State of the Art MHD Methods for Astrophysical Applications

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Plan of Attack

- Is NOT a survey of all possible methods!
- Highlight relatively new methods!
- Focus on tests and comparisons between the methods.
- Cover general idea of algorithms.
- Details are deferred to given references.

Outline

- Introduction to ideal Relativistic MagnetoHydroDynamics (RMHD).
- Two basic approaches: Non-conservative and Conservative
- Methods for preserving the Divergence Constraint
- Conclusions

Ideal Relativistic Magnetohydrodynamics

- Perfect fluid = Isotropic, inviscid, no thermal conductivity:
  - $T^{\mu\nu}_{\rm fluid} = (P + \rho + \rho\epsilon)\, u^\mu u^\nu + P g^{\mu\nu}$
  - $\nabla_\mu \left(\rho u^\mu\right) \;=\; \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \rho u^\mu\right) \;=\; 0$
- Electrically neutral, infinitely conducting fluid:
  - $T^{\mu\nu}_{\rm EM} = F^{\mu\alpha} F^{\nu}{}_{\alpha} - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}$
  - $\nabla_\mu F^{\mu\nu} = J^\nu$
  - $\nabla_\mu F^{*\mu\nu} = 0 \quad , \quad F^{*\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\kappa\lambda} F^{\kappa\lambda}$
  - Infinite conductance $\Rightarrow$ $F^{\mu\nu} u_\mu = 0$

Ideal RMHD (cont.)

$$F^{\mu\nu} u_\mu = 0 \qquad \nabla_\mu F^{*\mu\nu} = 0 \qquad \nabla_\mu F^{\mu\nu} = J^\nu \qquad v^j = u^i / u^t$$

$$b^\mu \equiv F^{*\mu\nu} u_\nu = \tfrac{1}{2} \varepsilon^{\mu\nu\kappa\lambda} F_{\kappa\lambda} u_\nu \qquad F^{*\mu\nu} = b^\mu u^\nu - b^\nu u^\mu$$

$$b^\mu u_\mu = 0 \qquad B^i \equiv F^{*it} \qquad b^t = B^i u^\mu g_{i\mu} \qquad b^i = \tfrac{1}{u^t} \left(B^i + b^t u^i \right)$$

$$\partial_t \left(\sqrt{-g} B^i \right) = -\partial_j \left[\sqrt{-g} \left(B^j v^i - B^i v^j \right) \right] \qquad \partial_j \left(\sqrt{-g} B^j \right) = 0$$

Ideal RMHD (cont.)

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Ideal RMHD (cont.)

$\nu := i$

$$F^{\mu\nu} u_\mu = 0 \qquad \boxed{\nabla_\mu F^{*\mu\nu} = 0} \qquad \nabla_\mu F^{\mu\nu} = J^\nu \qquad v^j = u^i / u^t$$

$$b^\mu \equiv F^{*\mu\nu} u_\nu = \frac{1}{2} \varepsilon^{\mu\nu\kappa\lambda} F_{\kappa\lambda} u_\nu \qquad \boxed{F^{*\mu\nu} = b^\mu u^\nu - b^\nu u^\mu}$$

$$b^\mu u_\mu = 0 \qquad B^i \equiv F^{*it} \qquad \boxed{b^t = B^i u^\mu g_{i\mu} \qquad b^i = \frac{1}{u^t} \left(B^i + b^t u^i \right)}$$

$$\boxed{\partial_t \left(\sqrt{-g} B^i \right) = -\partial_j \left[\sqrt{-g} \left(B^j v^i - B^i v^j \right) \right]} \qquad \partial_j \left(\sqrt{-g} B^j \right) = 0$$

Ideal RMHD (cont.)

$\nu = t$

$F^{\mu\nu} u_\mu = 0$ $\quad$ $\boxed{\nabla_\mu F^{*\mu\nu} = 0}$ $\quad$ $\nabla_\mu F^{\mu\nu} = J^\nu$ $\quad$ $v^j = u^i / u^t$

$$b^\mu \equiv F^{*\mu\nu} u_\nu = \frac{1}{2} \varepsilon^{\mu\nu\kappa\lambda} F_{\kappa\lambda} u_\nu \qquad \boxed{F^{*\mu\nu} = b^\mu u^\nu - b^\nu u^\mu}$$

$$b^\mu u_\mu = 0 \qquad B^i \equiv F^{*it} \qquad \boxed{b^t = B^i u^\mu g_{i\mu} \qquad b^i = \frac{1}{u^t} \left(B^i + b^t u^i \right)}$$

$$\partial_t \left(\sqrt{-g} B^i \right) = -\partial_j \left[\sqrt{-g} \left(B^j v^i - B^i v^j \right) \right] \qquad \boxed{\partial_j \left(\sqrt{-g} B^j \right) = 0}$$

Ideal RMHD (cont. II)

$$T^{\mu\nu}_{\rm MHD} = \left(\rho + \rho\epsilon + P + b^2\right) u^\mu u_\nu \; + \; \left(P + \tfrac{1}{2} b^2\right) \delta^\mu{}_\nu \; - \; b^\mu b_\nu$$

$$\nabla_\mu T^\mu{}_\nu = 0$$

$$\partial_t \sqrt{-g} \begin{bmatrix} \rho u^t \\ T^t{}_\mu \\ B^i \end{bmatrix} + \partial_j \sqrt{-g} \begin{bmatrix} \rho u^j \\ T^j{}_\mu \\ B^j v^i - B^i v^j \end{bmatrix} = \sqrt{-g} \begin{bmatrix} 0 \\ T^\alpha{}_\beta \Gamma^\beta{}_{\alpha\mu} \\ 0 \end{bmatrix}$$

$$\partial_t \left(\sqrt{-g}\mathbf{q}\right) \quad + \quad \partial_j \left(\sqrt{-g}\mathbf{F}^j\right) \quad = \quad \mathbf{S}(\mathbf{q}, g_{\mu\nu})$$

$$\partial_j \left(\sqrt{-g} B^j\right) \; = \; 0$$

Non-conservative Methods

$\partial_t \mathbf{q} + \partial_i \mathbf{f}^i = \mathbf{S}(\mathbf{q}, \partial_\mu q)$

- + Experienced methods from hyrdro. simulations;
- + Excellent for smooth flows;
- + Computationally efficient;
- + Use of upwind, monotonic methods for advection helps w/ shocks;
- − Requires use of artificial viscosity, $P \rightarrow P + Q$;
- − Still problems near shocks, especially as $v \rightarrow 1$;

Norman and Winkler, in *Astrophys. Radiation Hydro.*, 449 (1986)

ZEUS-like: FD/CT/MOC-CT Methods

FD = Finite Difference , MOC = Method of Characteristics

CT = Constrained Transport

- Wilson, in *7th Texas Symp. on Rel. Astrophys.*, 1975. [MHD, Schwarz. Accretion]
- Hawley, Smarr, Wilson, *ApJ* **277**, 296 (1984). [Hydro. Eq.s]
- Hawley, Smarr, Wilson, *ApJS* **55**, 211, (1984). [Hydro. Tests, Kerr Accretion]
- Evans and Hawley, *ApJ* **332**, 659 (1988). [CT, 2D MHD, Schwarzschild Accretion]
- Stone and Norman, *ApJS* **80**, 791 (1992). [ZEUS-2D, MOC-CT]
- ZEUS Code webpage, http://www.astro.princeton.edu/∼jstone/zeus.html
- Hawley, Stone, *Comput. Phys. Commun.* **89**, 127 (1995). [MOC-CT]
- De Villiers and Hawley, *ApJ* **589**, 458 (2003). [3D MHD, Kerr Accretion]

De Villiers and Hawley 2003 (DH)

- Non-conservative scheme:

$$\partial_t \mathbf{q} + \partial_i \left[v^i \mathbf{G}(\mathbf{q}, g_{\mu\nu}) \right] = \mathbf{S}(\mathbf{q}, \partial_\mu \mathbf{q}, g^{\mu\nu}, \partial_\alpha g^{\mu\nu})$$

- Evolution is broken down into several sub-steps: $\partial_t \mathbf{q} = L(\mathbf{q})$

