A Numerical Study of Relativistic Fluid Collapse

Final Defense

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Outline

- Theoretical Model of Non-equilibrium Neutron Stars
- Methods for their Numerical Simulation
- Parameter Space Survey and Dynamical Scenarios
- Type I Critical Behavior
- Type II Critical Behavior
- Conclusion

Theoretical Setting

$f = f(r, t)$

- Dynamic, spherically-symmetric systems

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$$G_{ab} = 8\pi T_{ab}$$

- Time-dependent spacetime governed by Einstein's Eq.

Fluid Equations of Motion

Local Conservation of Baryons Equation : $\nabla_\mu J^\mu = 0$

Local Conservation of Energy Equation : $\nabla_\mu T^\mu{}_\nu = 0$

$$\underbrace{\frac{\partial}{\partial t}\begin{bmatrix} D \\ S \\ \tau \end{bmatrix}}_{\mathbf{q}} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\alpha}{a} \underbrace{\begin{bmatrix} Dv \\ Sv + P \\ v\,(\tau + P) \end{bmatrix}}_{\mathbf{f}} \right) = \underbrace{\begin{bmatrix} 0 \\ \Sigma \\ 0 \end{bmatrix}}_{\boldsymbol{\psi}}$$

$$v = \frac{a u^r}{\alpha u^t}\,, \quad W^2 = \frac{1}{1 - v^2}\,, \quad D = a \rho_\circ W\,, \quad S = (\rho + P)\, W^2 v\,, \quad \tau = S/v - D - P$$

- $\Sigma = \Sigma(\alpha, a, \mathbf{q}) \neq \Sigma(\alpha, a, \mathbf{q}, \partial_r \mathbf{q}, \partial_t \mathbf{q}) \Rightarrow$ EOM are hyperbolic!
- Relativistic Ideal gas Equation of State : $P = (\Gamma - 1)\, \rho_\circ \epsilon$, $\Gamma =$ constant

Metric Equations

Slicing Condition :

$$\frac{\alpha'}{\alpha} = a^2 \left[4\pi r \left(Sv + P\right) + \frac{1}{2r}\left(1 - 1/a^2\right)\right]$$

Hamiltonian Constraint :

$$\frac{a'}{a} = a^2 \left[4\pi r \left(\tau + D\right) - \frac{1}{2r}\left(1 - 1/a^2\right)\right]$$

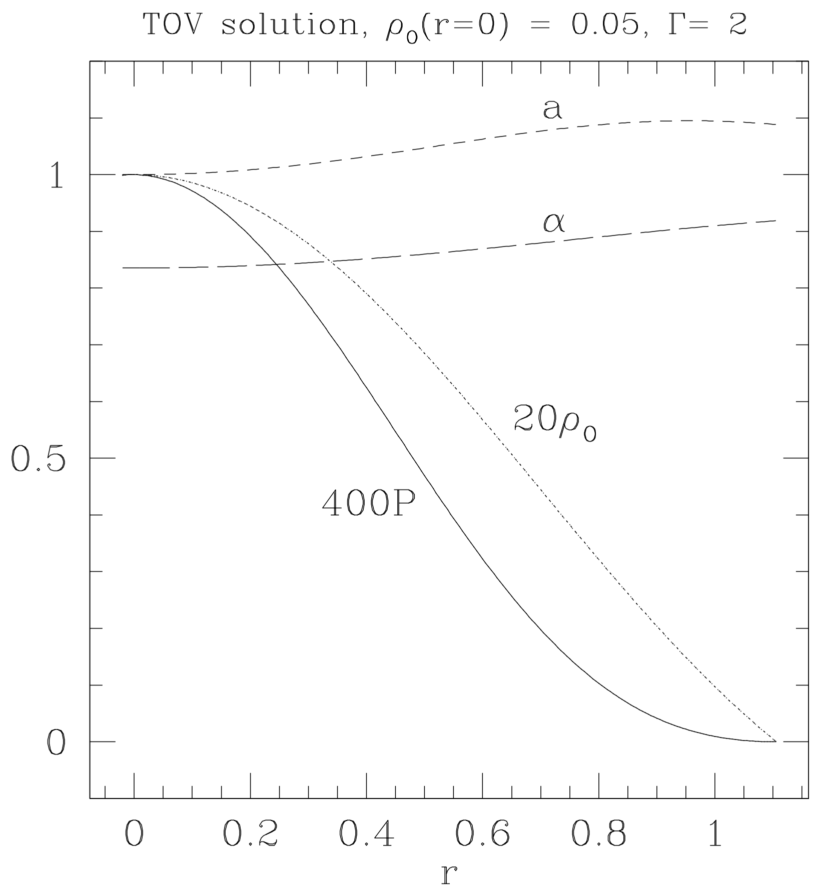
Mass Aspect Function :

$$m(r,t) = \frac{r}{2}\left(1 - 1/a^2\right)$$

Mass of Spherical Shell :

$$\frac{dm}{dr} = 4\pi r^2 \left(\tau + D\right)$$

Neutron Star Model



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  Static, spherical solutions to Einstein's Eq. w/ perfect fluid;

