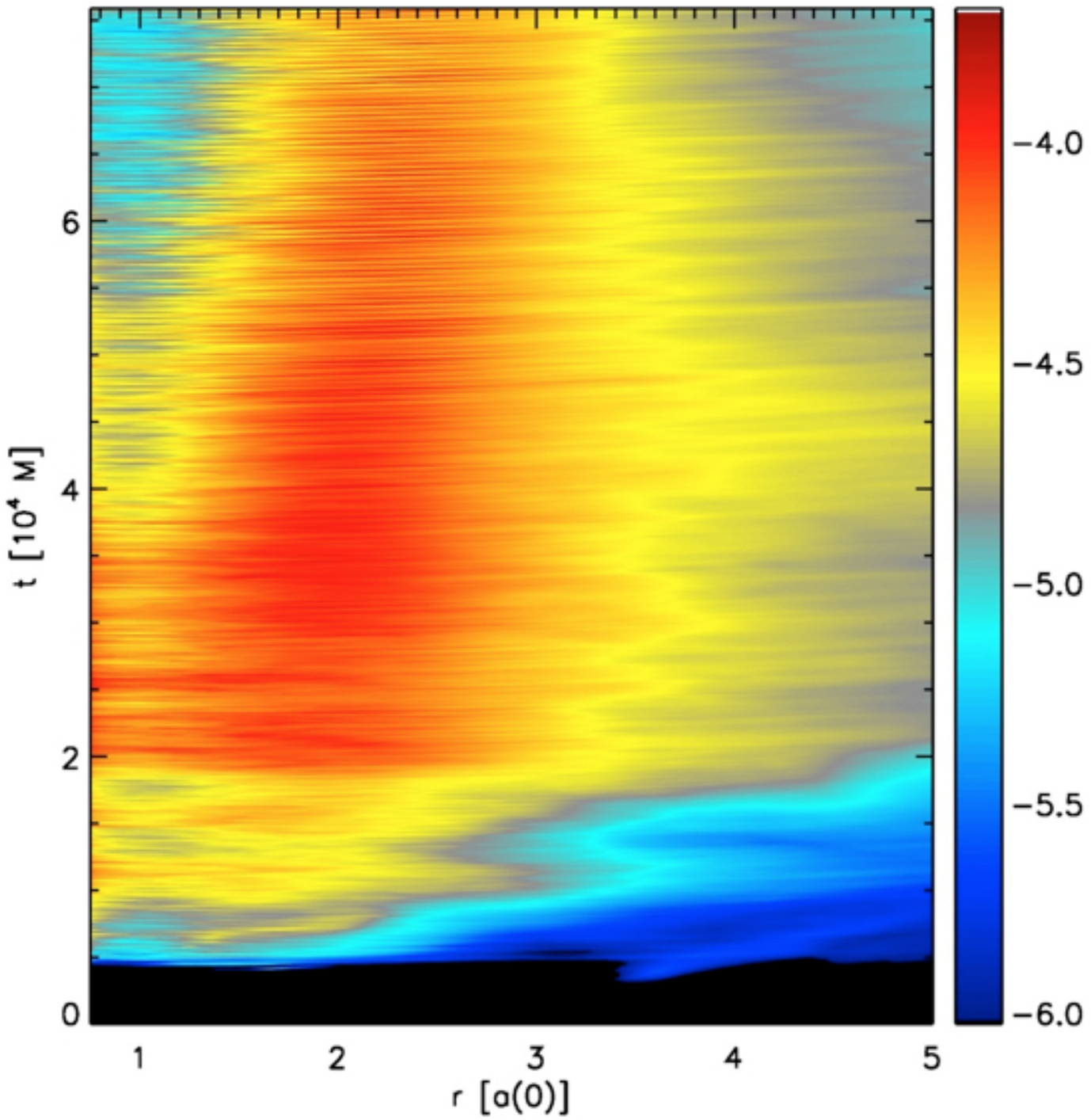
Luminosity





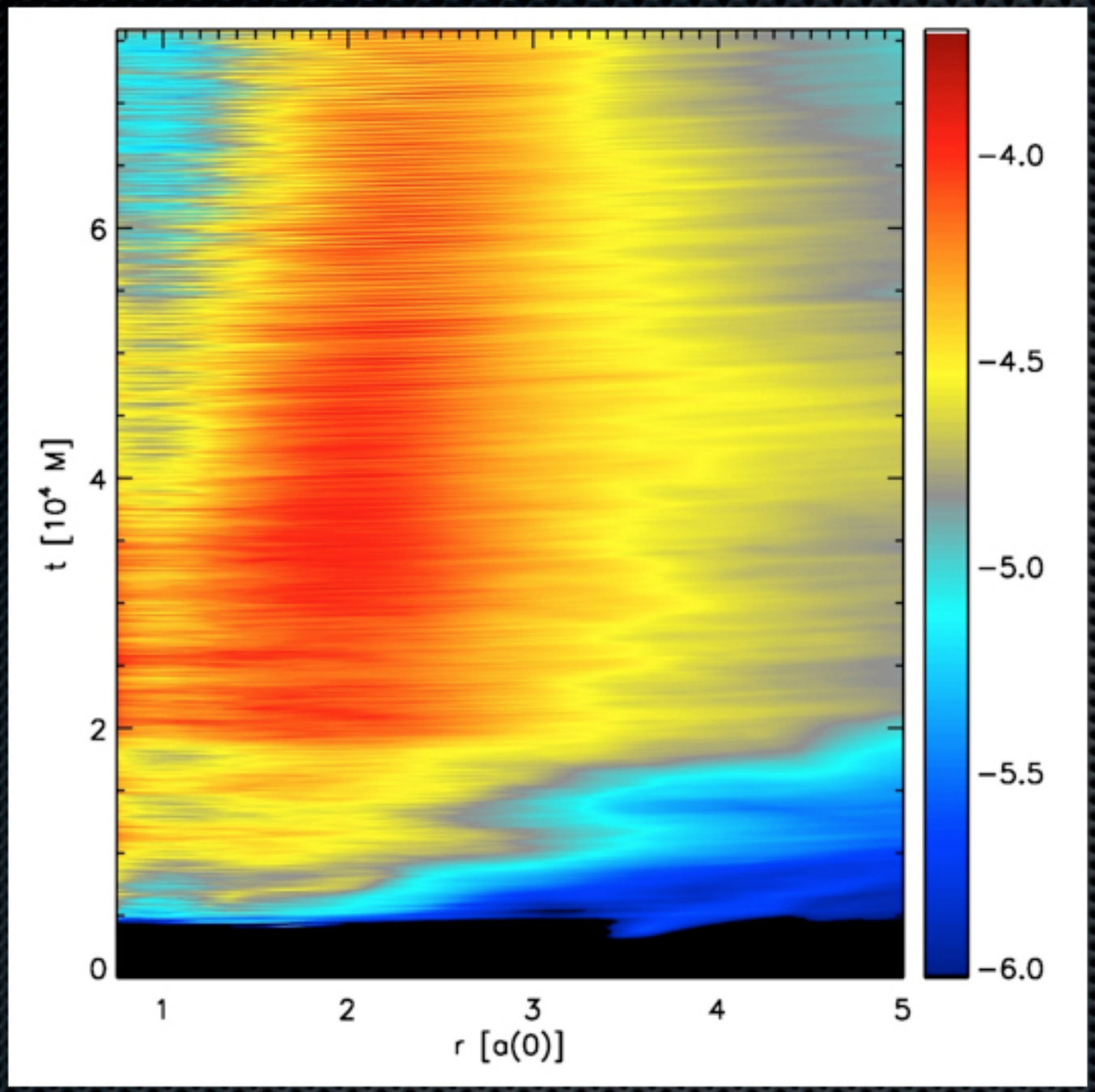
Luminosity



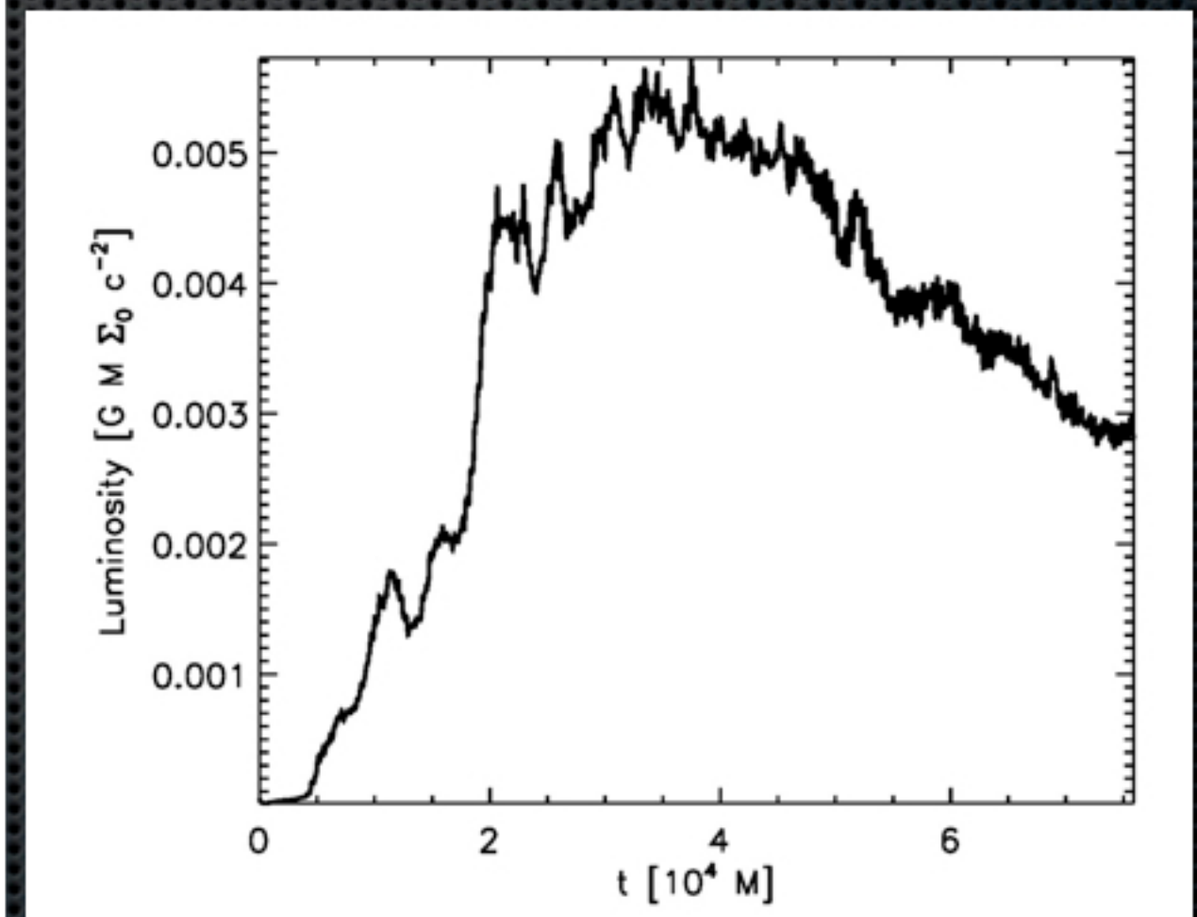


Variability

Full view



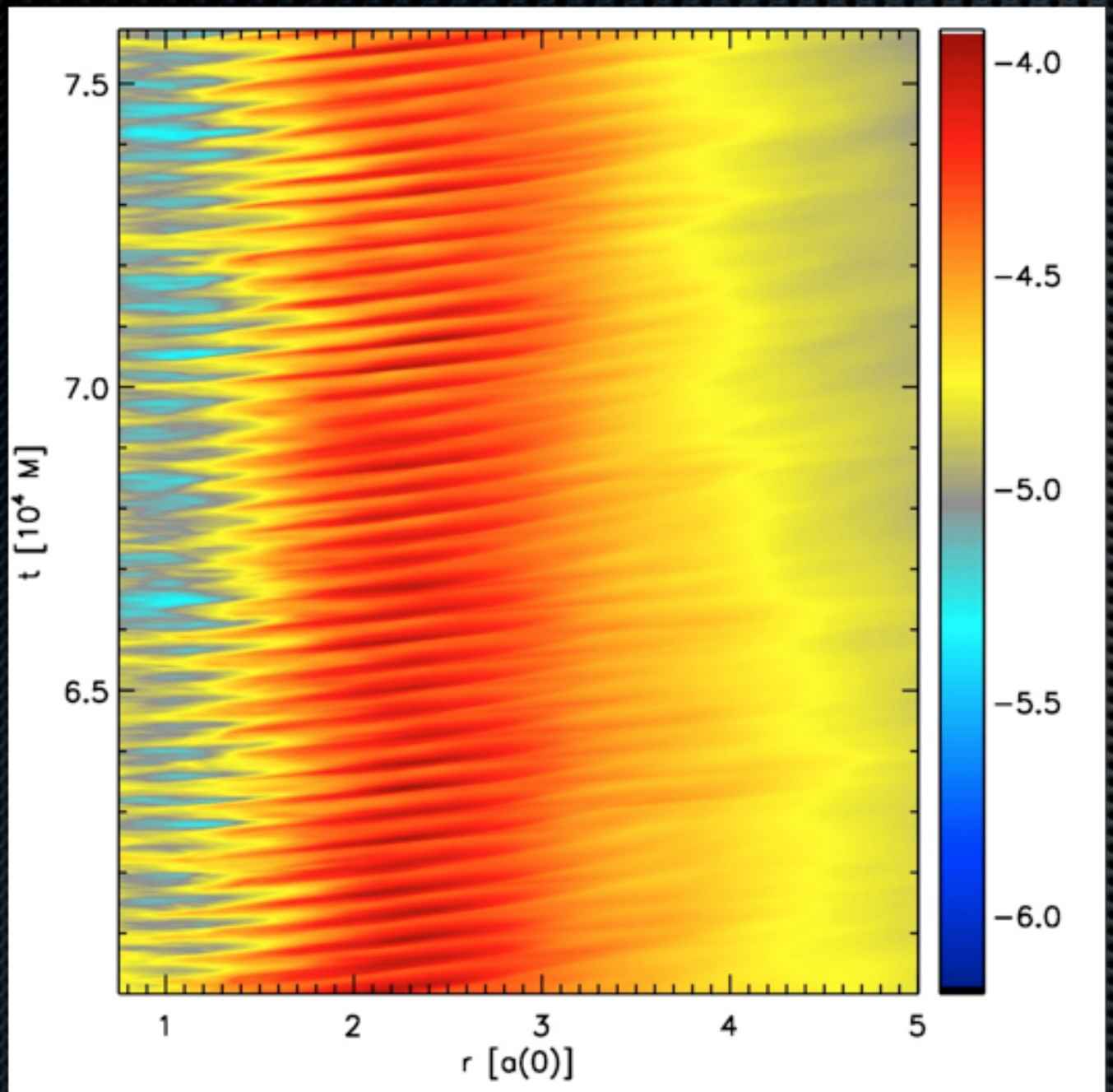
(r , t) space