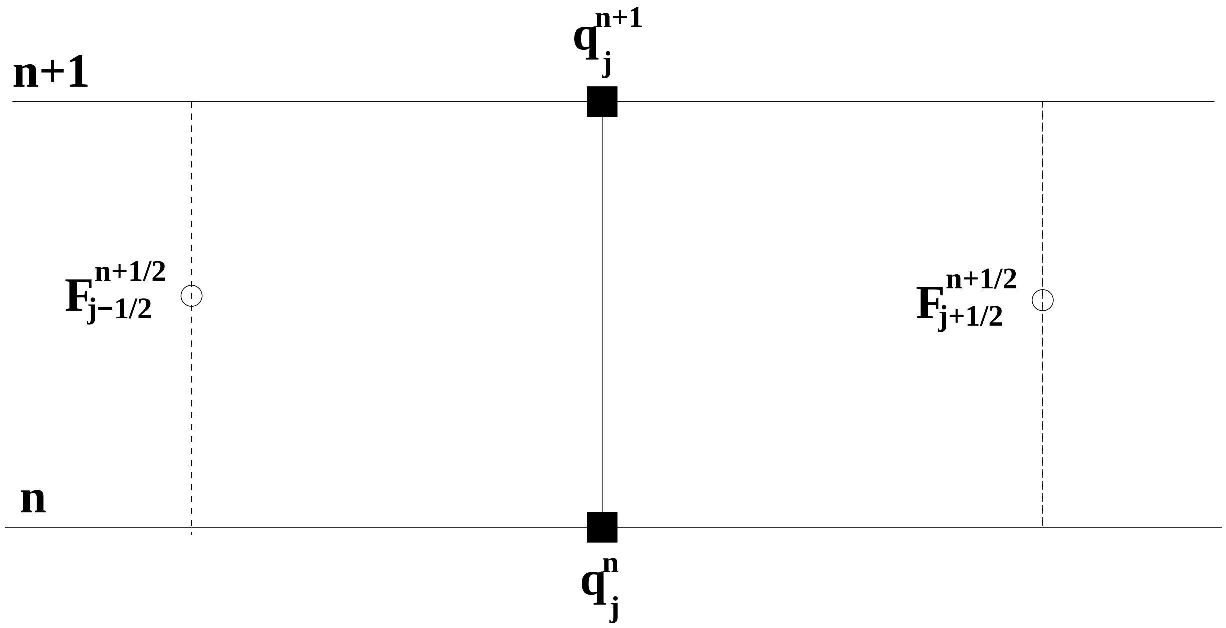
$$\begin{aligned} & \mathbf{MOC-CT} && : \ \text{Finds } b^\mu \\ \mathbf{q}_1 &= \mathbf{q}_0 + \Delta t L_0(\mathbf{q}_0) && : \ \text{Transport } L_0 \propto \partial_i(\sqrt{\gamma} v^i H), H = \{D, E, S_j\} \\ \mathbf{q}_2 &= \mathbf{q}_1 + \Delta t L_1(\mathbf{q}_1) && : \ \text{Source } L_1 \propto \partial_\mu \{W, P, g_{\nu\lambda}, b^\nu\}\ ,\ \mathbf{q} \end{aligned}$$

- Staggered mesh: vectors at cell faces , scalars at cell centers;
- New version of MOC-CT for $\partial_i B^i = 0$;
- Full 3D MHD in Boyer-Lindquist coordinates;
- L_0 found via monotonic upwind scheme (van Leer 1977);

Conservative Methods: Finite Volume

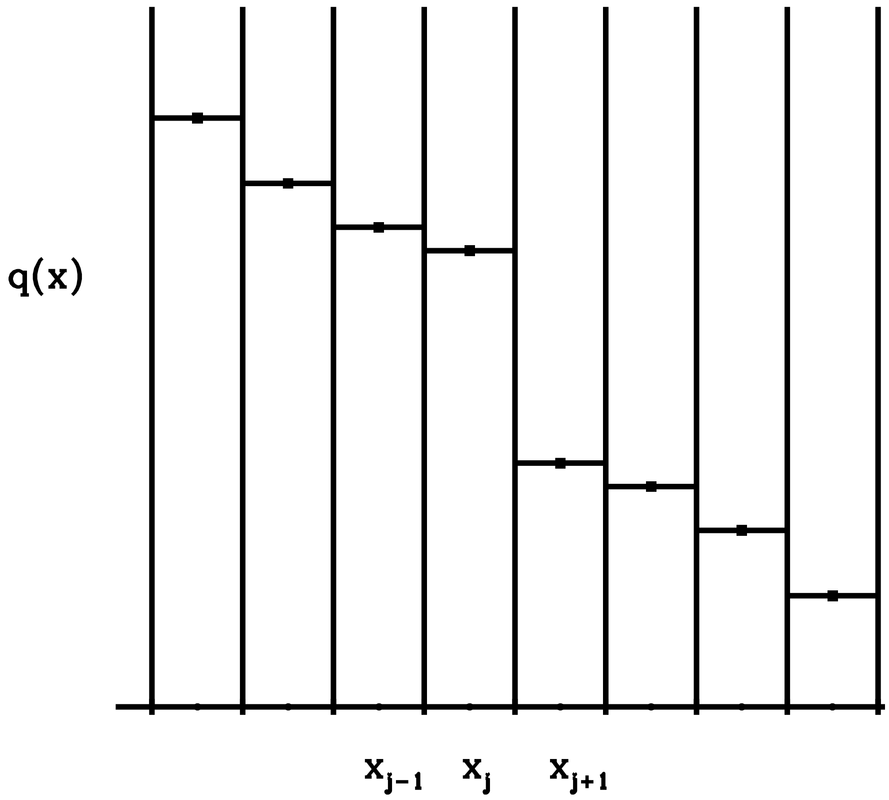
$$\partial_t \mathbf{q} + \partial_x \mathbf{f} = \mathbf{S}(\mathbf{q})$$

$$\bar{\mathbf{q}}_j^{n+1} - \bar{\mathbf{q}}_j^{n} = -\Delta t \left(\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2} \right)^{n+1/2} + \Delta t \, \bar{\mathbf{S}}_j^{n+1/2}$$



Conservative Methods: Godunov Schemes

$$\partial_t \mathbf{q} + \partial_i \mathbf{f}^i = \mathbf{S}(\mathbf{q})$$

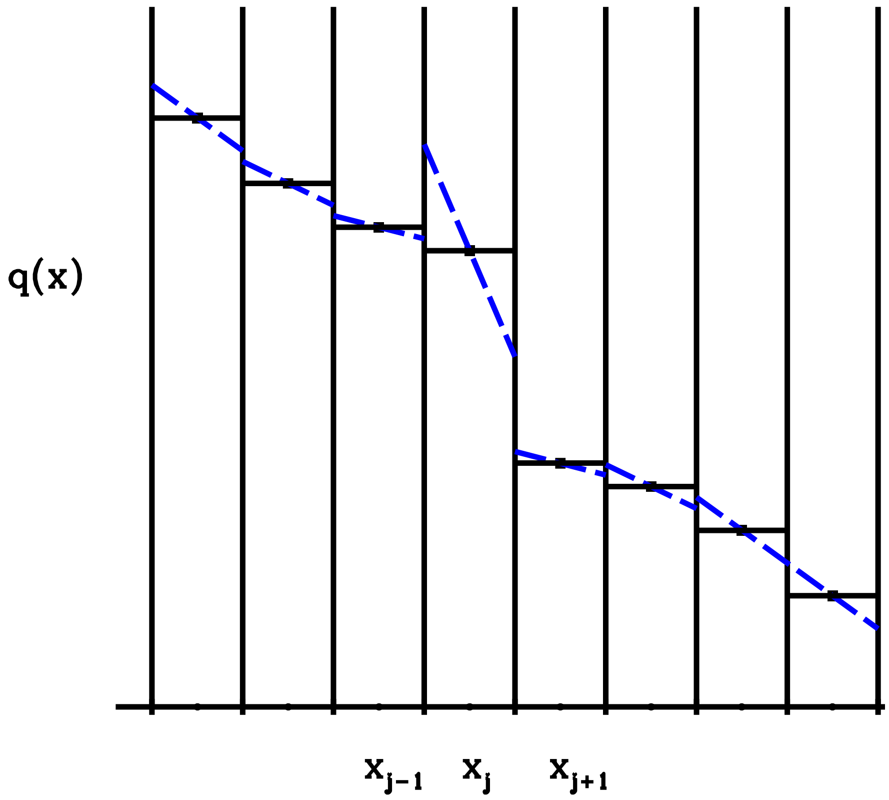


- Assume piecewise-constant data,

$$\mathbf{q} \rightarrow \bar{\mathbf{q}}$$

Conservative Methods: Godunov Schemes

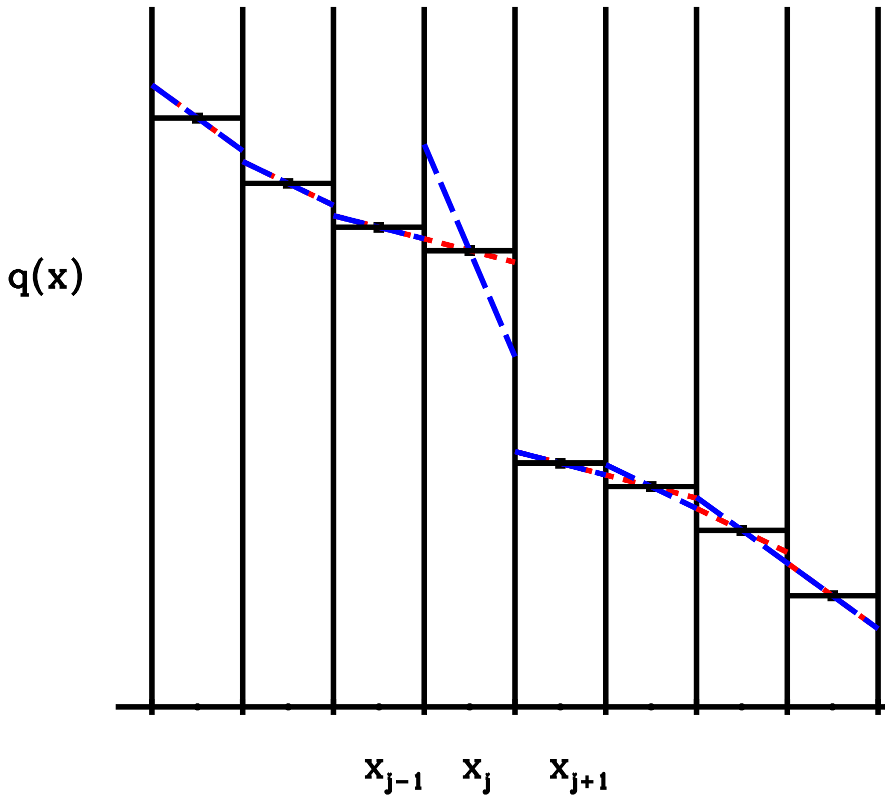
$$\partial_t \mathbf{q} + \partial_i \mathbf{f}^i = \mathbf{S}(\mathbf{q})$$