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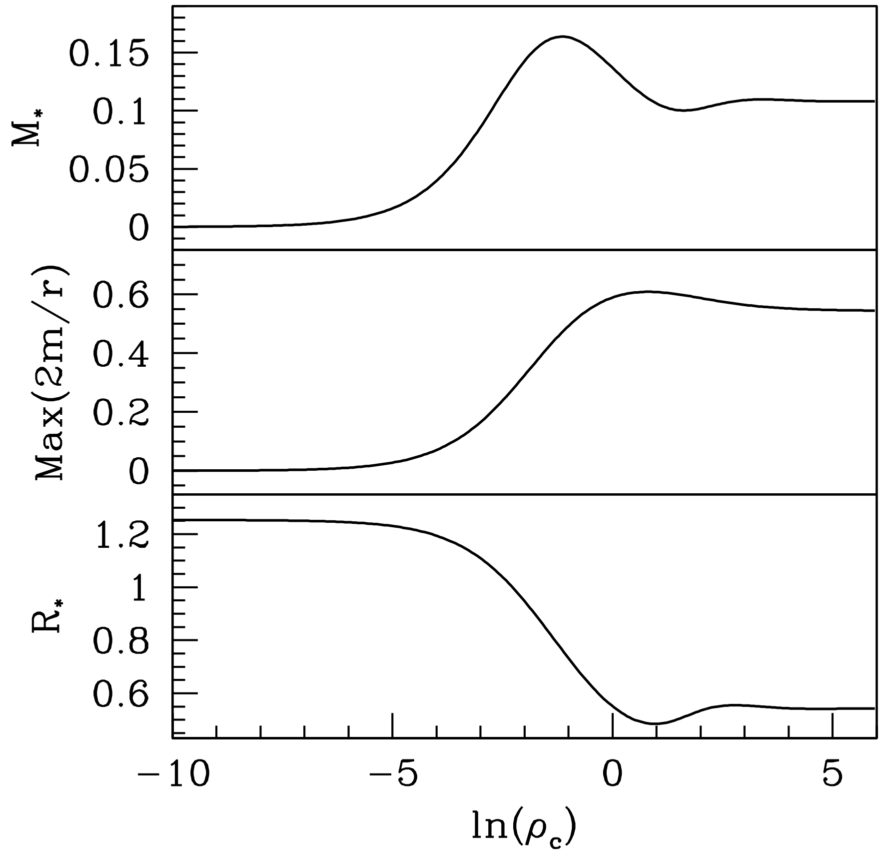

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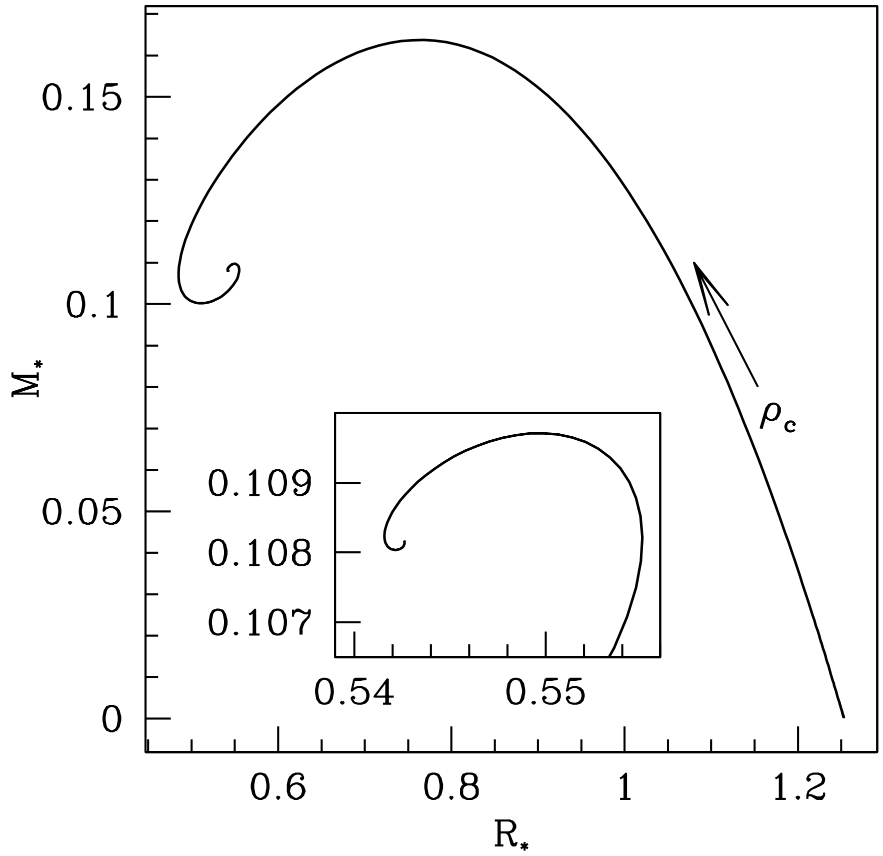

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- Stable & Unstable Solutions