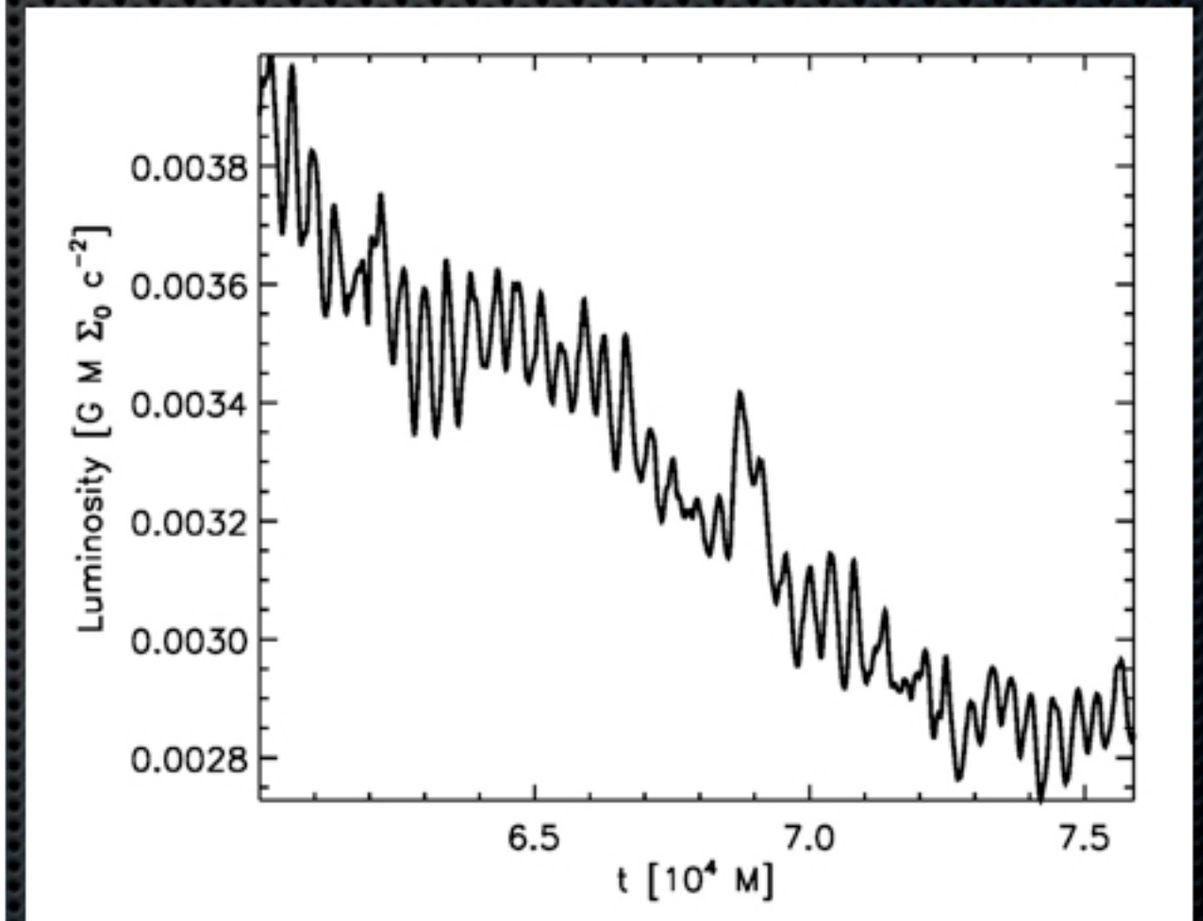


integrated over radius

Variability

Window



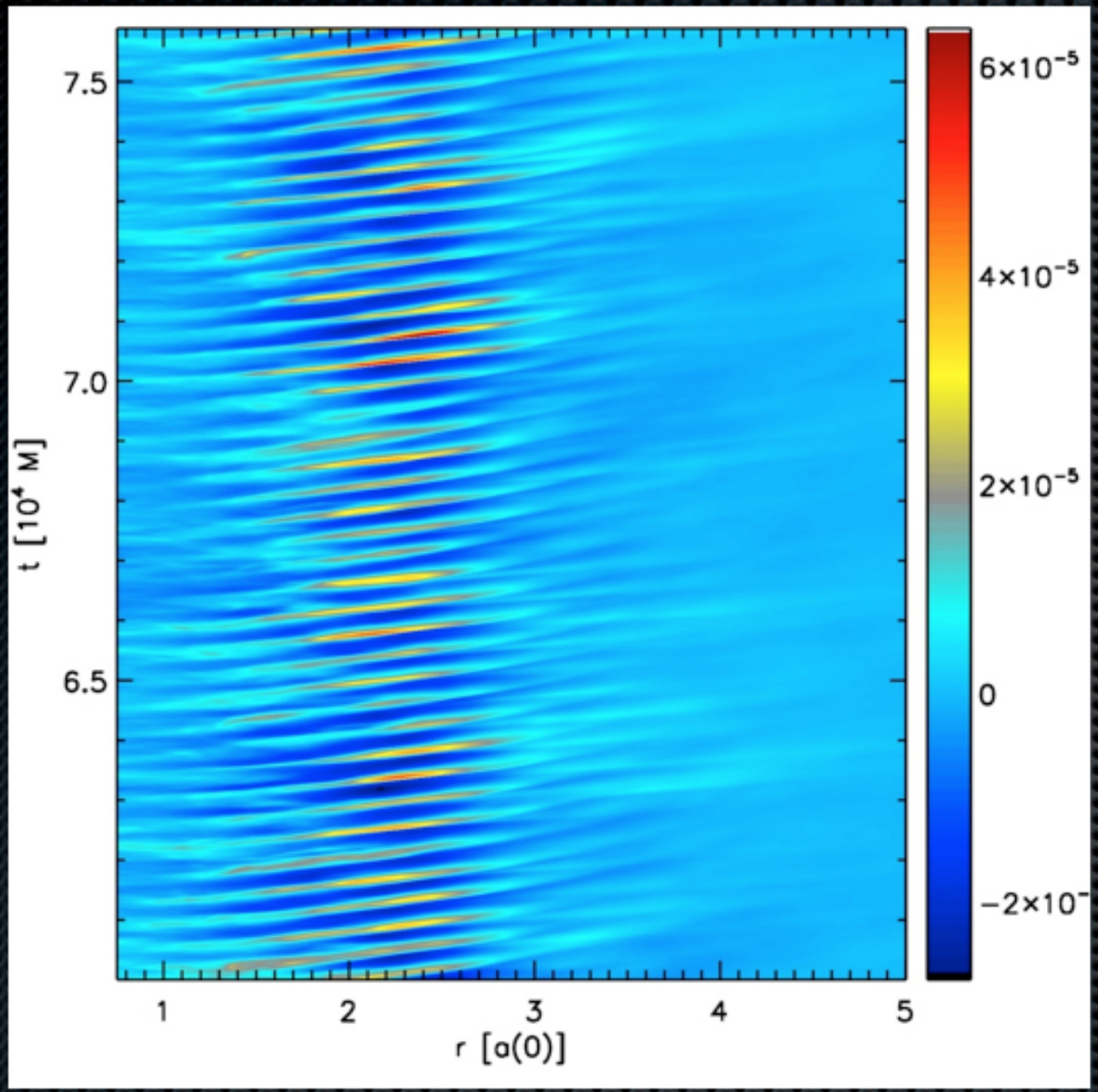


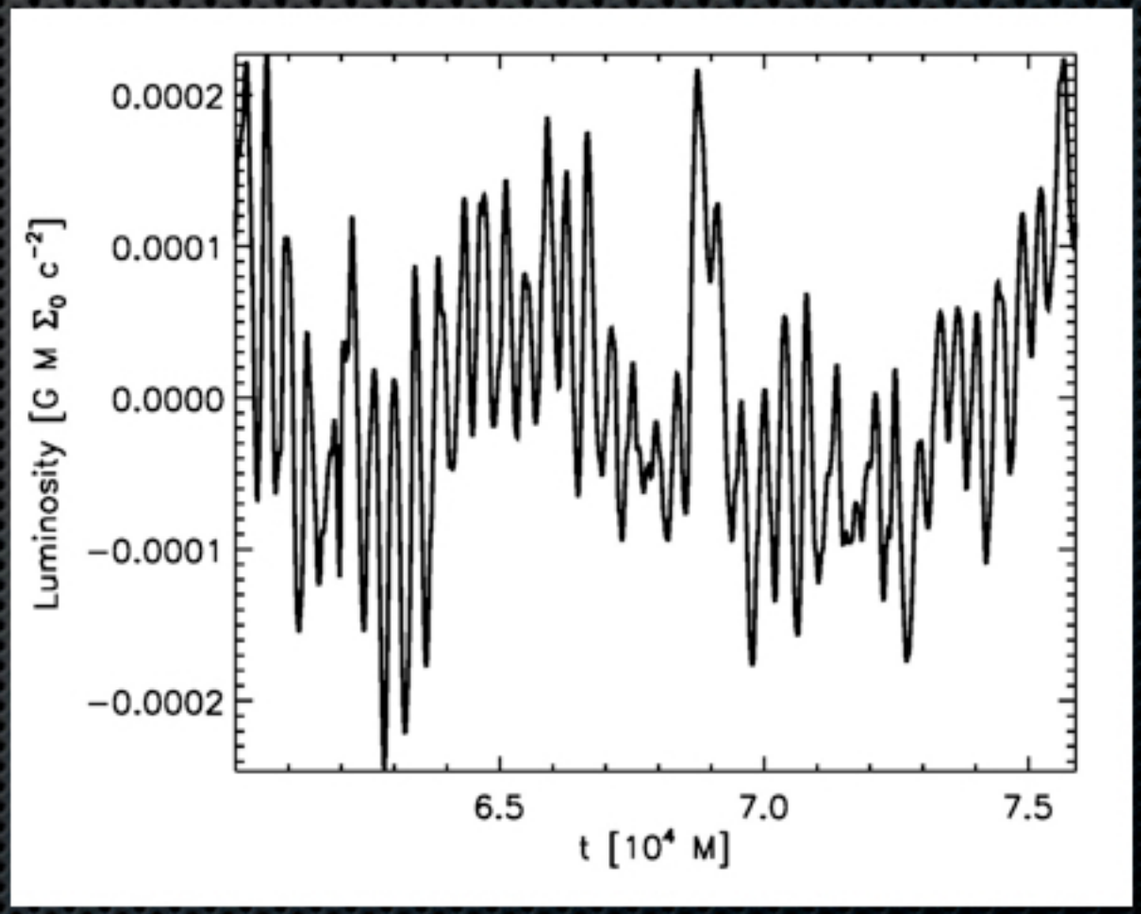
(r , t) space

integrated over radius

Variability

Remove secular trend
(linear fit)