- Assume piecewise-constant data,
  $$\mathbf{q} \rightarrow \bar{\mathbf{q}}$$
- For higher resolution, interpolate for $\bar{\mathbf{q}}^L$ and $\bar{\mathbf{q}}^R$

Conservative Methods: Godunov Schemes

$$\partial_t \mathbf{q} + \partial_i \mathbf{f}^i = \mathbf{S}(\mathbf{q})$$



- Assume piecewise-constant data,
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- For higher resolution,
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- Need to use monotonic
  (aka slope-limiting) interpolation;

Conservative Methods: Godunov Schemes

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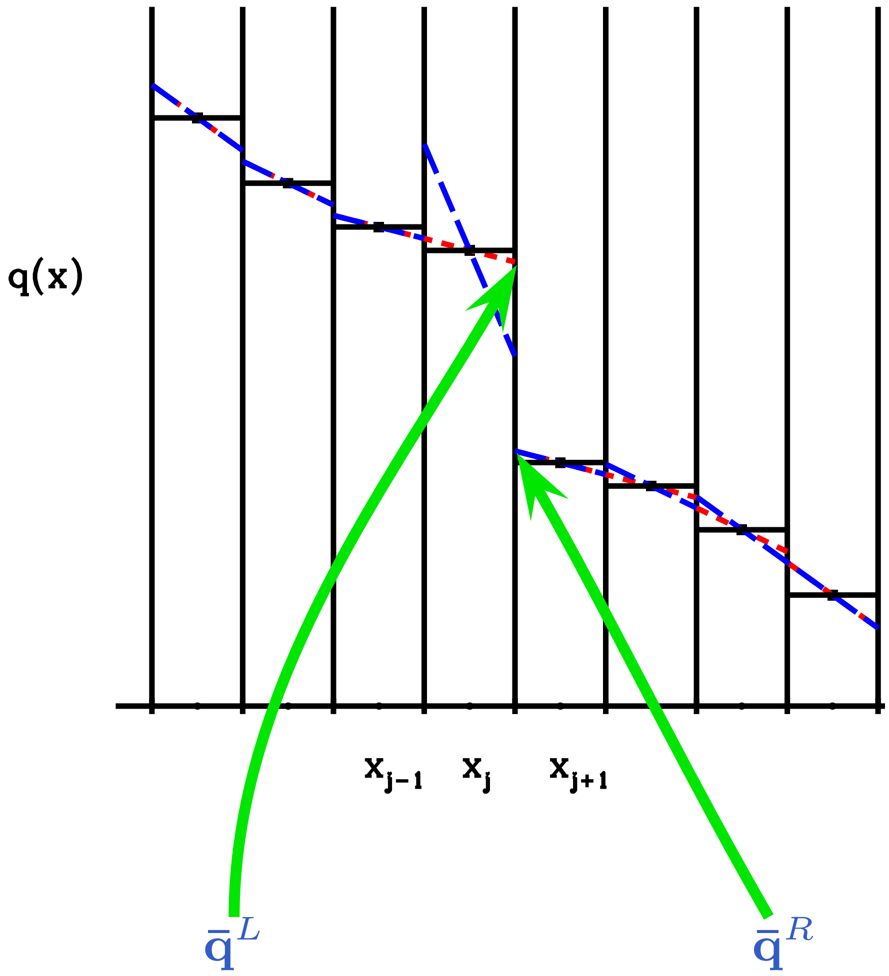
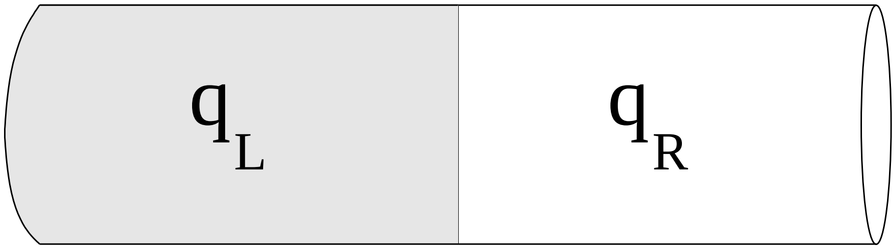


- Assume piecewise-constant data,
  $\mathbf{q} \rightarrow \bar{\mathbf{q}}$
- For higher resolution,
  interpolate for $\bar{\mathbf{q}}^L$ and $\bar{\mathbf{q}}^R$
- Need to use monotonic
  (aka slope-limiting) interpolation;