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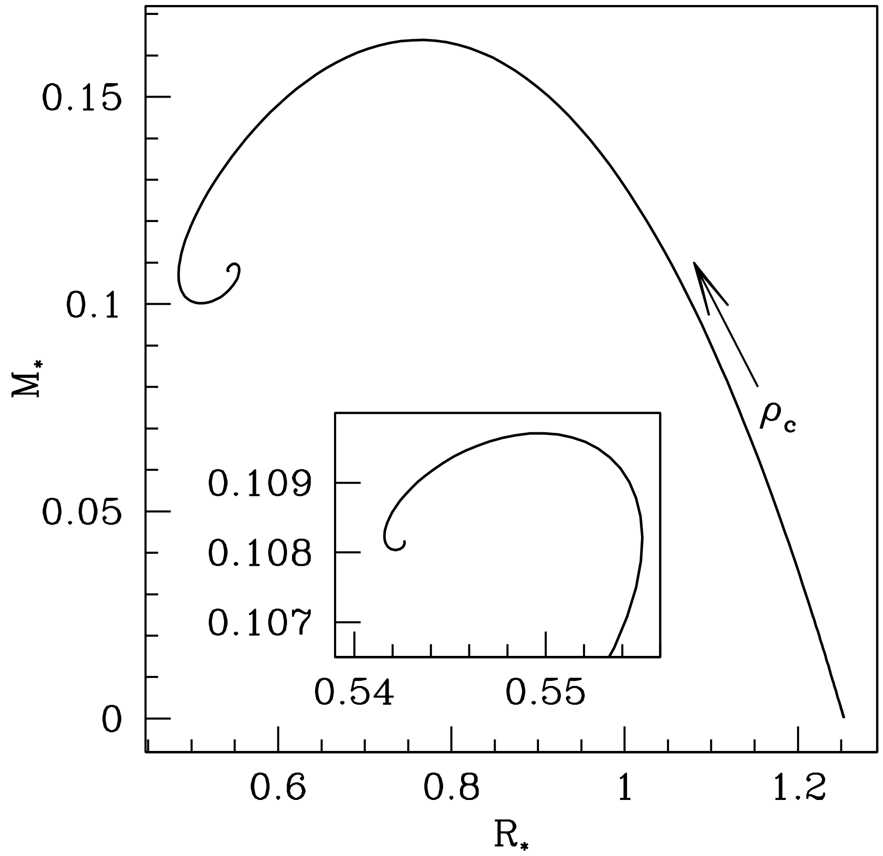
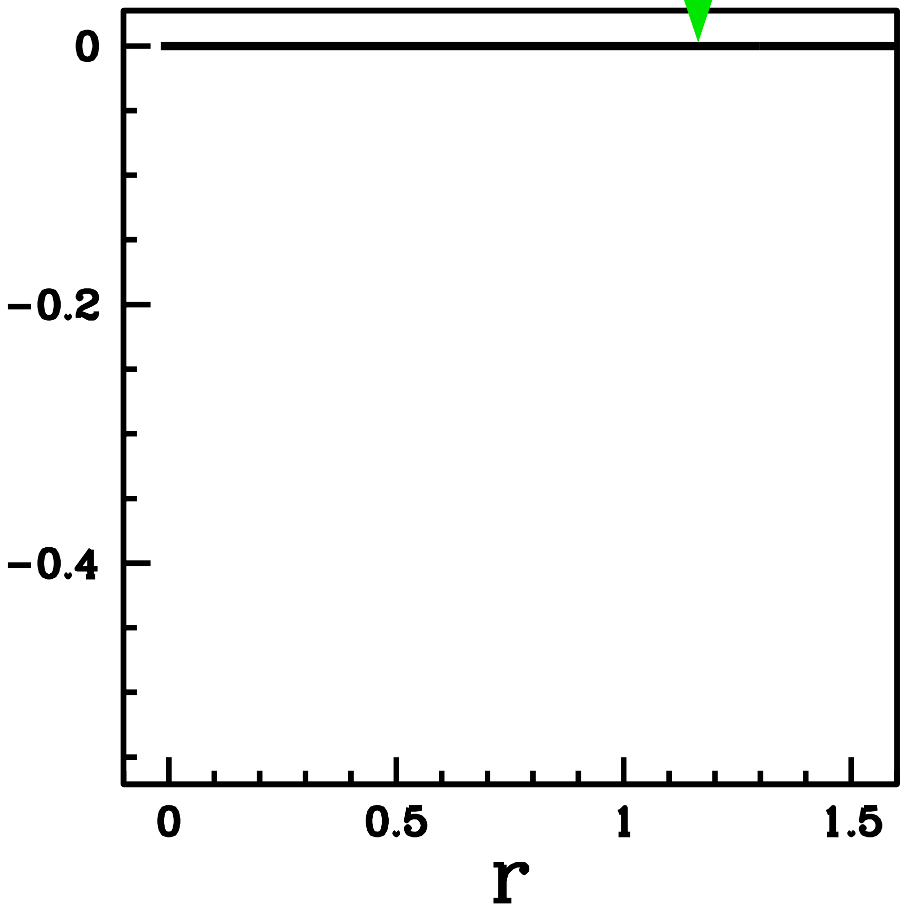