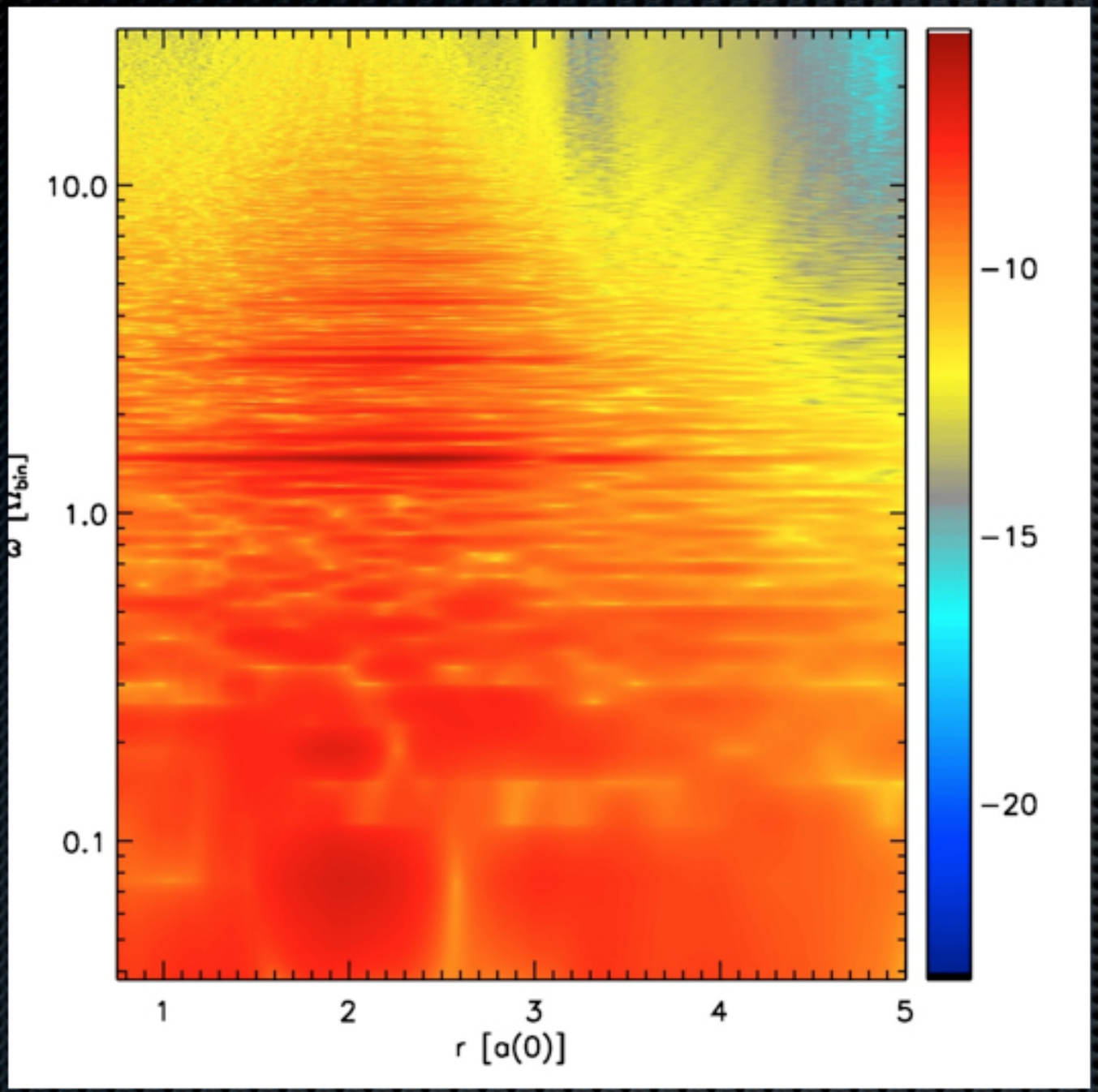


(r , t) space

integrated over radius

Variability

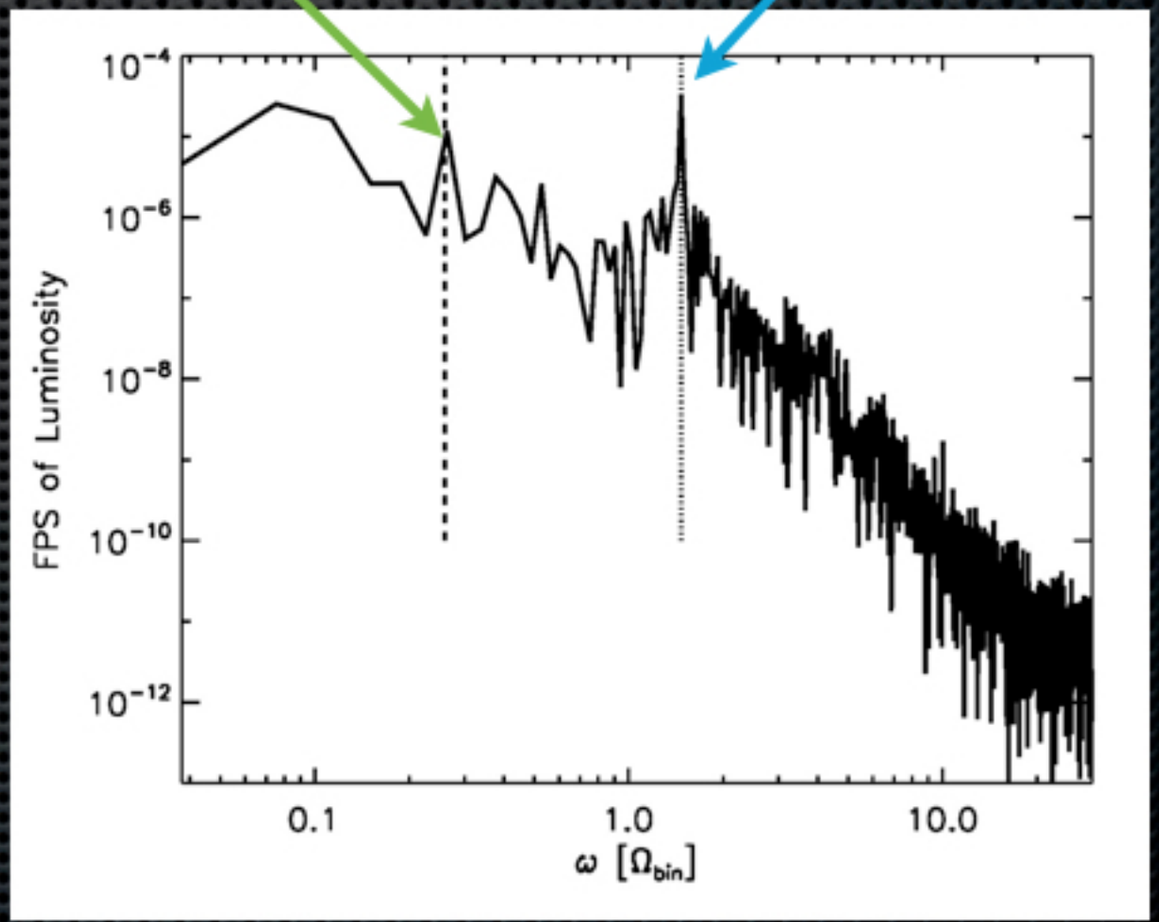
FFT



$r_{\rm lump} \simeq 2.5a$

$\Omega_K(r_{\rm lump})$

$1.47\Omega_{\rm bin}$



(r ,ω) space

integrated over radius

Variability

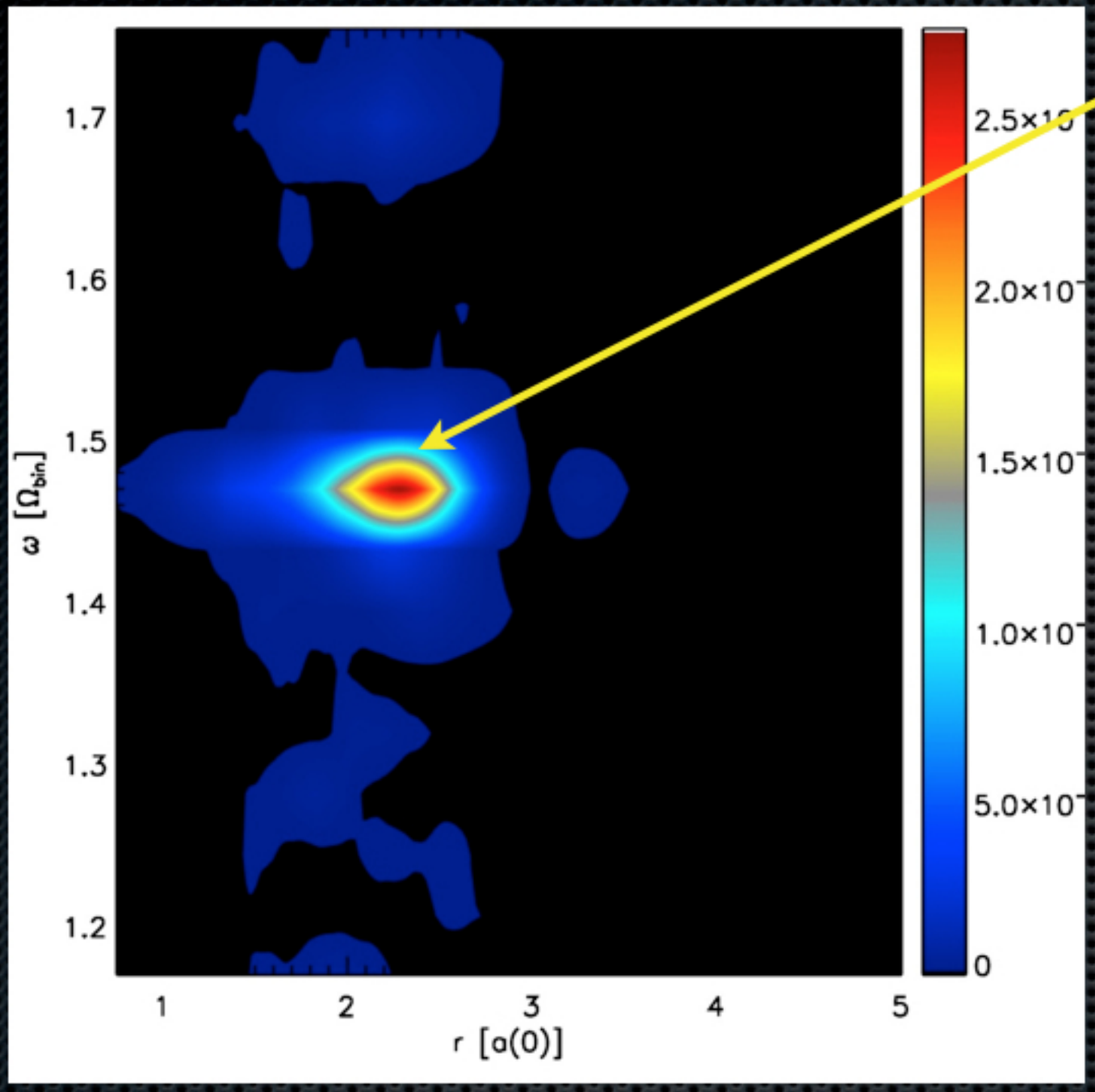
FFT close-up

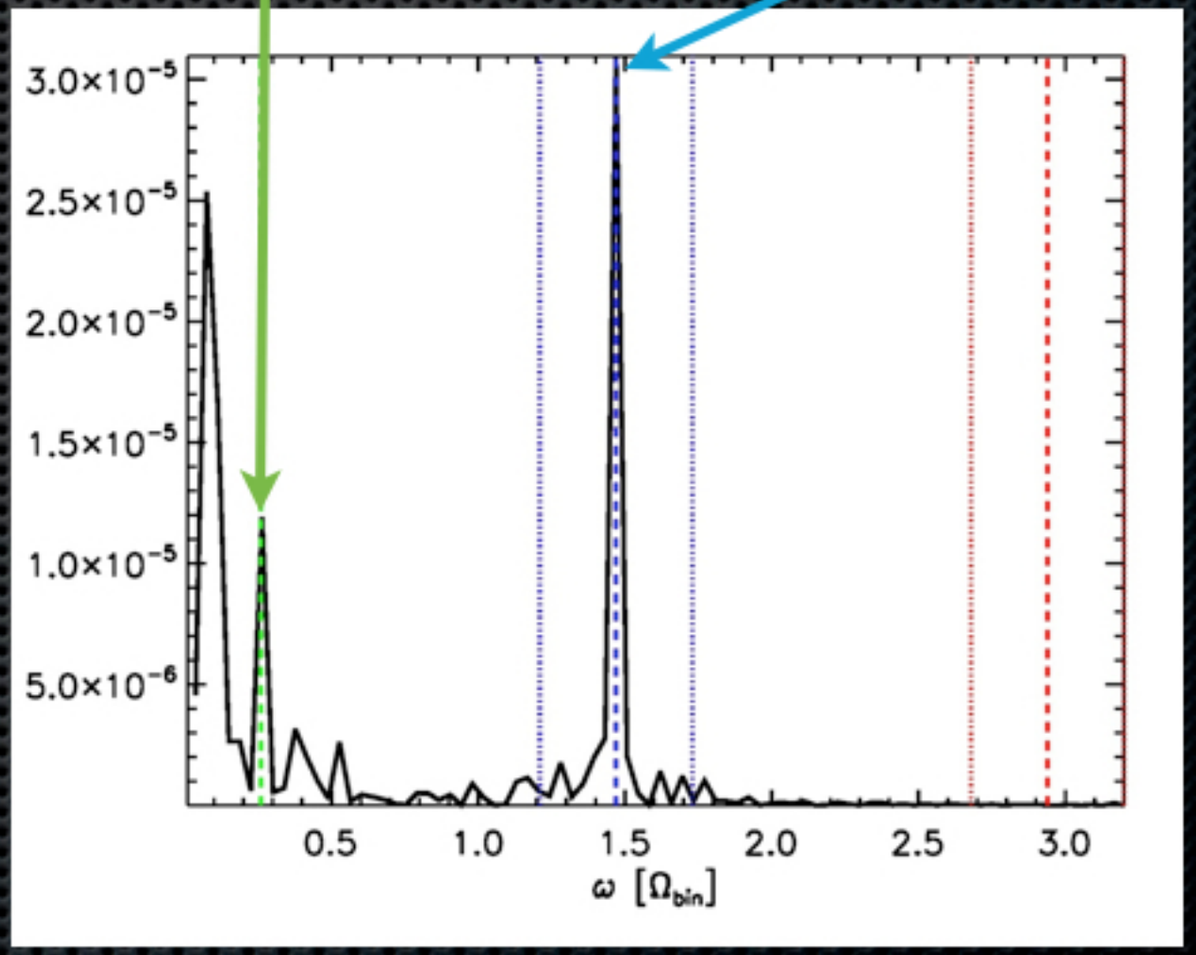
$r_{\rm peak} \simeq 2.3a$

$r_{\rm lump} \simeq 2.5a$

$\Omega_K(r_{\rm lump})$

$1.47\Omega_{\rm bin}$



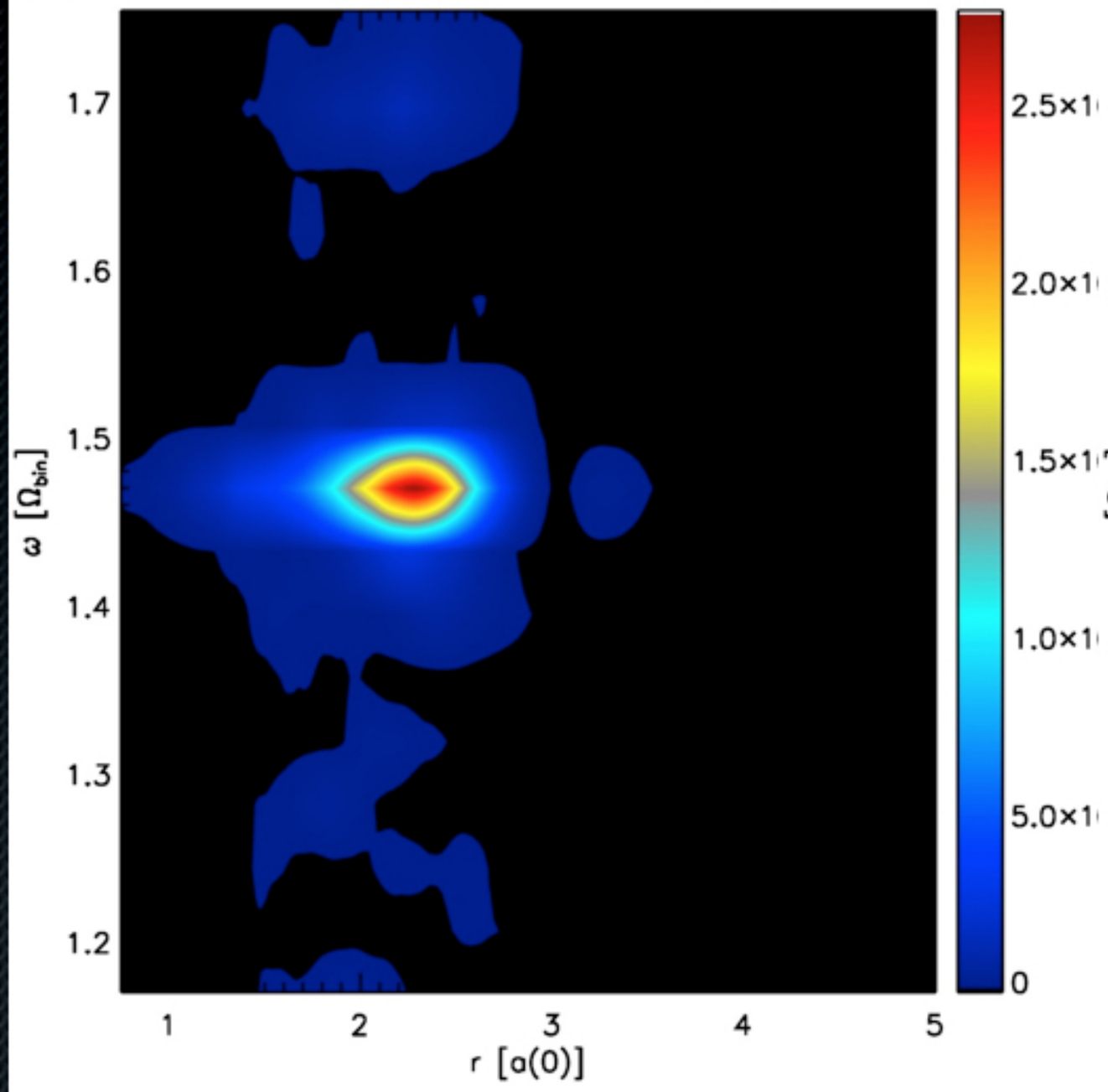


(r ,ω) space

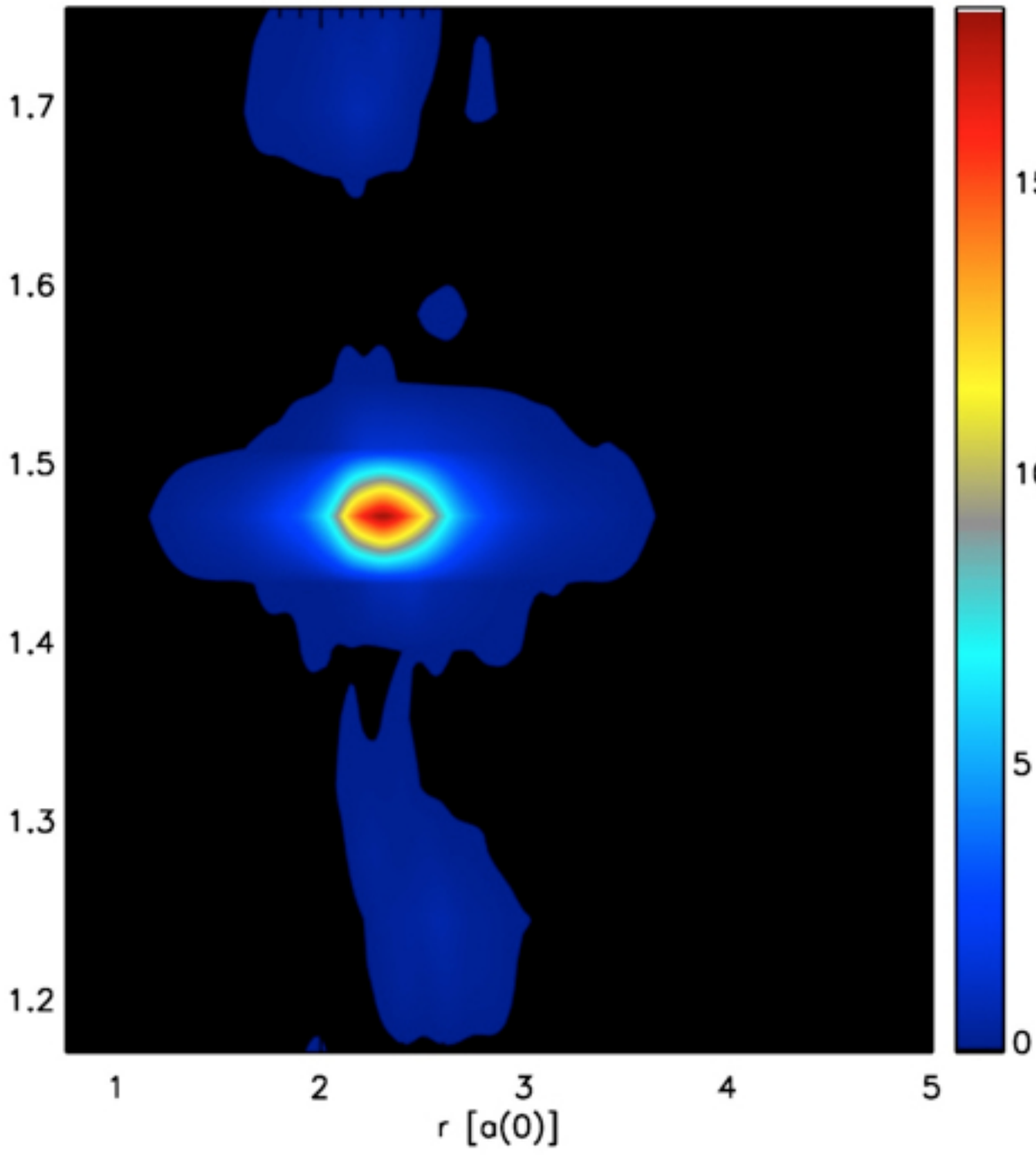
integrated over radius

Variability

FFT close-up



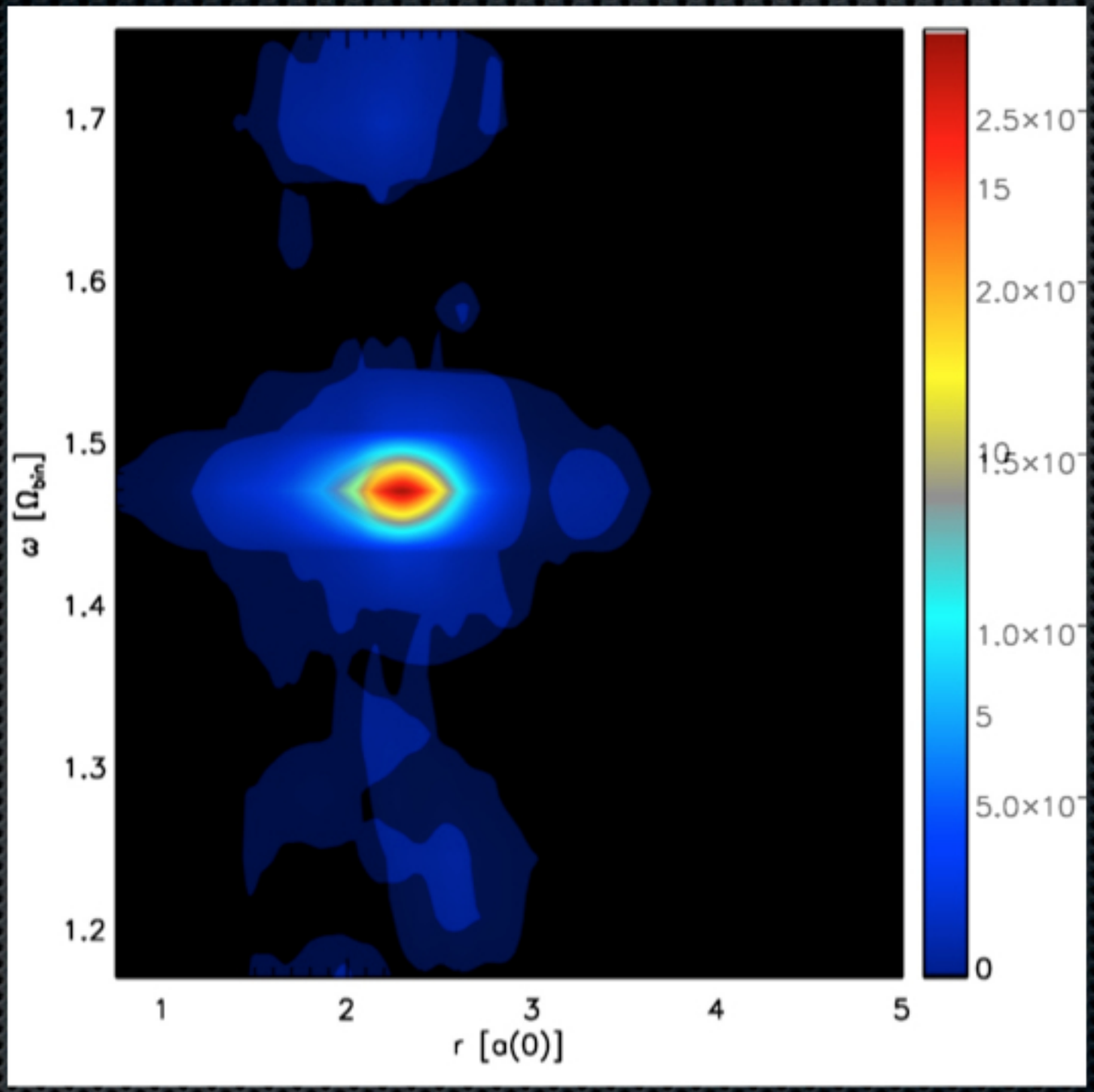
Luminosity



Surface Density

Variability