Riemann Problem



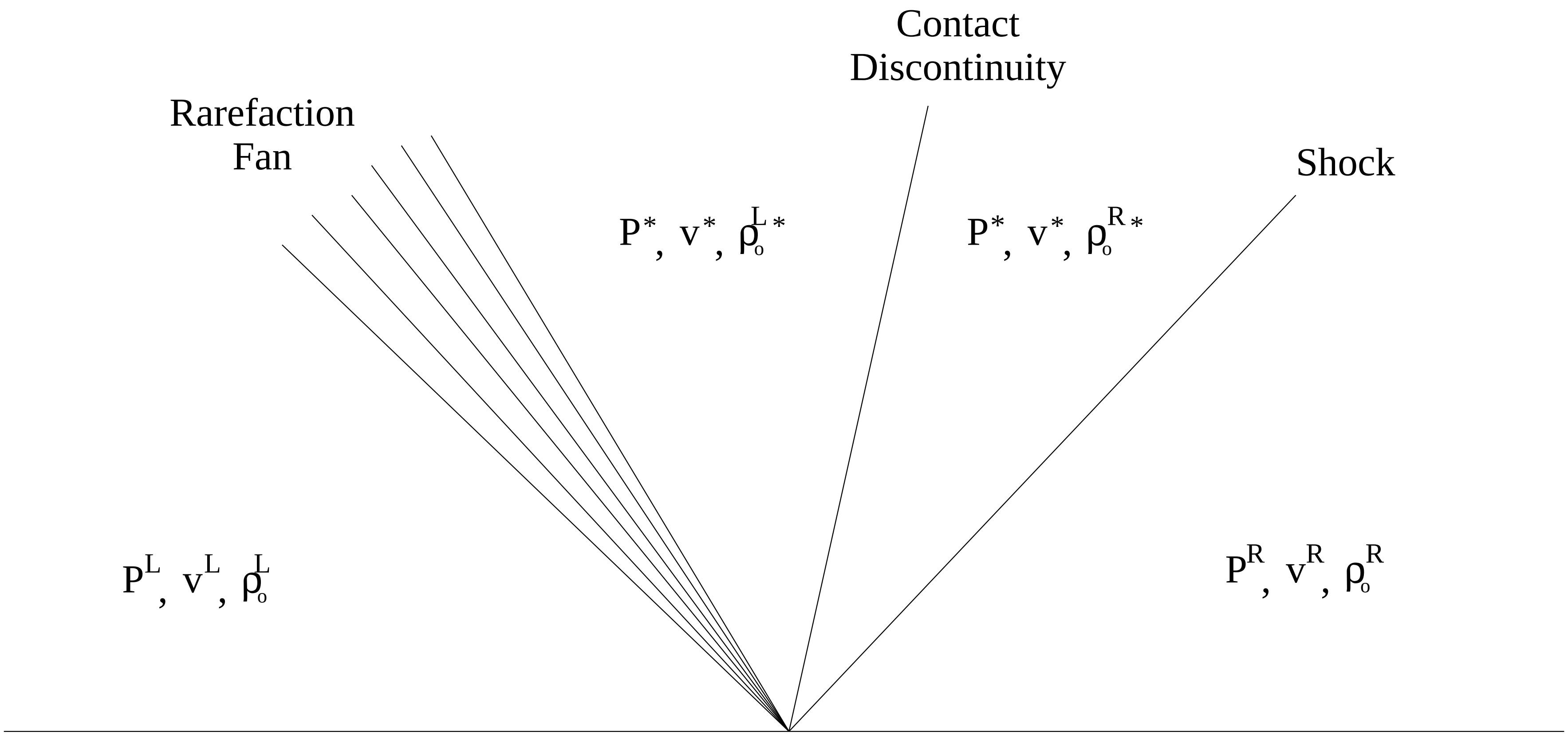
- Find $\mathbf{q}(x, t)$ given

$$\partial_t \mathbf{q} + \partial_x \mathbf{f} = 0$$

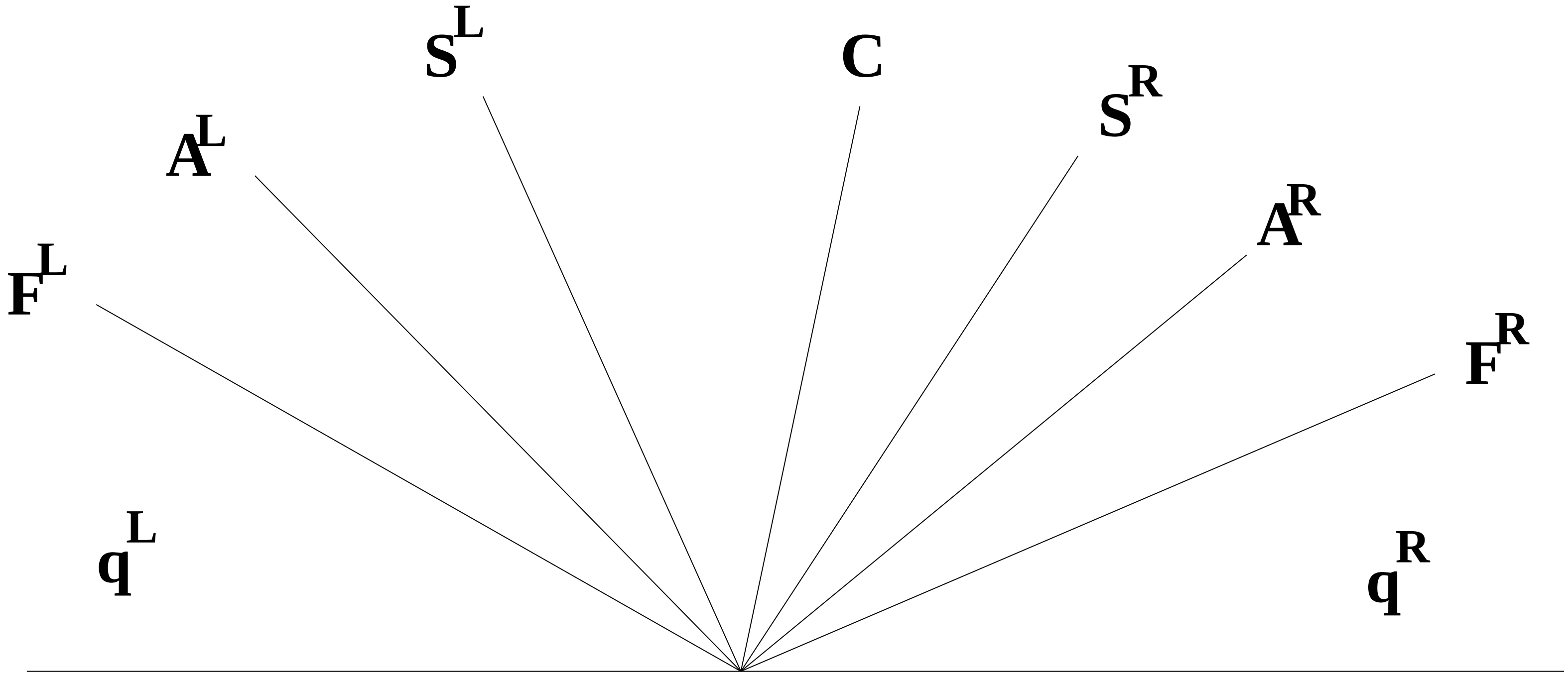
$$\mathbf{q}(x, t = 0) \; = \begin{cases} \mathbf{q}^L & \text{for } x < 0 \\ \mathbf{q}^R & \text{for } x > 0 \end{cases}$$

- Riemann solution $\Rightarrow \mathbf{F}^{n+1/2}$.

Hydrodynamic Riemann Solution



MHD Riemann Solution



$F^{L,R}$ = Fast magnetosonic wave

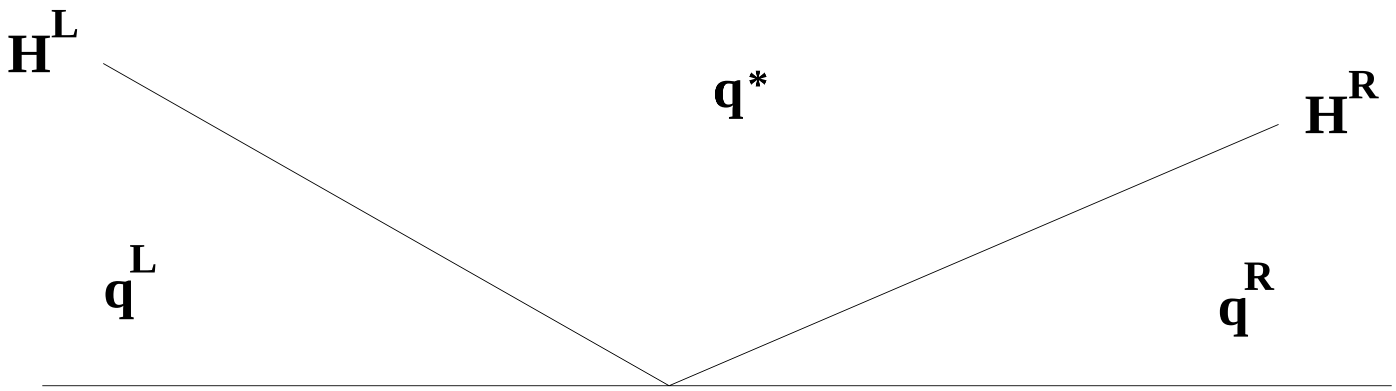
$S^{L,R}$ = Slow magnetosonic wave

$A^{L,R}$ = Alfvenic wave

C = Contact Discontinuity

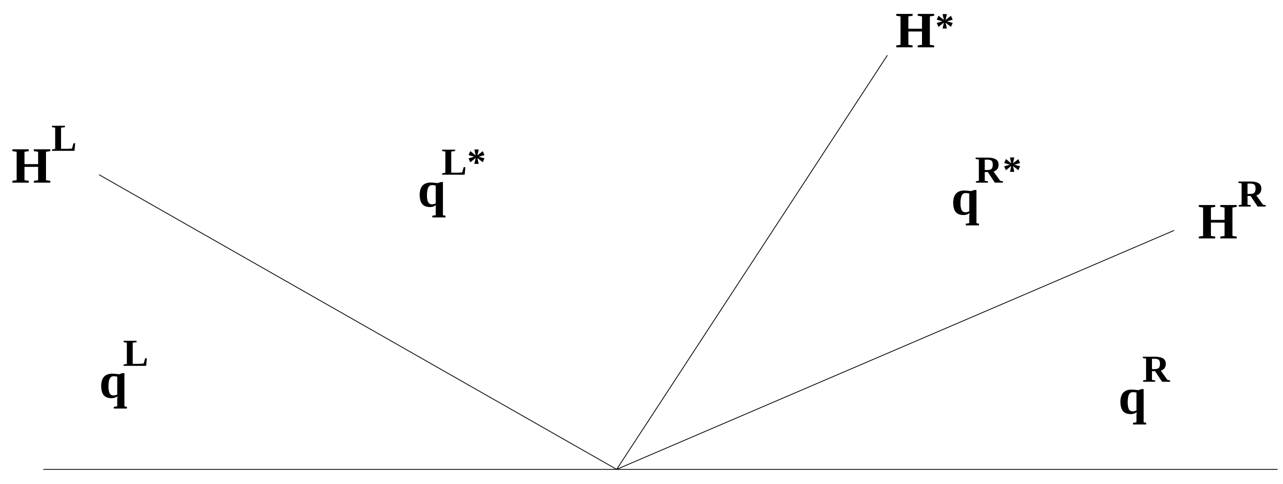
Approximate Riemann Solvers: HLL

$$\partial_t \mathbf{q} \; + \; \mathbf{A}^i \partial_i \mathbf{q} \; = \; 0$$



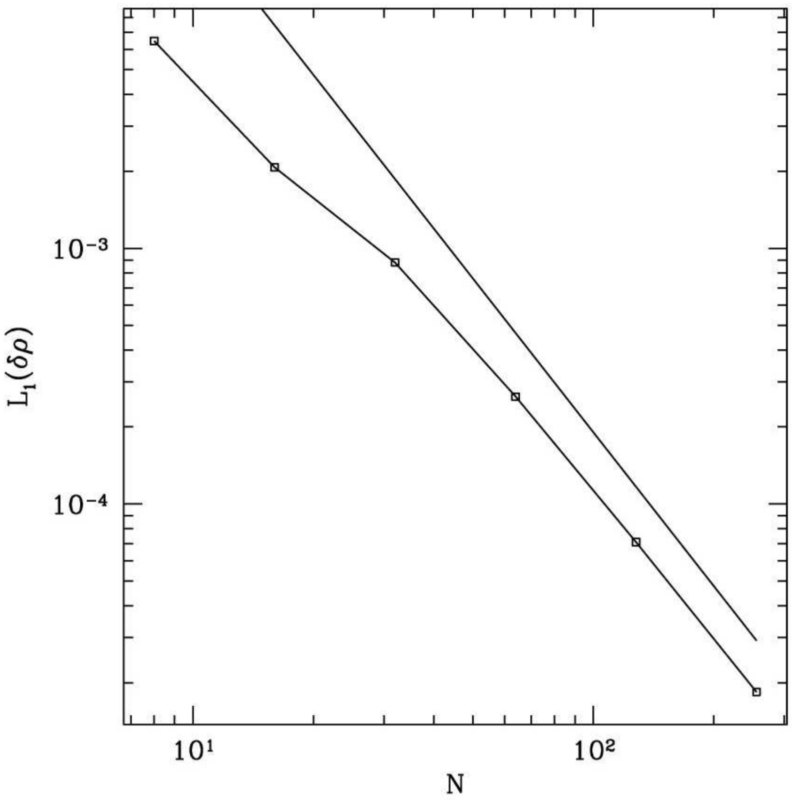
$$\mathbf{F}^{n+1/2}_{j+1/2} \; = \; \begin{cases} \mathbf{f}(\mathbf{q}^L) & \text{if } 0 \geq H^L \\ \frac{1}{\left(H^R - H^L\right)} \left[H^R \mathbf{f}^L - H^L \mathbf{f}^R + H^L H^R \left(\mathbf{q}^R - \mathbf{q}^L \right) \right] & \text{if } H^L \leq 0 \leq H^R \\ \mathbf{f}(\mathbf{q}^R) & \text{if } H^R \leq 0 \end{cases}$$