- Tolman-Oppenheimer-Volkoff (TOV) solutions:
  Static, spherical solutions to Einstein's Eq. w/ perfect fluid;
- Parameterized by $\rho_c = \rho_\circ(0,0)$
- Stable & Unstable Solutions
- Isentropic State Equations:
  $$P = K\rho_\circ^\Gamma \ , \ P = (\Gamma - 1)\,\rho_\circ\epsilon$$
  $$\Gamma = 2$$

Velocity-Driven Collapse



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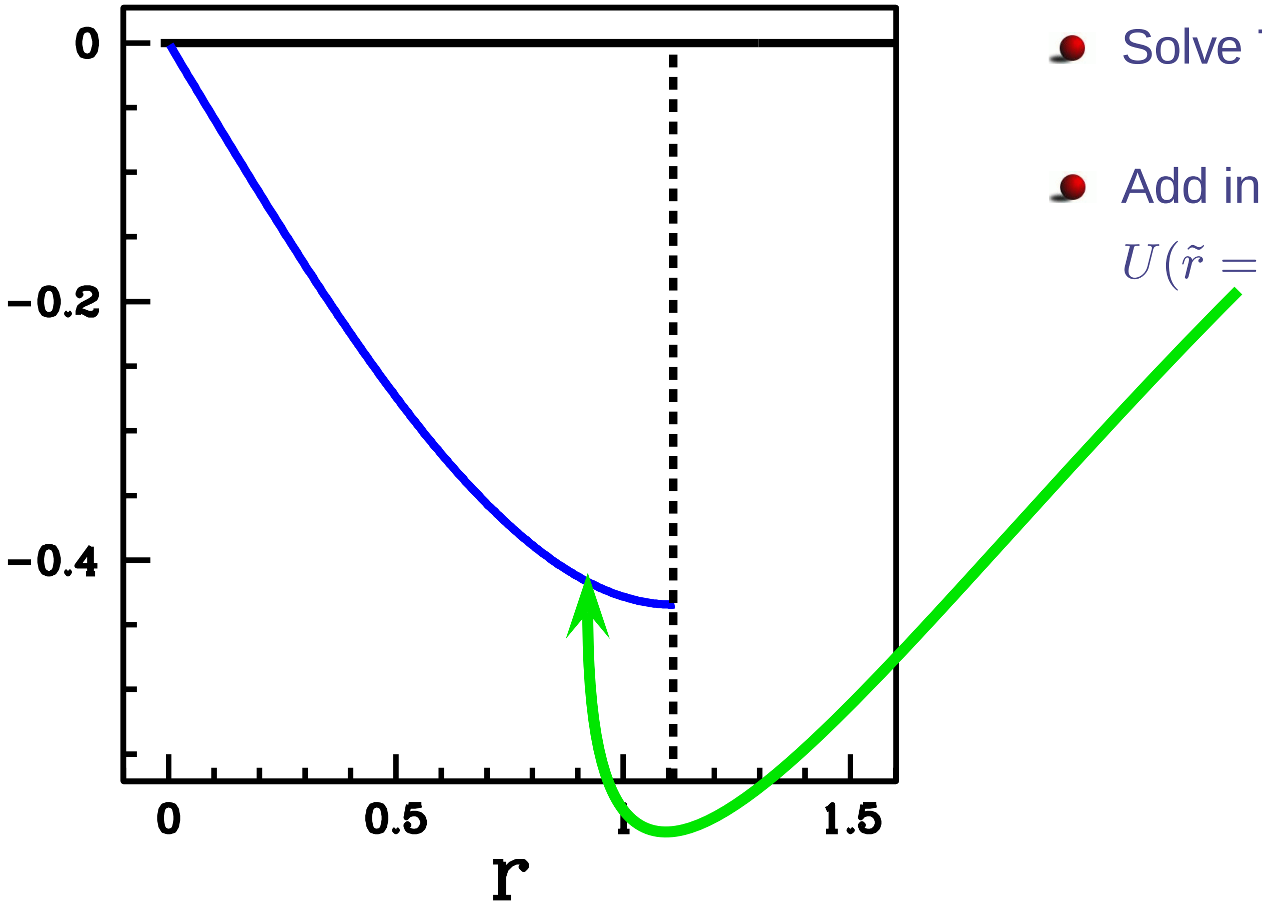

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- Add in-going coordinate velocity:
  $U(\tilde{r} = r/R_*) = \frac{u^r}{u^t} = p\,\tilde{r}\left(\tilde{r}^2 - b\right)$