FFT close-up



Luminosity

Surface Density

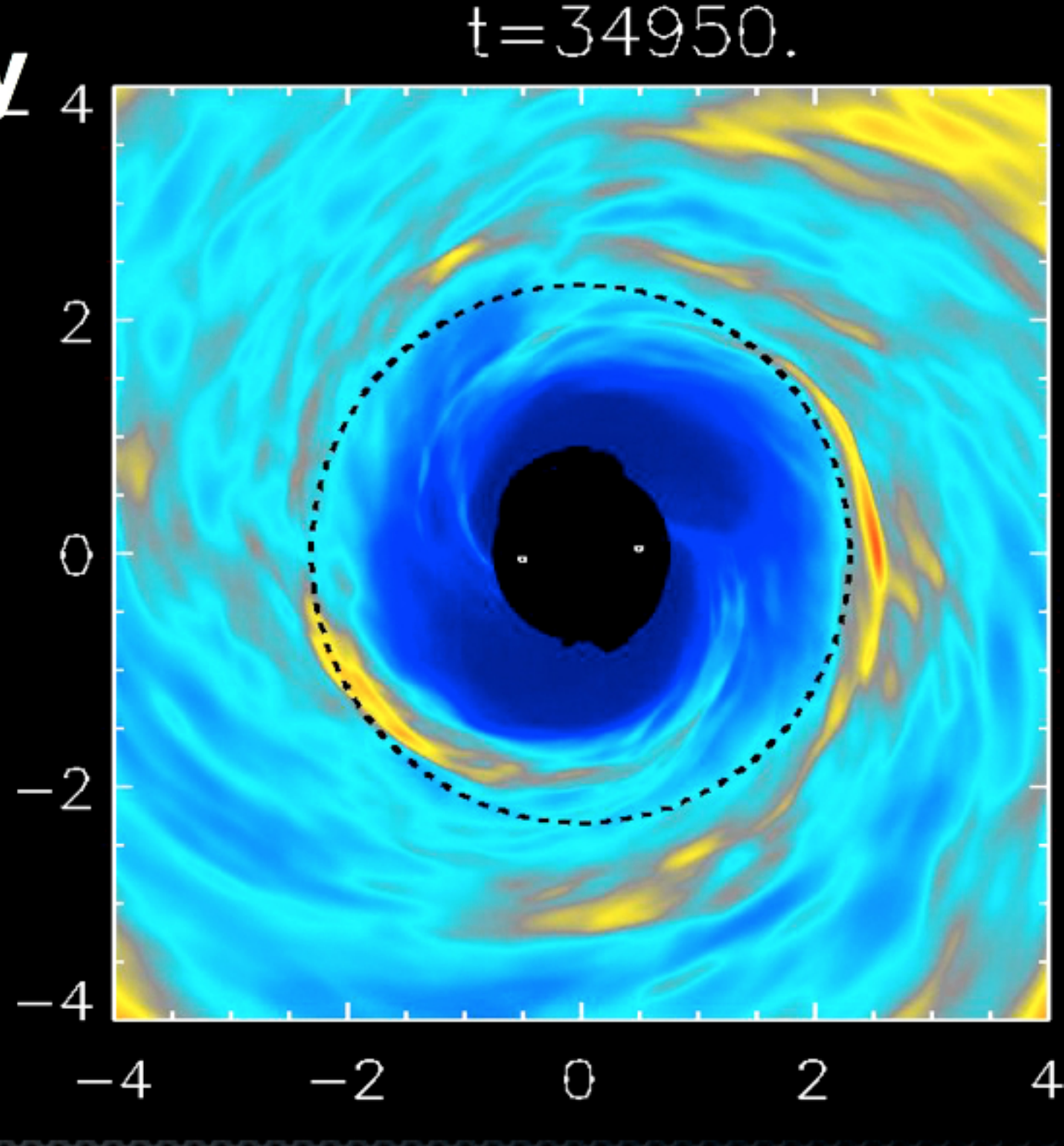
Origin of Variability

$$\omega_{\rm peak} = 2\left(\Omega_{\rm bin} - \Omega_{\rm lump}\right)$$

??

$$1 < \frac{\omega_{\rm peak}}{\left(\Omega_{\rm bin} - \Omega_{\rm lump}\right)} < 2$$

$$0 < \frac{M_2}{M_1} < 1$$



May be obfuscated by "low-pass" filter of disk's opacity:

$$0.16\left(\frac{\alpha}{0.3}\right) \lesssim f_{\rm supp} \lesssim 0.32\left(\frac{\alpha}{0.3}\right)$$

--> Ray-tracing may help determine quality of signal

Conclusions & Future Directions

- Demonstrated consistency with prior work in the Newtonian regime;
- Investigated the evolution of the disk with separation evolution;