Approximate Riemann Solvers: HLLC

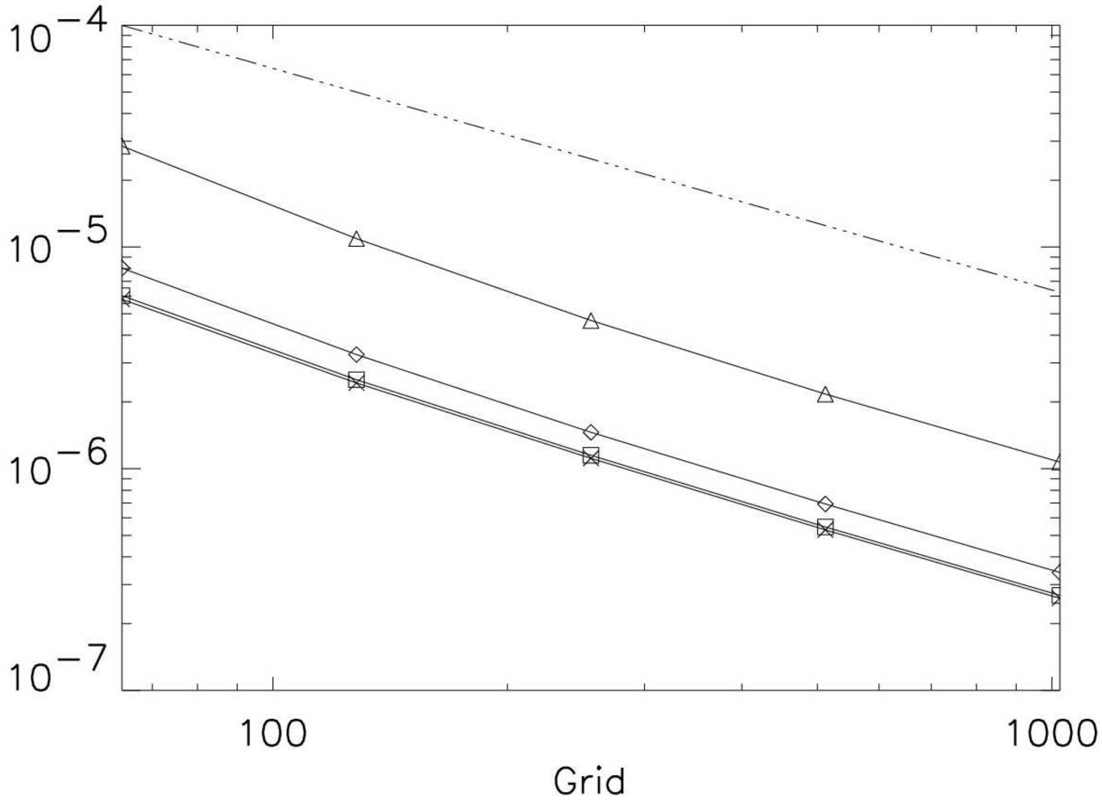


$$\mathbf{F}^{n+1/2}_{j+1/2} = \begin{cases} \mathbf{f}(\mathbf{q}^L) & \text{if } 0 \geq H^L \\ \mathbf{f}(\mathbf{q}^L) + H^L\left(\mathbf{q}^{L*} - \mathbf{q}^L\right) & \text{if } H^L \leq 0 \leq H^* \\ \mathbf{f}(\mathbf{q}^R) + H^R\left(\mathbf{q}^{R*} - \mathbf{q}^R\right) & \text{if } H^L \leq 0 \leq H^* \\ \mathbf{f}(\mathbf{q}^R) & \text{if } H^R \leq 0 \end{cases}$$

Magnetized Bondi Flow



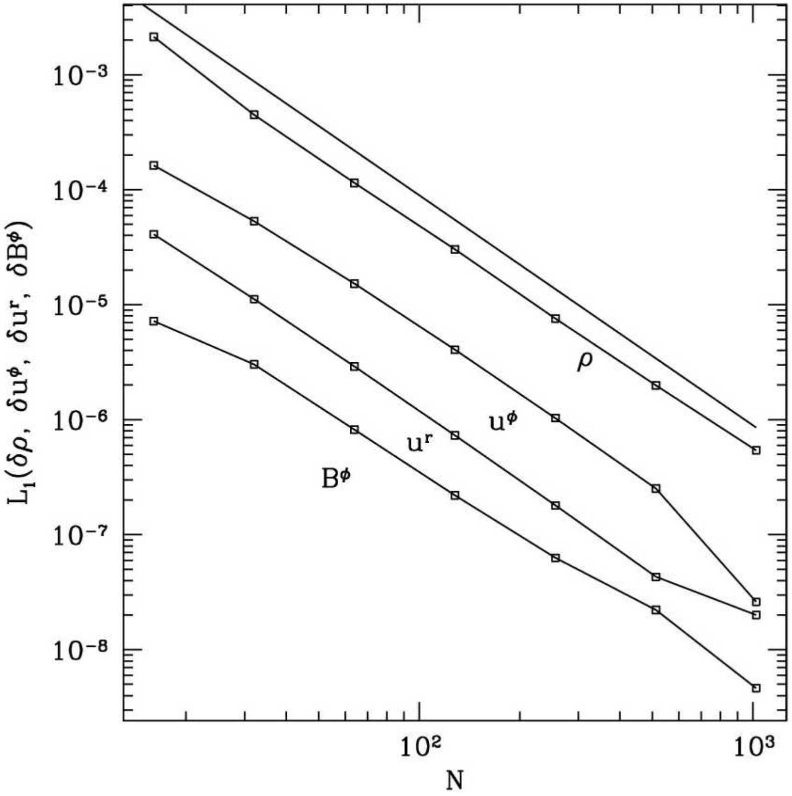
HARM



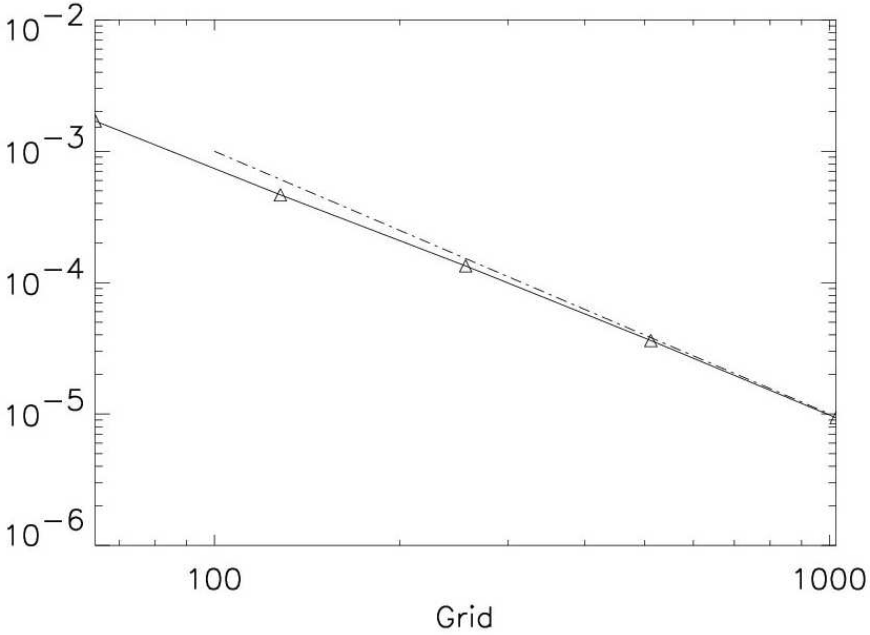
DH 2003

Gammie Inflow

- Steady-state, equatorial, magnetized inflow in Kerr inside marginally stable orbit;



HARM



B^{ϕ} DH 2003

Slow Shock

- Komissarov, *MNRAS* **303**, 343 (1999).