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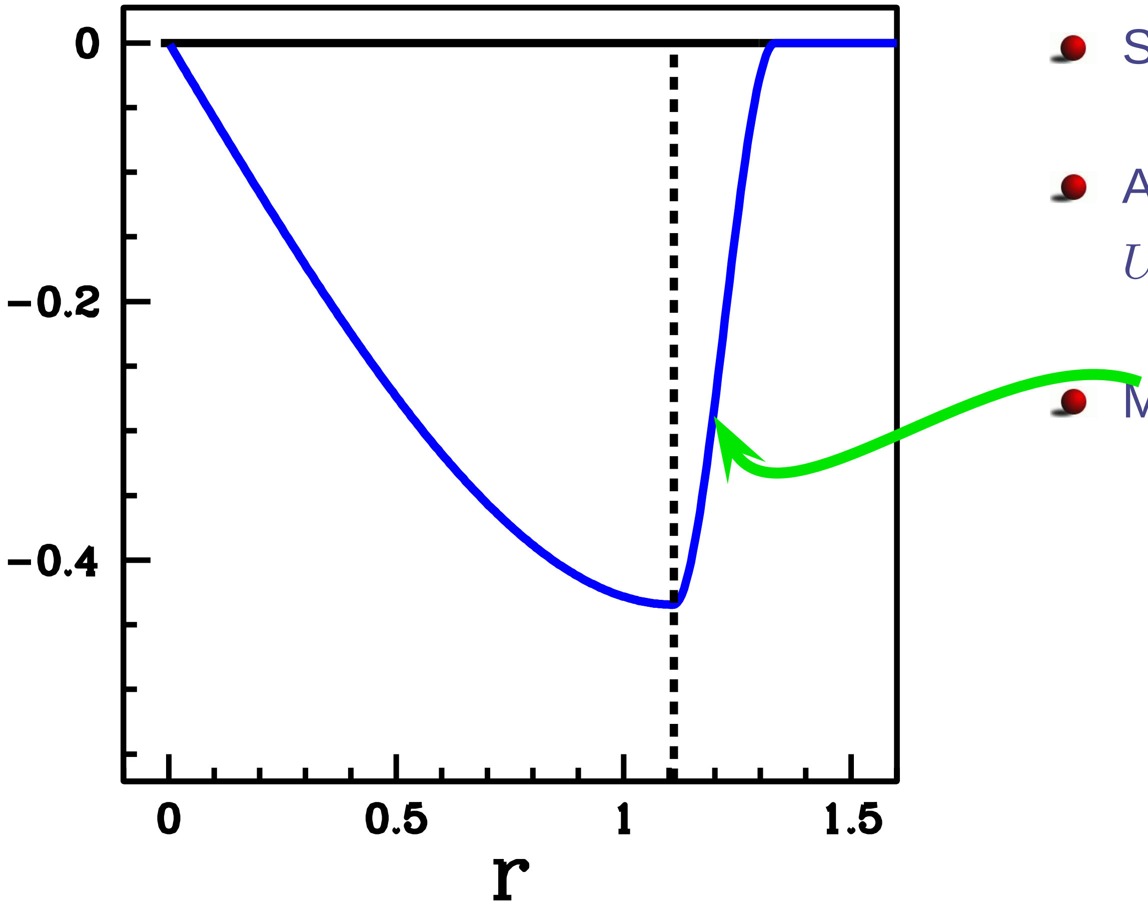

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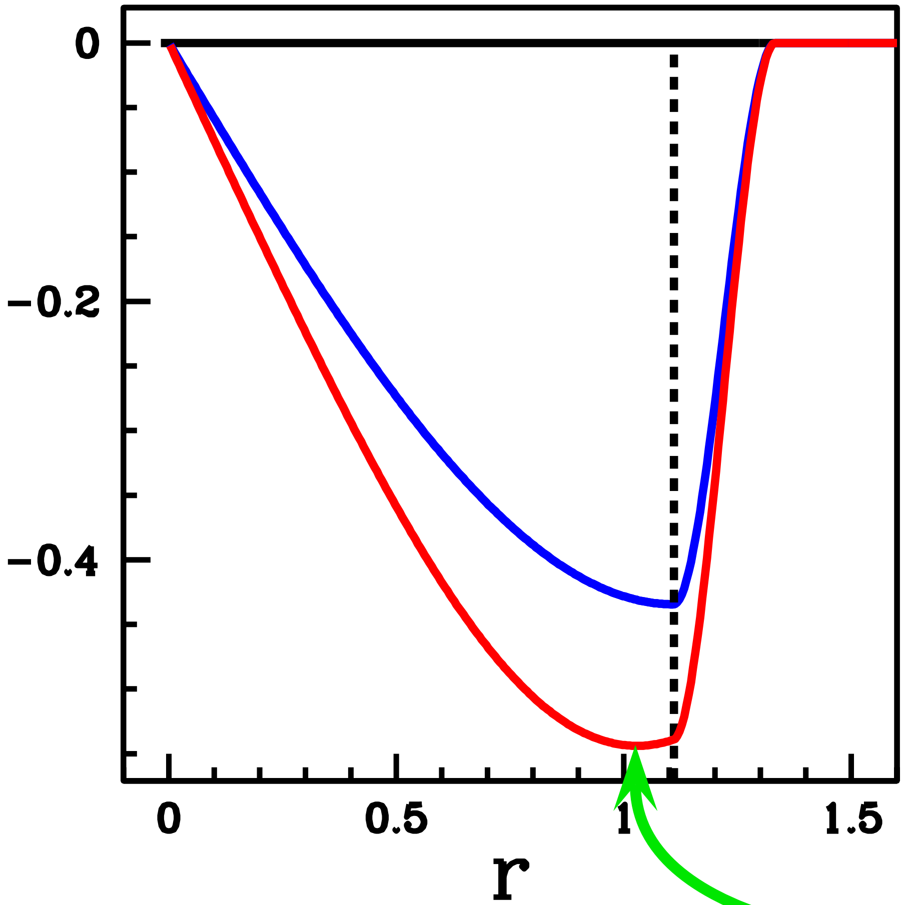