  - Separation time is consistent with alpha-disk theory using our sim's data;
  - Found that a disk can follow the binary to very small separations;

- Measured the luminosity from self-consistent (though ad-hoc) cooling rate;
  - Flux profile follows inspiral inward;
  - Strong periodic signal found from binary-lump interaction;
  - Opacity may blur the signal....? --> Need to ray-trace

- Need to calculate dynamics in immediate vicinity of black holes:
  - How does lump evolve after merger?
  - Does lump form in the first place, or is the m=1 mode from MHD turb.?
  - Does $r < r_{gap}$ dominate flux?

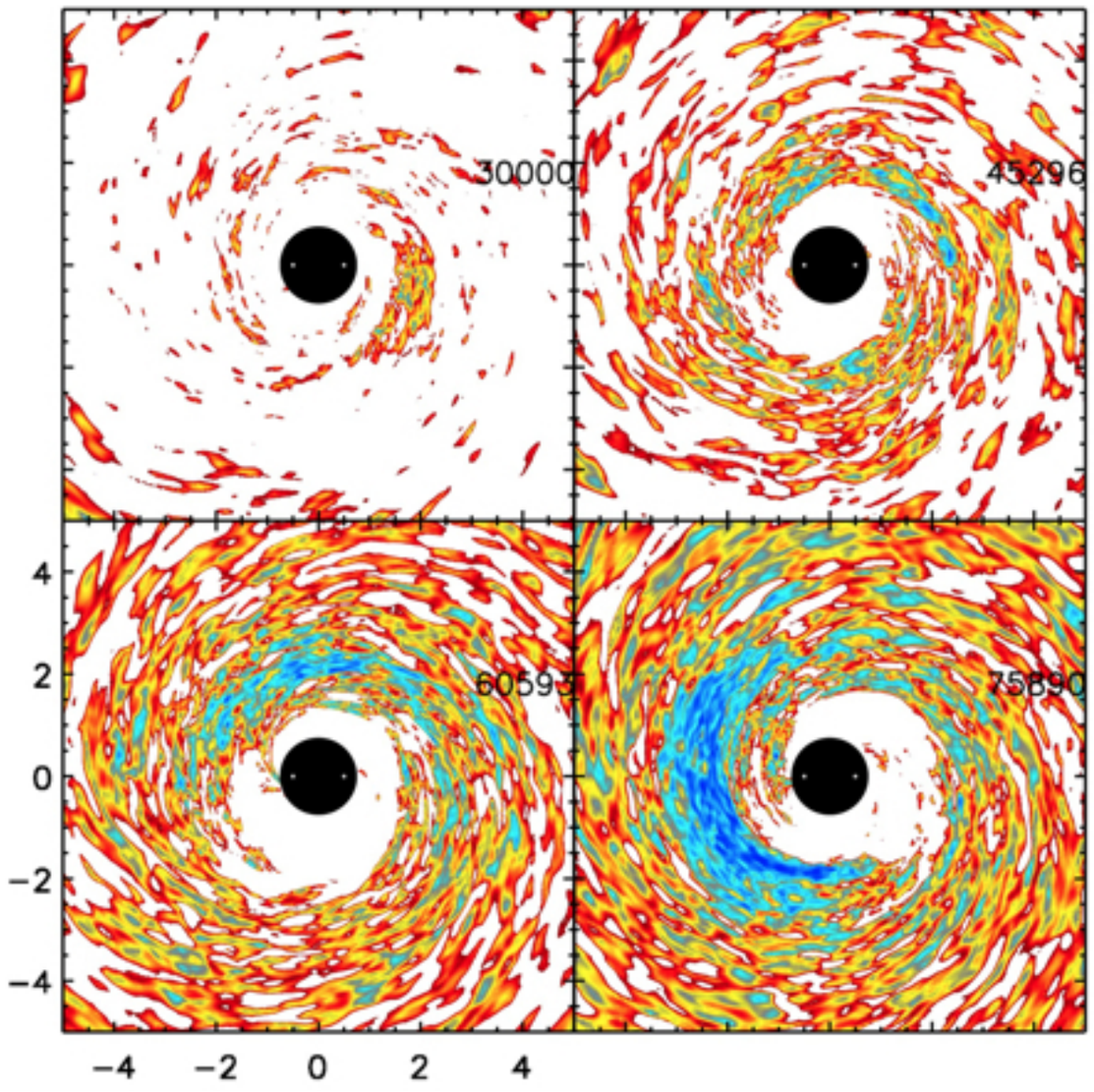
- Does a lump form with unequal mass or precessing binaries?