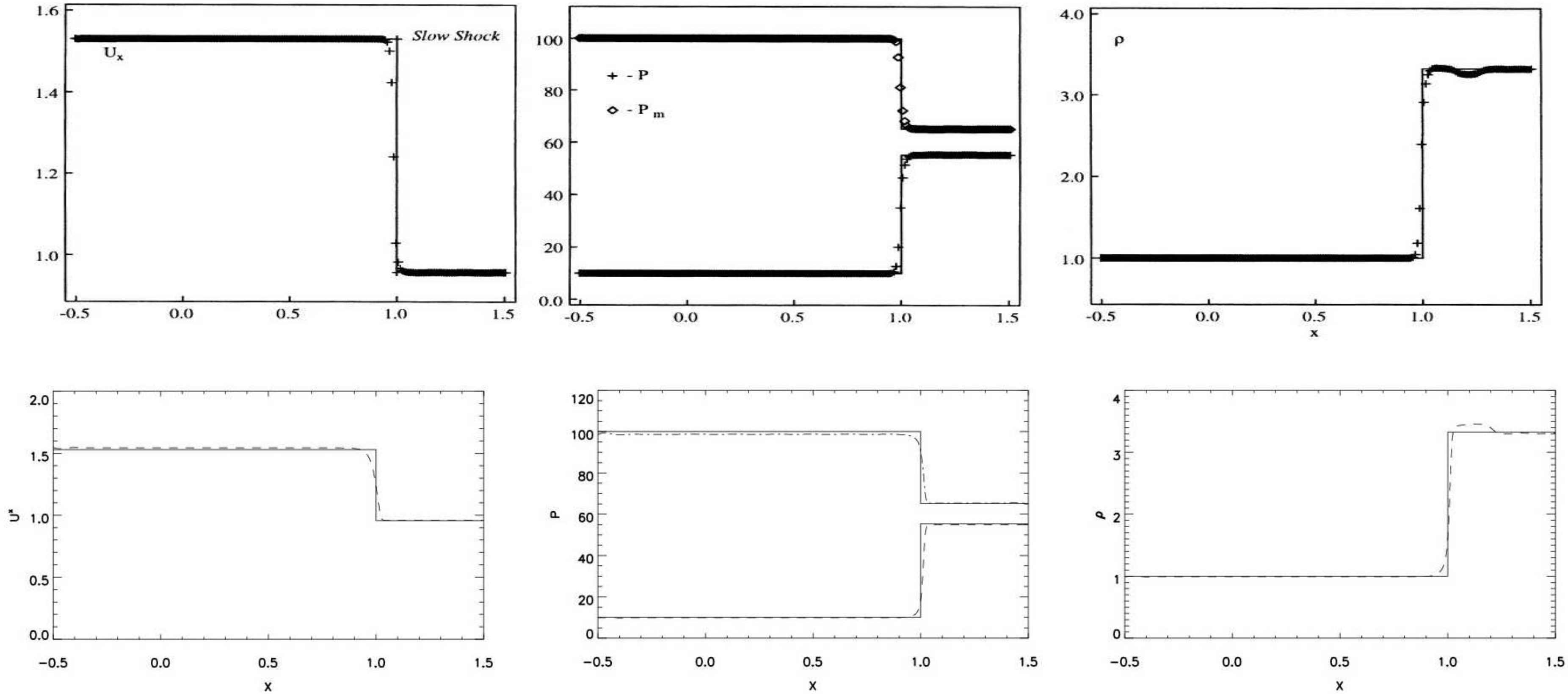
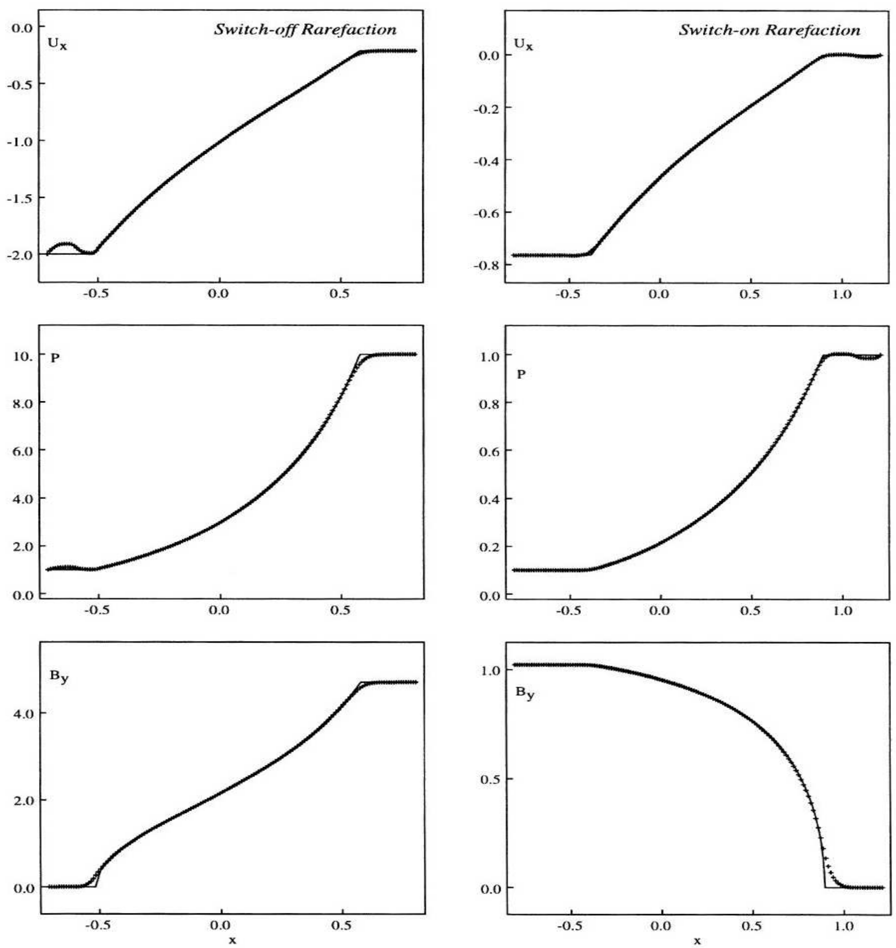


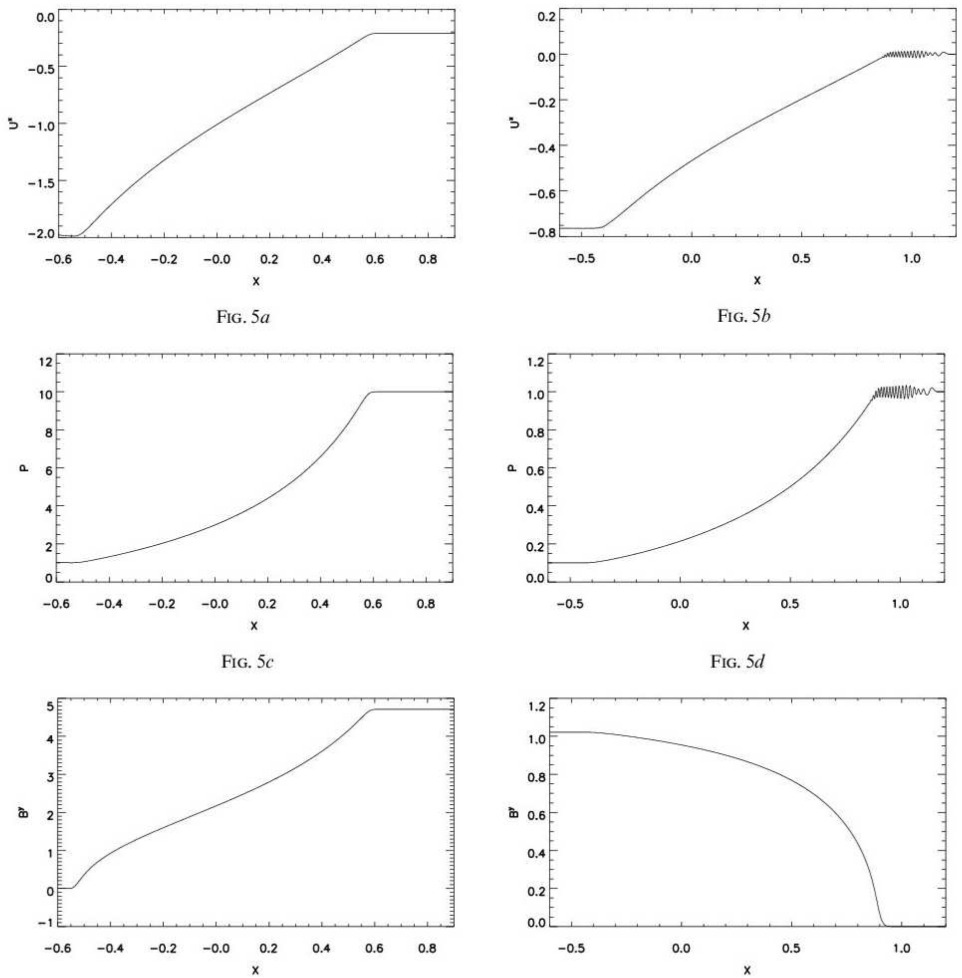
FIG. 3a FIG. 3b FIG. 3c

DH 2003

Rarefaction



Komissarov $N = 150$



DH 2003 $N = 2048$

Qualities of Conservative GRMHD Schemes

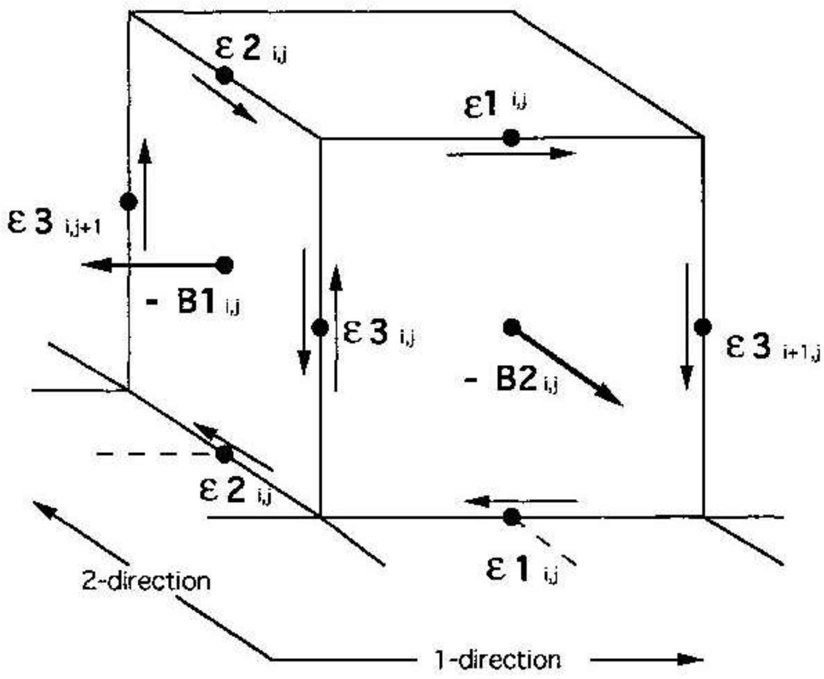
- \+ Improved behavior in shock tests
- \+ Equivalent behavior in relativistic cases
- \+ Better behavior for ultra-relativistic flows
- − Still developmental
- − Worse behavior in “floored” regions
- − Worse behavior when $P_{\mathrm{mag}} \gg P$
- − Conservative $\rightarrow$ Primitive variable calculation

  $\Rightarrow$ Slower code!