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- Solve $(\alpha' = \ldots)$ and $(a' = \ldots)$ and find $v = aU/\alpha$

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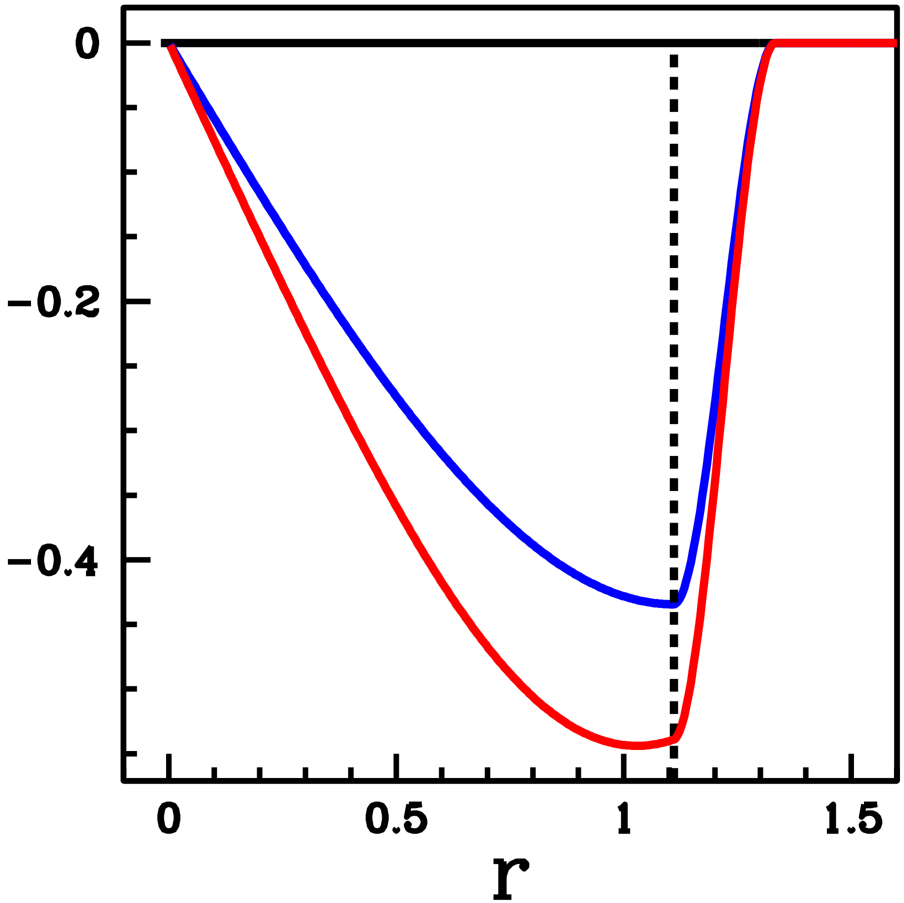