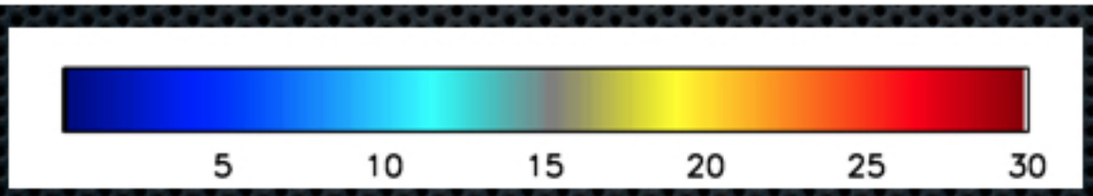
Extra Slides

MRI Resolution

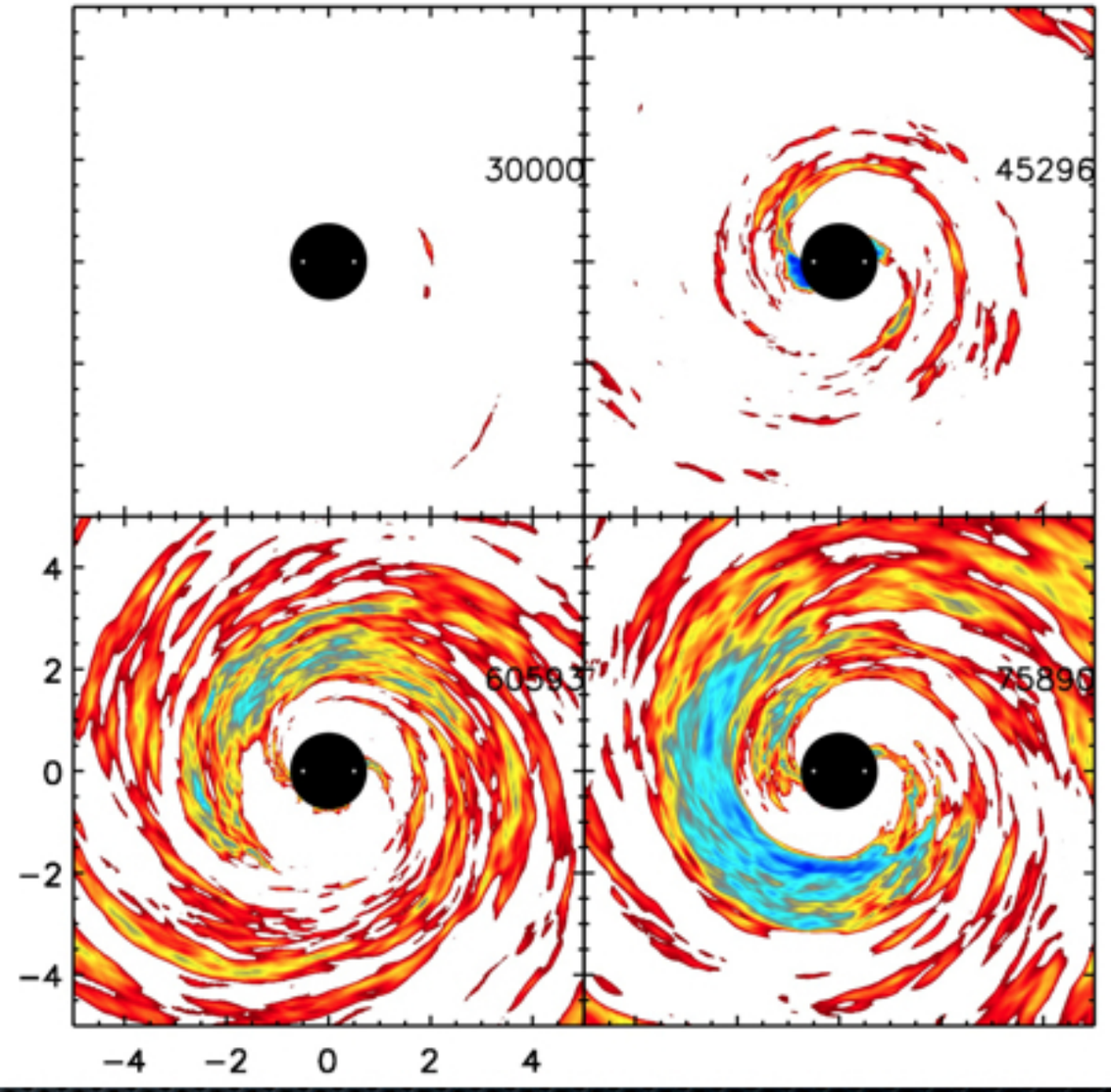
$$Q^i = \frac{2\pi \left| b^i \right|}{\Delta x^i \Omega(r) \sqrt{\rho h + 2 p_m}}$$

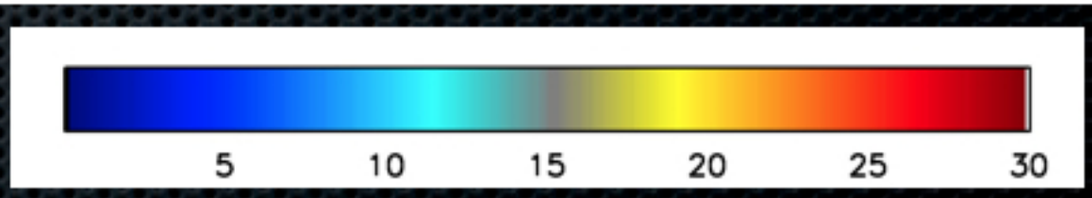
Sano++ 2004 Noble++ 2010 Guan, Gammie 2010 Sorathia++ 2010, 2011
Hawley++ 2011