Other Schemes

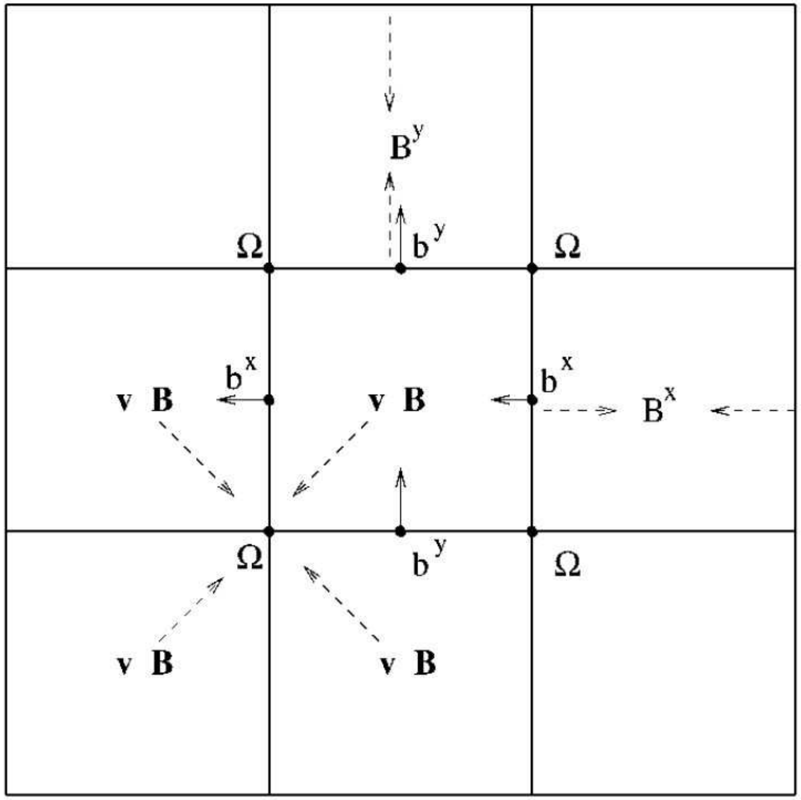
- Del Zanna et al., *AA* **400**, 397 (2003).
  - 3rd-order CENO with HLL
  - SRMHD
- Balsara, *ApJS* **132**, 83 (2001).
  - Roe-type Approx. Riemann Solver
  - Interpolates characteristic variables
  - SRMHD
- ATHENA, http://www.astro.princeton.edu/ jstone/athena.html
  - Roe-type Approx. Riemann Solver
  - Not relativistic

Maintianing the Divergence Constraint



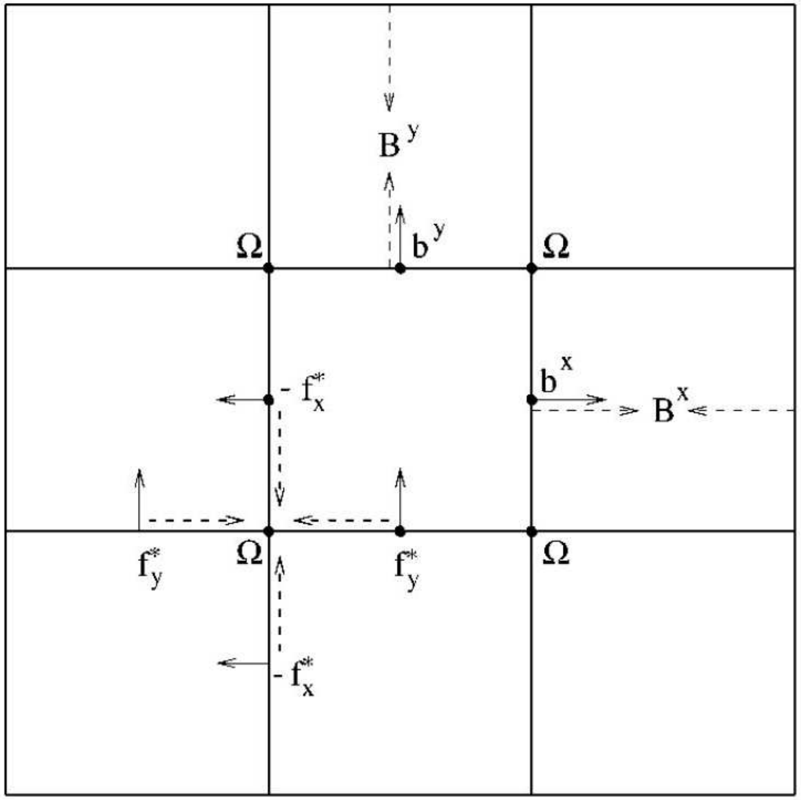
- Toth, *J. Comp. Phys.*, **161**, 605 (2000).
  TVD-Lax-Friedrich
- $\partial_i B^i \neq 0$ leads to:
  - Non-perpendicular Lorentz forces to B^i
  - Inconsistency in MHD
  - Instabilities and spurious effects
- $\mathcal{E}^z = v^x B^y - v^y B^x = f^x$
- CT : Evans and Hawley 1988
- MOC-CT: Hawley and Stone 1995
- MOC-CT2: De Villiers and Hawley 2003

Flux/Field CT/CD for Conserv. Methods



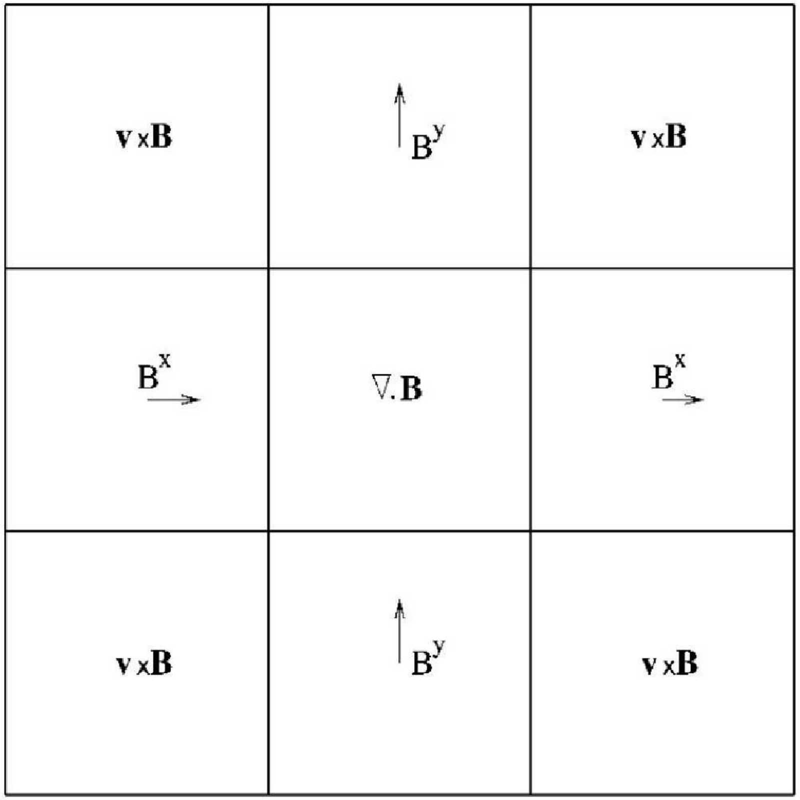
- Field-CT: Interp. predicted B^i and v^i for $\mathcal{E}_{j+1}^{n+1/2}$;

Flux/Field CT/CD for Conserv. Methods



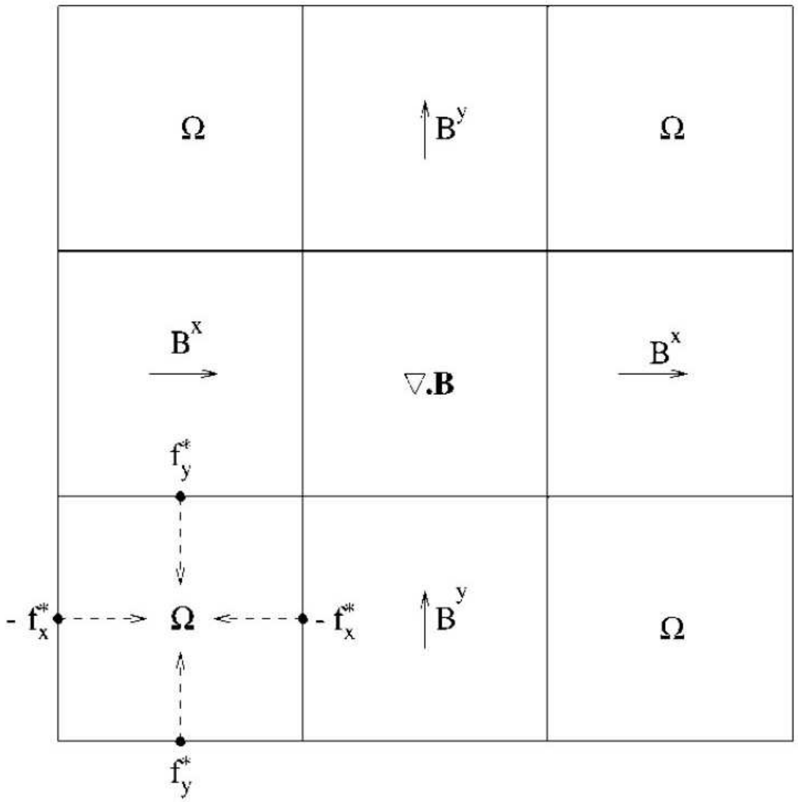
- Field-CT: Interp. predicted B^i and v^i for $\mathcal{E}_{j+1}^{n+1/2}$;
- Flux-CT: Interp. f^{i*} and $f^{i\,n}$ for $\mathcal{E}_{j+1/2}^{n+1/2}$;

Flux/Field CT/CD for Conserv. Methods



- Field-CT: Interp. predicted B^i and v^i for $\mathcal{E}_{j+1}^{n+1/2}$;
- Flux-CT: Interp. f^{i*} and $f^{i\,n}$ for $\mathcal{E}_{j+1/2}^{n+1/2}$;
- Field-CD: $\mathcal{E}_j^{n+1/2} = \frac{1}{2}\left(\mathcal{E}_j^n + \mathcal{E}_j^*\right)$;

Flux/Field CT/CD for Conserv. Methods



- Field-CT: Interp. predicted B^i and v^i for $\mathcal{E}_{j+1}^{n+1/2}$;
- Flux-CT: Interp. $f^{i\,*}$ and $f^{i\,n}$ for $\mathcal{E}_{j+1/2}^{n+1/2}$;
- Field-CD: $\mathcal{E}_j^{n+1/2} = \frac{1}{2}\left(\mathcal{E}_j^n + \mathcal{E}_j^*\right)$;
- Flux-CD: $f^{i\,*}$ for $\mathcal{E}_j^{n+1/2}$;

8-Wave Formulation

K. G. Powell, P. L. Roe, et al., *J. Comput. Phys.* **154**, 284 (1999).
K. G. Powell, ICASE Report No. 94-24, Langley, VA, 1994.