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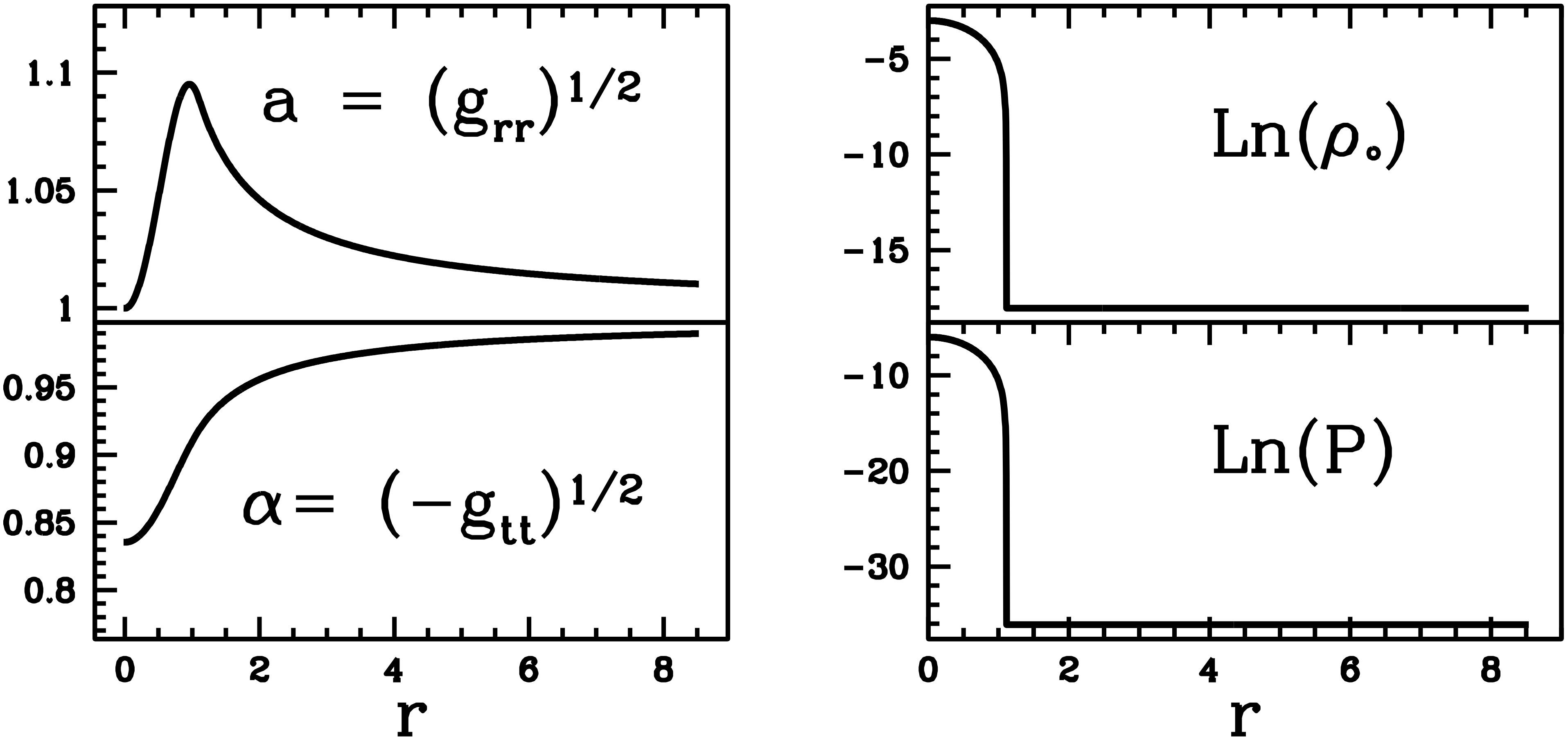
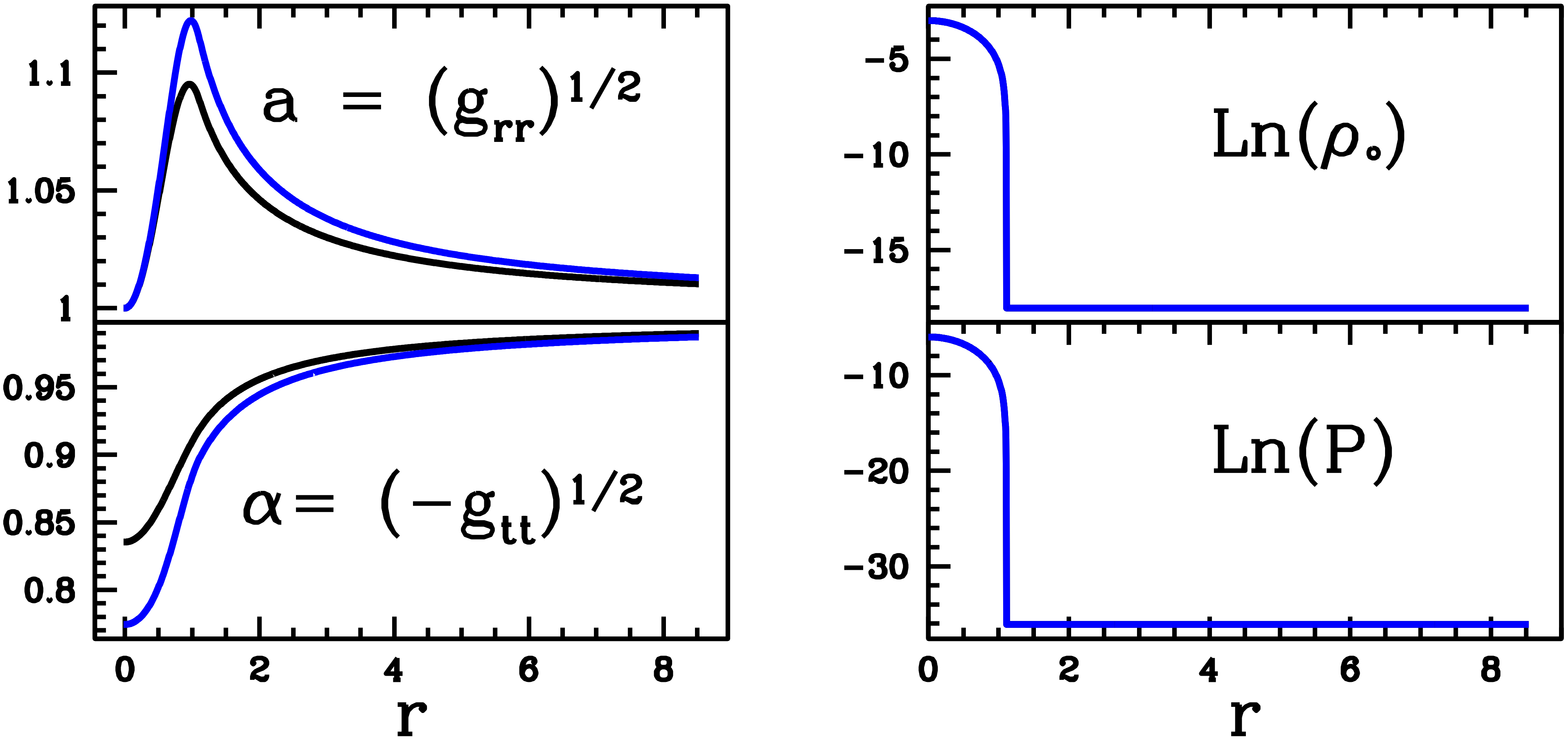
- Match to U = 0
- Solve $(\alpha' = \ldots)$ and $(a' = \ldots)$ and find $v = aU/\alpha$
- Tune to vary amount of kinetic energy