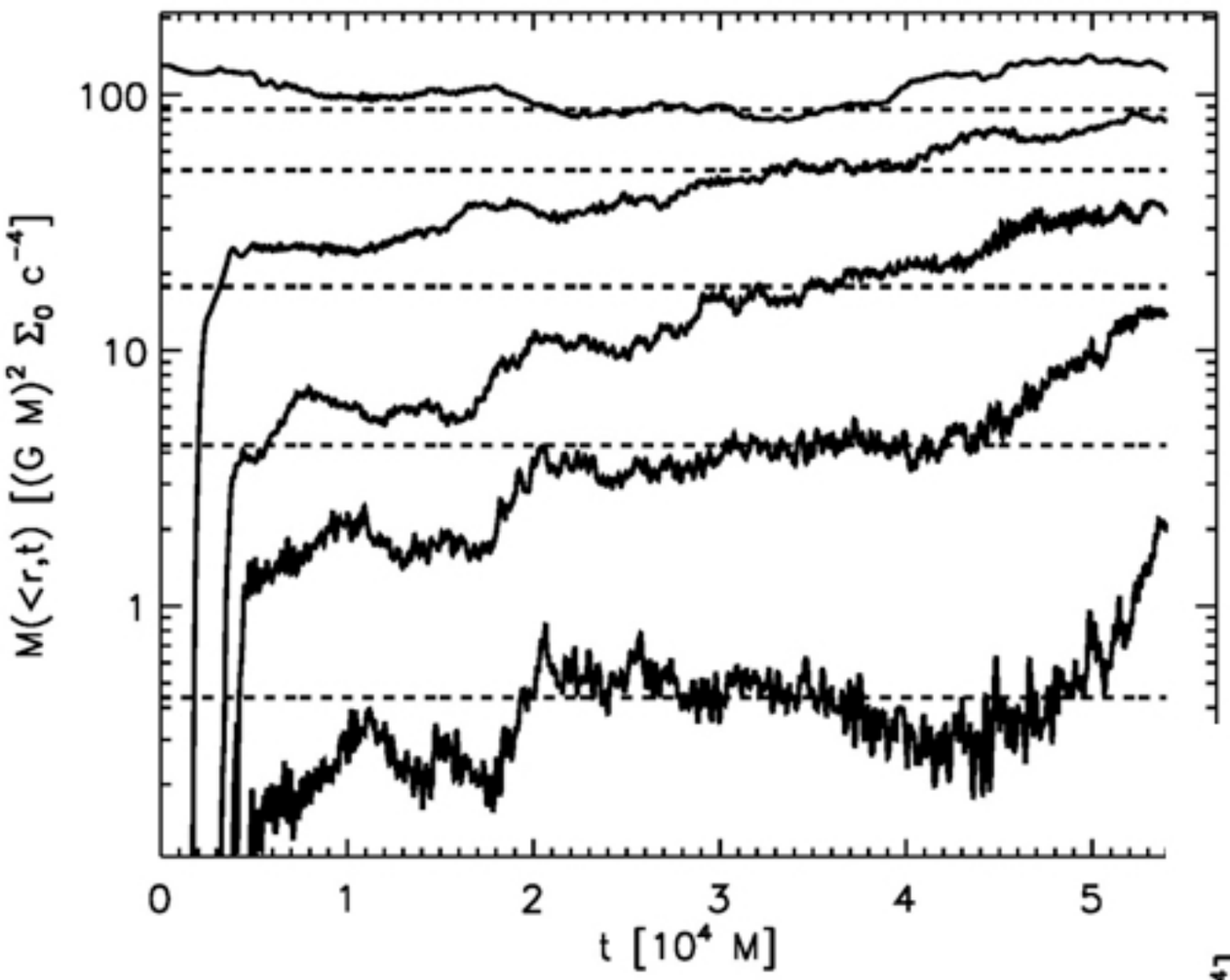


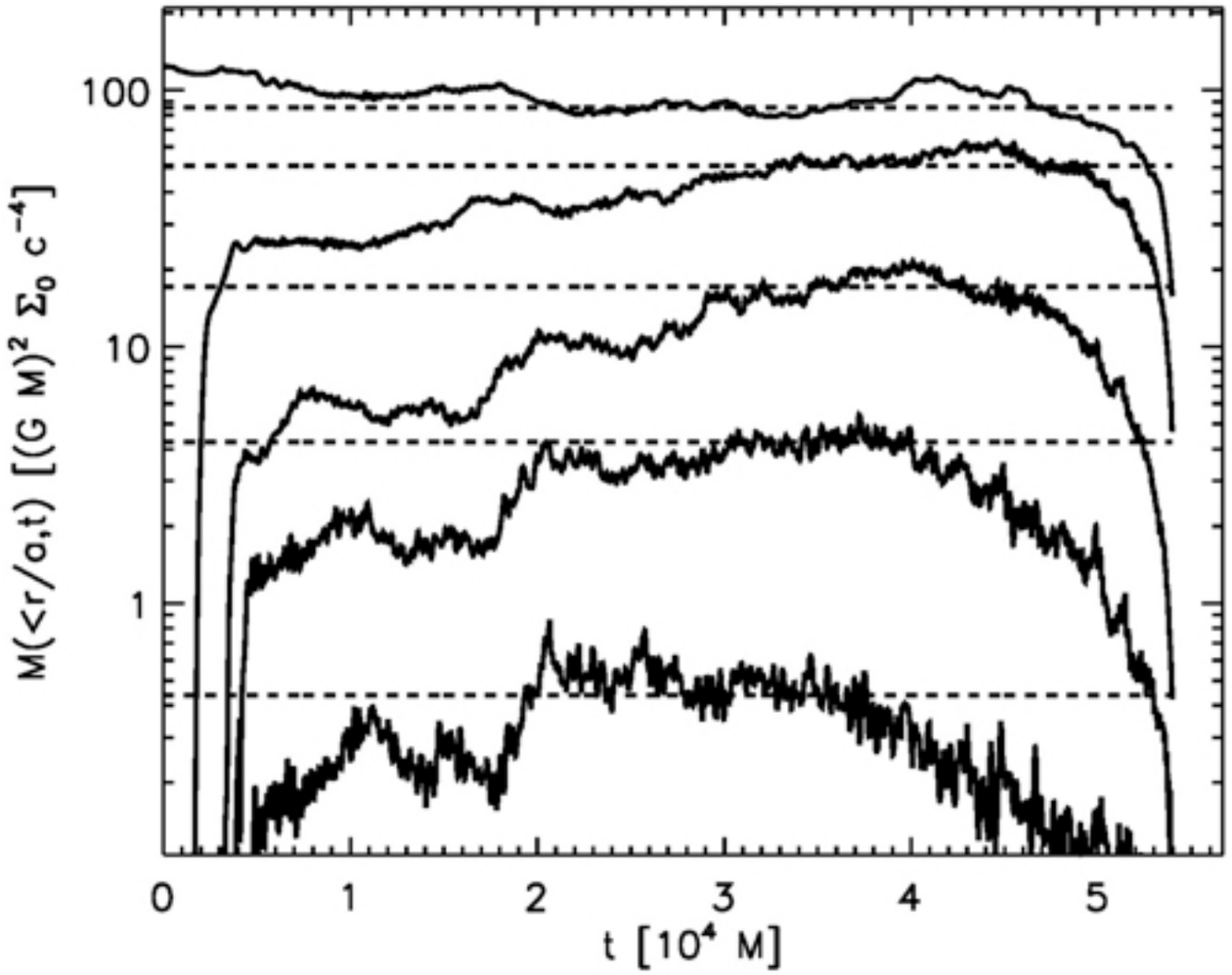
$Q^\theta > 10$



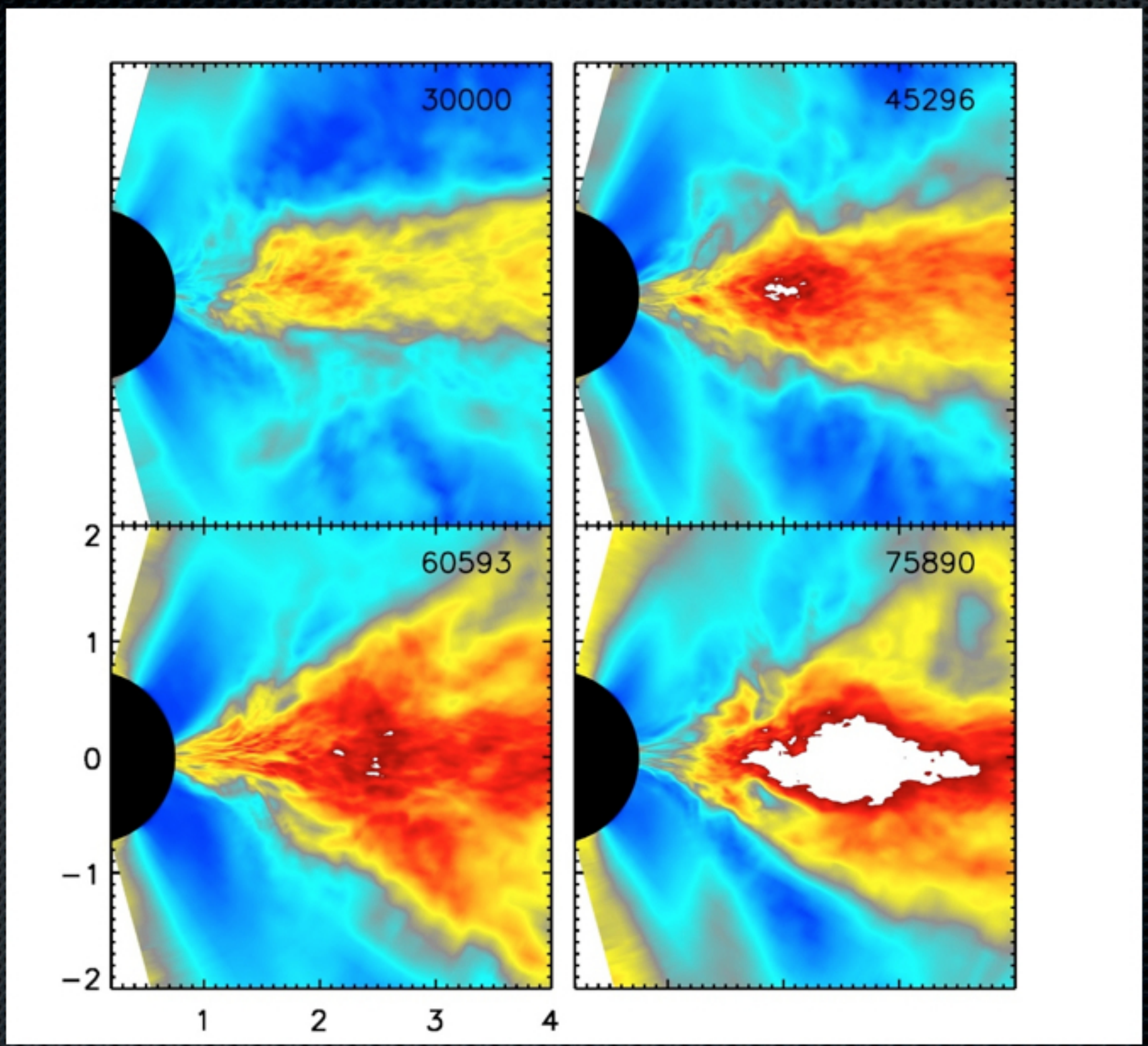


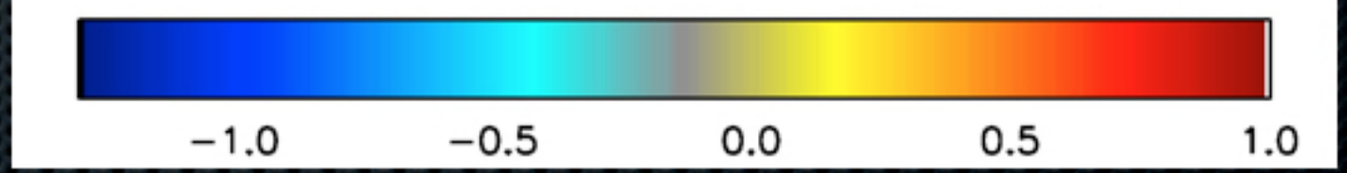
$Q^\phi > 25$


100
10
1
M(<r,t) [(G M)² Σ₀ c⁻⁴]
0
1
2
3
4
5
t [10⁴ M]

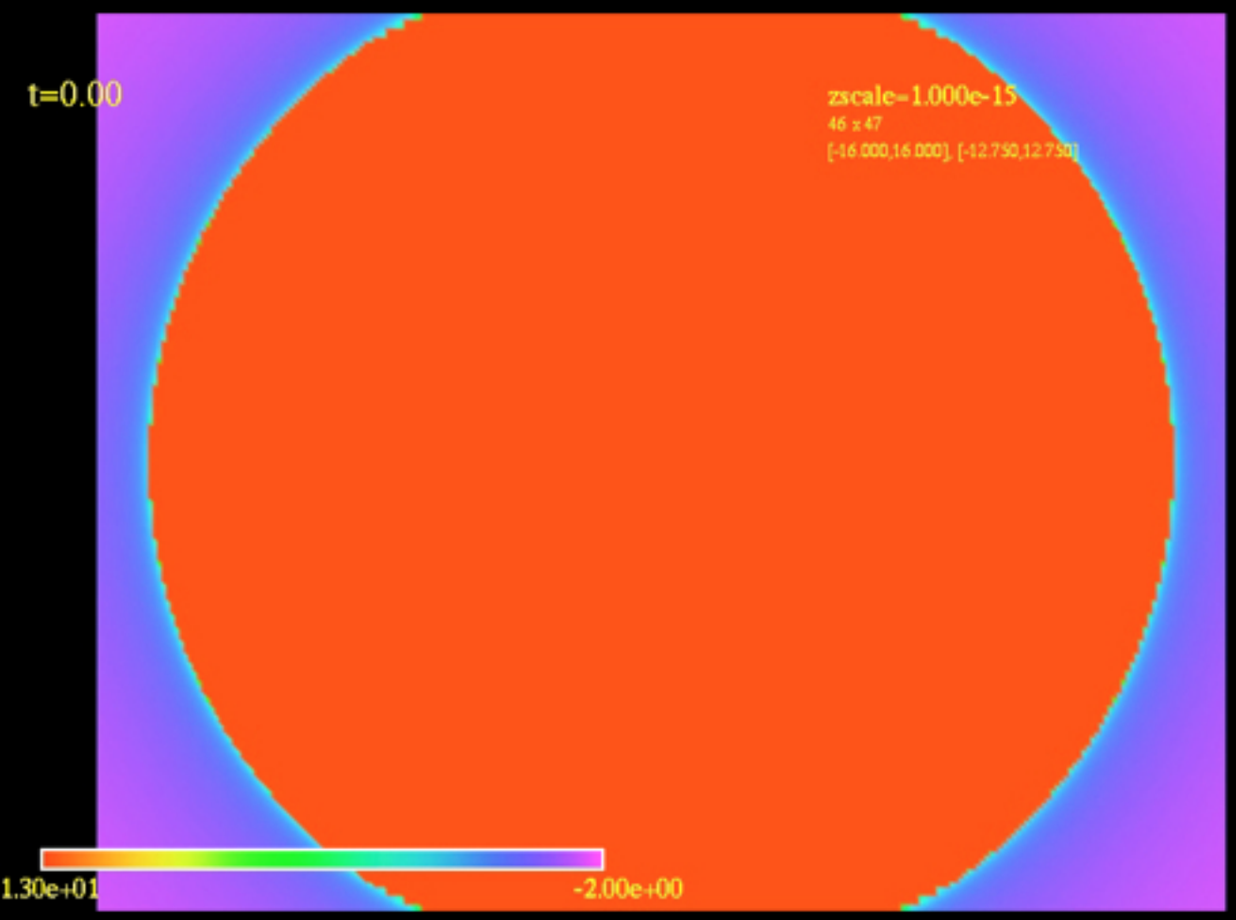

100
10
1
M(<r/a,t) [(G M)² Σ₀ c⁻⁴]
0
1
2
3
4
5
t [10⁴ M]