- MHD equations can be derived w/o assuming constraint
  $\Rightarrow\ \exists$ source terms $\propto \partial_i B^i$;

- With $\partial_i B^i$ source terms, conserves divergence constraint along flow lines:
  $\partial_t \left(\partial_i B^i\right) + \partial_j \left(v^j\ \partial_i B^i\right) = 0$

  + Supposed to preserve initially constrained data along flows;
  + Any monopoles are transported away *hopefully* off the grid;
  + Increases robustness, G. Tóth and D. Odstrěil, *J. Comput. Phys.* **128**, 82 (1996).
  - Formulation becomes non-conservative $\Rightarrow$ problems near strong shocks;

Divergence Cleaning Schemes

J. U. Brackbill and D. C. Barnes, *J. Comput. Phys.* **35**, 426 (1980).
C.-D. Munz, *J. Comput. Phys.* **161**, 484 (2000).
A. Dedner, et al., *J. Comput. Phys.* **175**, 645 (2002).
S. S. Komissarov, Appendix C of *astro-ph/0402403*.

$$\partial_t B^i - \partial_j \left(v^i B^j - v^j B^i \; + \; g^{ij} \Psi \right) = 0$$

$$\mathcal{D}(\Psi) + \partial_i B^i = 0$$

$$\Rightarrow \partial_t \mathcal{D}(\Psi) = \partial_i \partial^i \Psi$$

$\mathcal{D}(\Psi) = 0$	$\Rightarrow$	$\partial_i \partial^i \Psi \; = \; 0$	Elliptic
$\mathcal{D}(\Psi) = \frac{1}{p} \Psi$	$\Rightarrow$	$\partial_t \Psi \; = \; p \, \partial_i \partial^i \Psi$	Parabolic
$\mathcal{D}(\Psi) = \frac{1}{h^2} \, \partial_t \Psi$	$\Rightarrow$	$\partial_{tt} \Psi \; = \; h^2 \, \partial_i \partial^i \Psi$	Hyperbolic
$\mathcal{D}(\Psi) = \frac{1}{h^2} \, \partial_t \Psi \; + \; \frac{1}{p^2} \, \Psi$	$\Rightarrow$	$\partial_{tt} \Psi \; + \; \frac{h^2}{p^2} \, \partial_t \Psi \; = \; h^2 \, \partial_i \partial^i \Psi$	Mixed

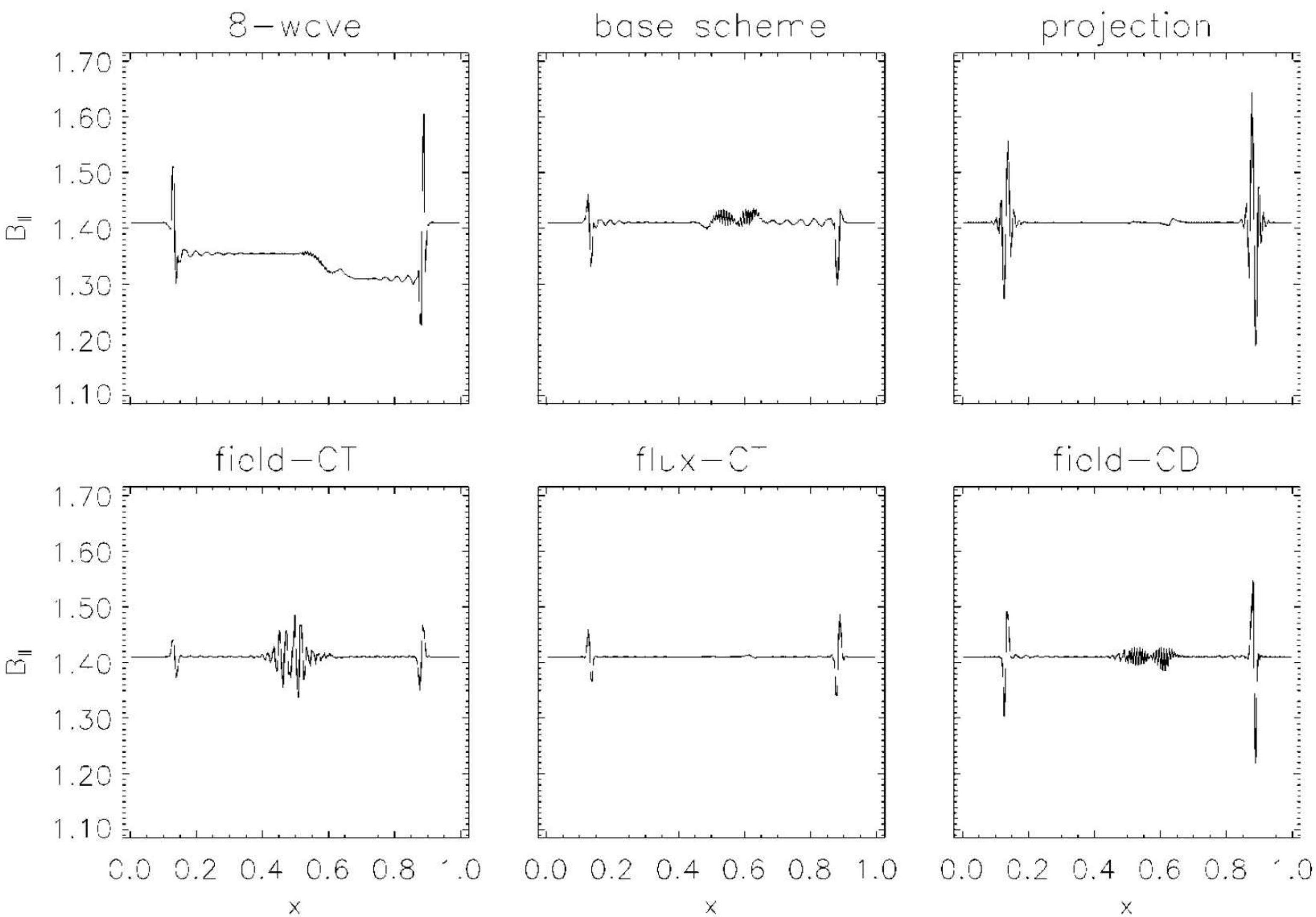
Discretization Differences in Constraint

TABLE IV

Divergence B in the 2D Rotated Shock Tube Test

	$\|\nabla \cdot \boldsymbol{B}_{j,k}\|$		$\|\nabla \cdot \boldsymbol{B}_{j+1/2,k+1/2}\|$	
	Max	Avrg	Max	Avrg
Base scheme	141.5	3.43	48.9	3.27
8-wave	142.5	3.62	57.0	1.91
Projection	0.3	0.01	130.9	4.73
Field-CD	10^{-12}	10^{-13}	84.2	3.81
Flux-CD	10^{-12}	10^{-13}	68.5	3.91
Field-CT	65.9	5.63	10^{-12}	10^{-13}
Flux-CT	73.5	2.09	10^{-12}	10^{-13}
Tr-flux-CT	102.8	2.95	10^{-12}	10^{-13}

2D Shock Tube Test



2D Shock Tube Test

TABLE III

Numerical Errors in the 2D Shock Tube Test for $\alpha = 63.4^\circ$ and $N = 256$

	$\delta\rho$	$\delta v_\parallel$	$\delta v_\perp$	δp	$\delta B_\parallel$	$\delta B_\perp$	$\bar{\delta}$
Field-CD	0.0074	0.0175	0.0936	0.0052	0.0046	0.0102	0.0231
Flux-CD	0.0075	0.0175	0.0965	0.0052	0.0036	0.0107	0.0235
Projection	0.0076	0.0177	0.0948	0.0055	0.0062	0.0093	0.0235
Flux-CT	0.0075	0.0176	0.0996	0.0052	0.0016	0.0098	0.0235
Base scheme	0.0075	0.0178	0.1006	0.0055	0.0037	0.0078	0.0238
Tr-flux-CT	0.0075	0.0177	0.1020	0.0054	0.0020	0.0089	0.0239
Field-CT	0.0075	0.0174	0.1214	0.0059	0.0043	0.0178	0.0291
8-wave	0.0076	0.0180	0.1027	0.0056	**0.0413**	0.0092	0.0307

Orzag-Tang Vortex