Plasma Beta parameter = pgas / pmag

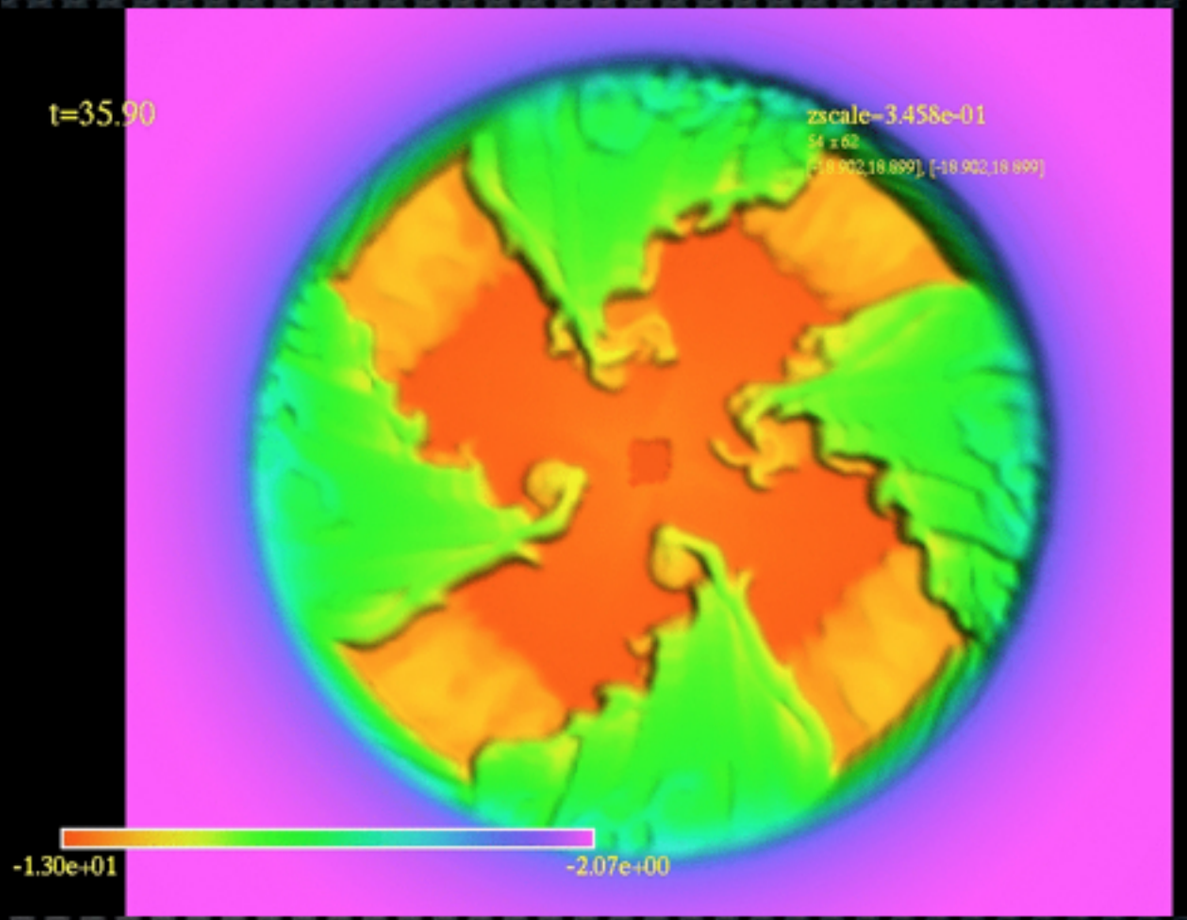




Disks with ET



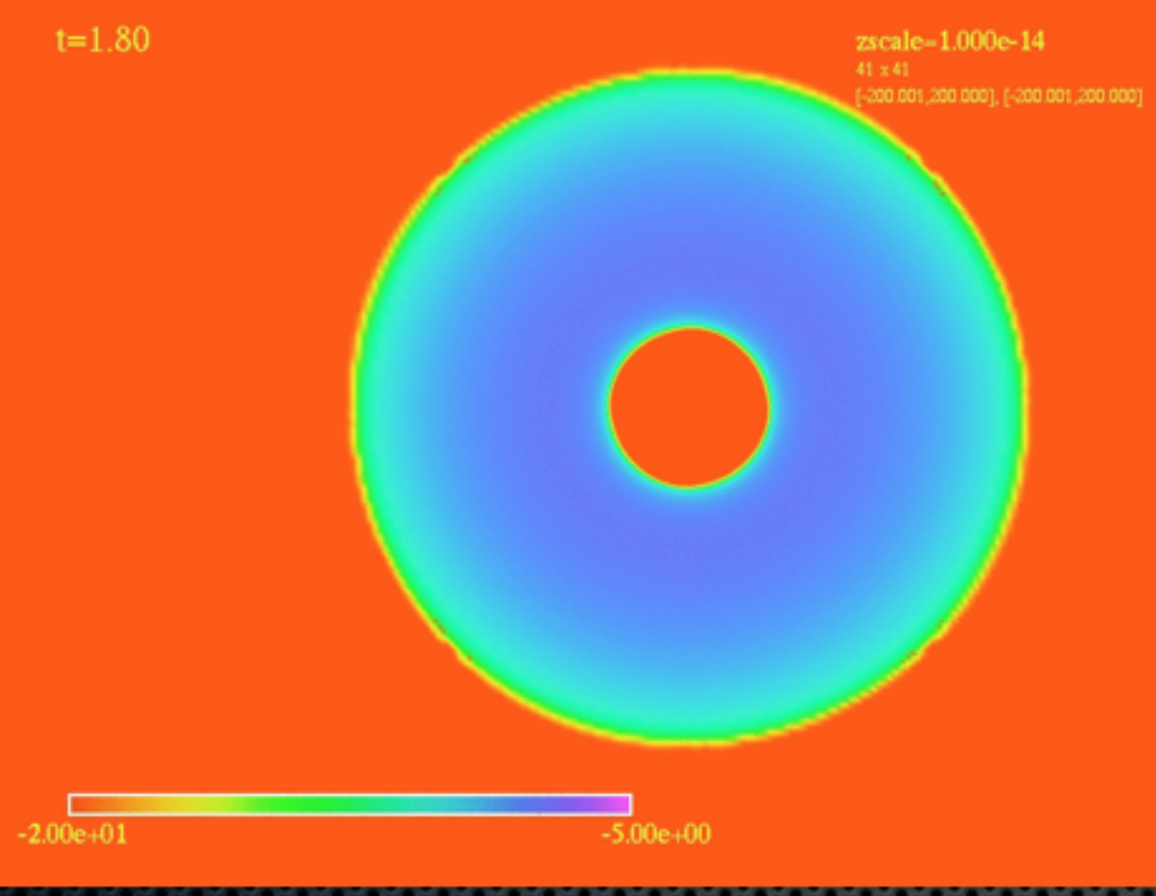
BBH



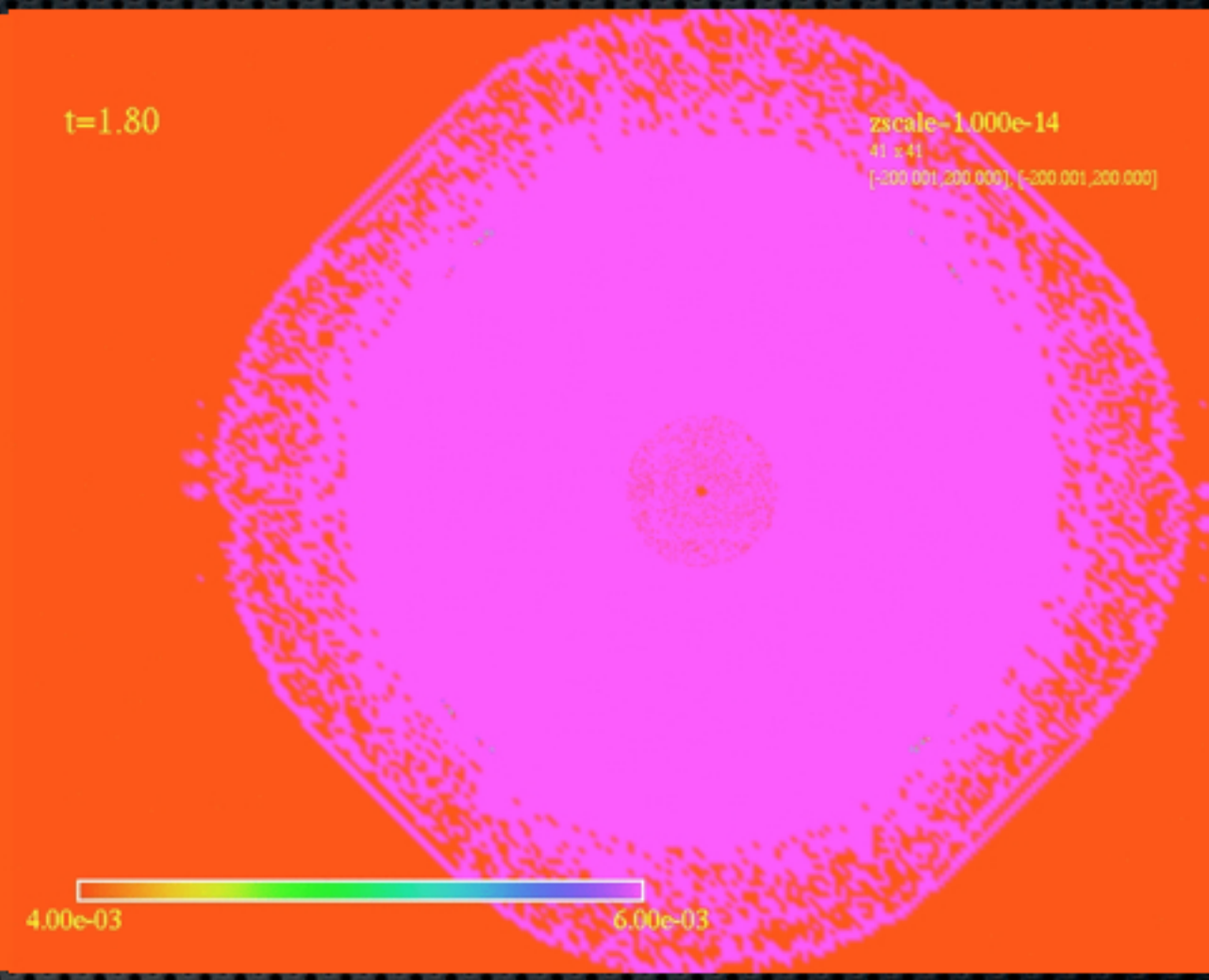
UBH

Viscous effects from Cartesian grid

Disks with ET



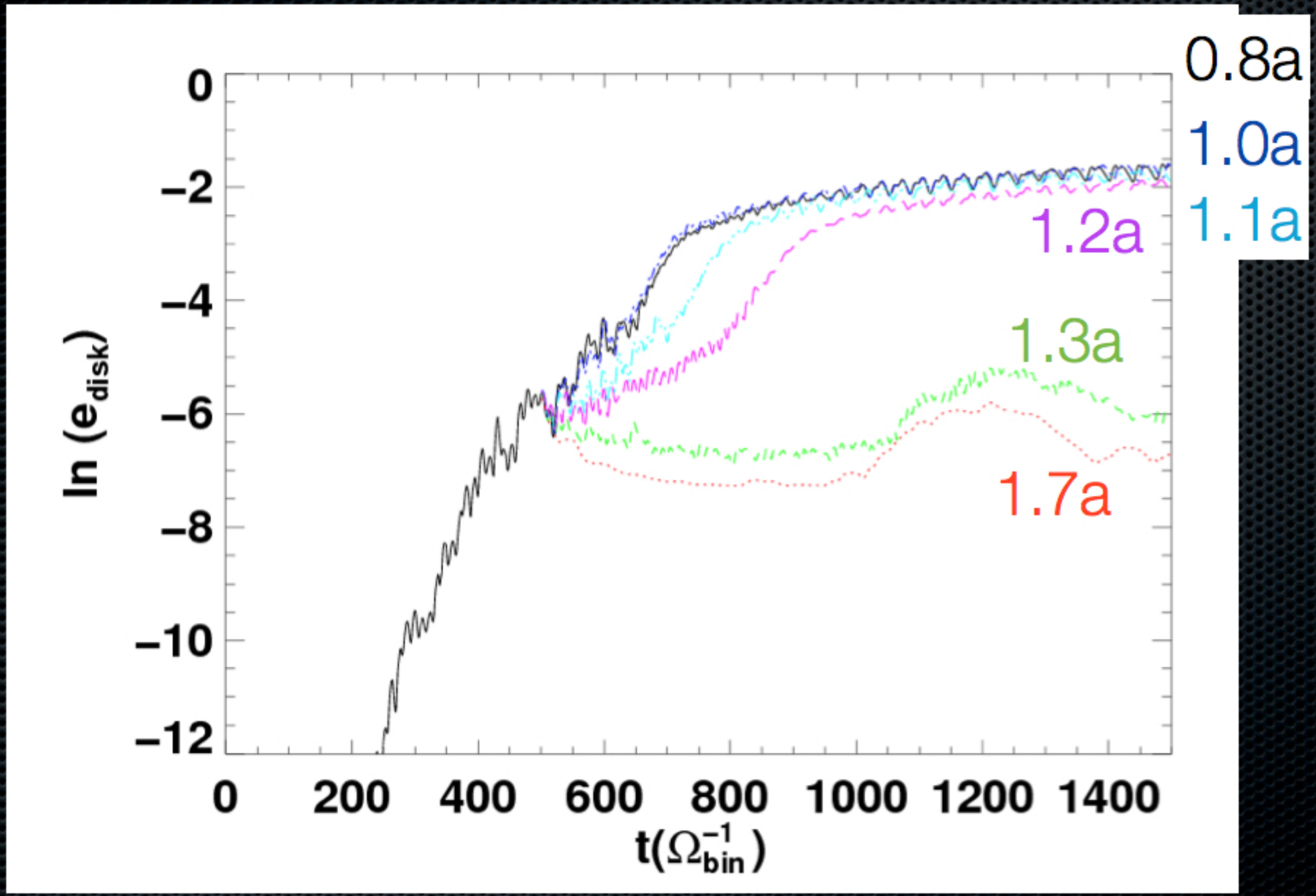
Emissivity/Cooling Rate



Entropy

Dissipative effects from AMR boundaries

Surveying effects of r_{in}



Shi++2012

Resolution Constraints: MRI

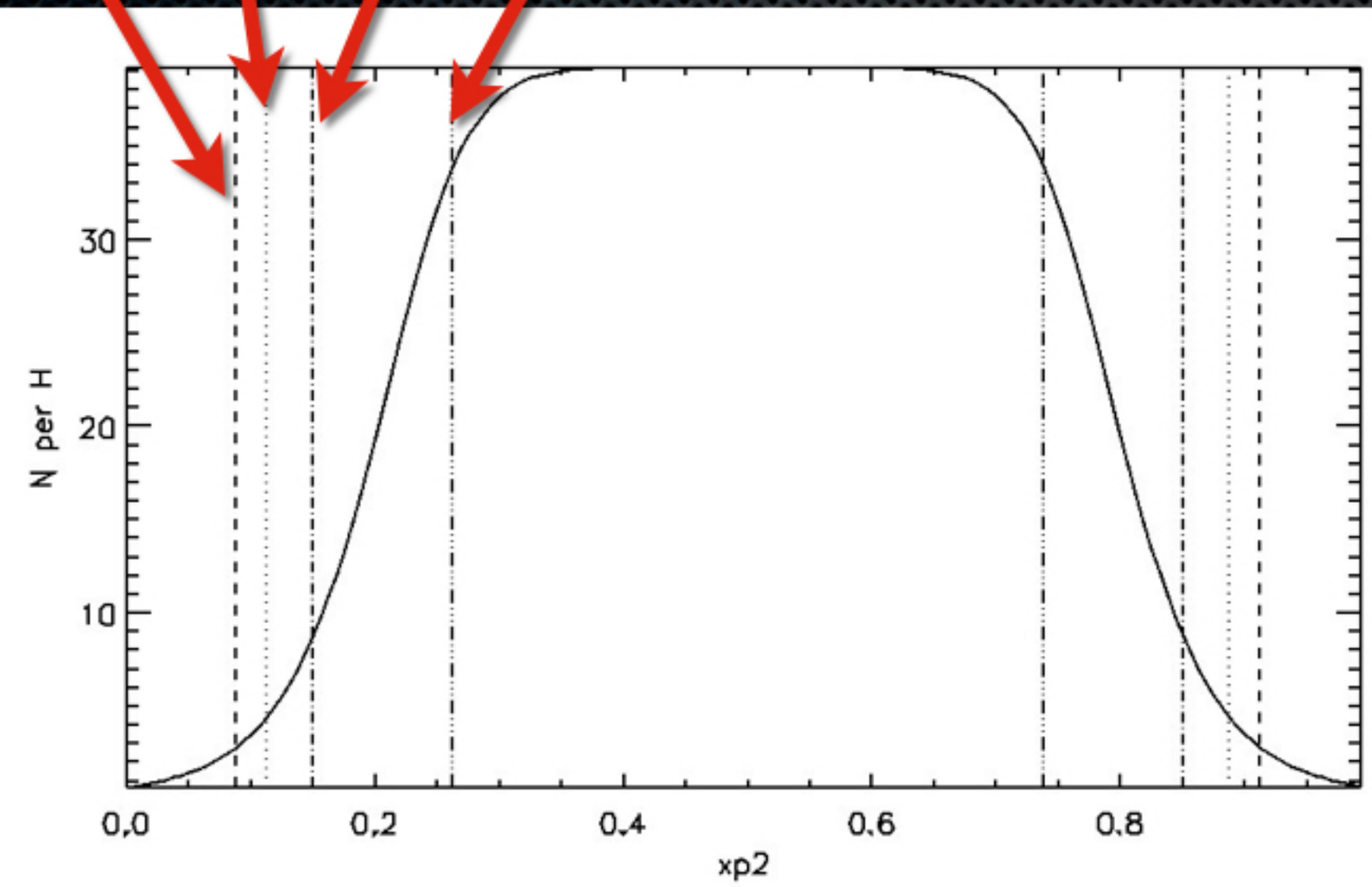
Sano++ 2004 Noble++ 2010 Guan, Gammie 2010 Sorathia++ 2010, 2011

Hawley++ 2011

$$N_\phi \simeq 1000\ (0.1\ R/H)\ (\beta/100)^{1/2}\ (Q_\phi/10)$$

$$N_z \simeq 16\,(\beta/100)^{1/2}\left(\langle v_A^2\rangle/\langle v_{Az}^2\rangle\right)^{1/2}(Q_z/10)$$

4H/r 3H/r 2H/r H/r



$$Q_z = \lambda_{MRI}/\Delta z = \frac{2\pi|v_{az}|}{\Omega\Delta z},$$

$$\beta_z/\beta \simeq 50$$

$$\beta \simeq 10$$

$$N_\theta > 36 \text{ per } H/r$$

$$\theta = \frac{\pi}{2}\left[1 + h_s\left(2x^2-1\right) + (1-h_s-2\theta_c/\pi)\left(2x^2-1\right)^n\right]$$

$$N_\theta = 160,\ h_s = 0.13,\ n = 9,\ \theta_c = 10^{-15}$$