Initial Data : TOV Solution



Initial Data : TOV + In-going Velocity



Minimally-Coupled Massless Scalar Field

- Einstein-massless-Klein-Gordon (EMKG) scalar field

$$T_{ab}^{\mathrm{scalar}} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \left(\nabla_c \phi \nabla^c \phi \right)$$

$$\nabla^a \nabla_a \phi = 0$$

- Coupled only through the geometry

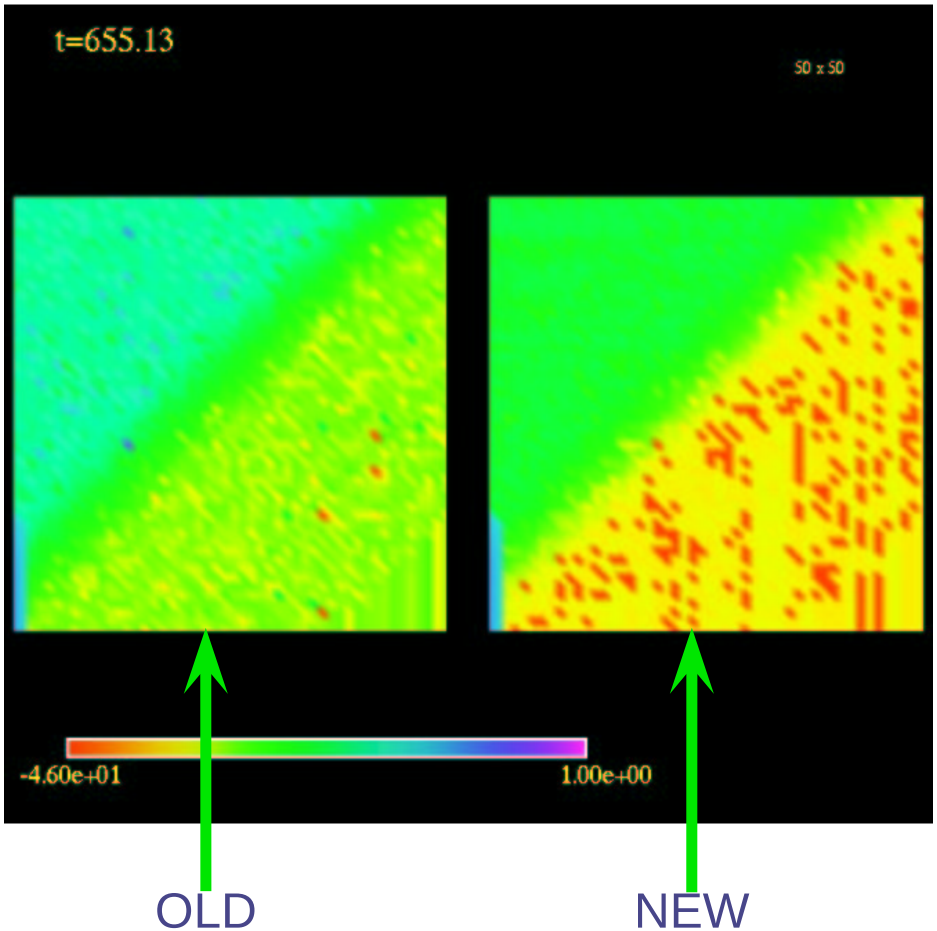
$$T_{ab} = T_{ab}^{\mathrm{scalar}} + T_{ab}^{\mathrm{fluid}} \quad , \quad G_{ab} = 8\pi T_{ab}$$

$$\frac{dm}{dr} = \frac{dm_{\mathrm{scalar}}}{dr} + \frac{dm_{\mathrm{fluid}}}{dr}$$

Numerical Methods

- **High-resolution shock-capturing methods:**
  - Conservative, finite volume methods, e.g. solves differences of integral equations;
  - Shocks propagate at correct speeds;
  - Resolve shocks with very little Gibbs phenomenon near discontinuities;
  - 2^{nd}-order accuracy in smooth regions;
- **Adaptive, non-uniform discretization:**
  - $\Delta r(r) \propto e^{r} \;\; \longrightarrow$ concentrates points near origin ;
  - Automatically adds points near origin when needed;

Advances in Numerical Techniques I



- Primitive Variable Calculation:

$$\begin{aligned} D &= a\rho_\circ W \\ S &= (\rho_\circ + \rho_\circ \epsilon + P)\, W^2 v \\ \tau &= S/v - D - P \end{aligned}$$

  - Solve for $P, v, \rho_\circ$

    $\rightarrow$ Finding minimum of non-linear function

- $\texttt{Err}(w) = \ln\left[\left(w_{\rm calc} - w_{\rm exact}\right)/w_{\rm exact}\right]$

$$\texttt{Err}(P), \quad \texttt{Err}(v), \quad \texttt{Err}(\rho_\circ)$$

Advances in Numerical Techniques II

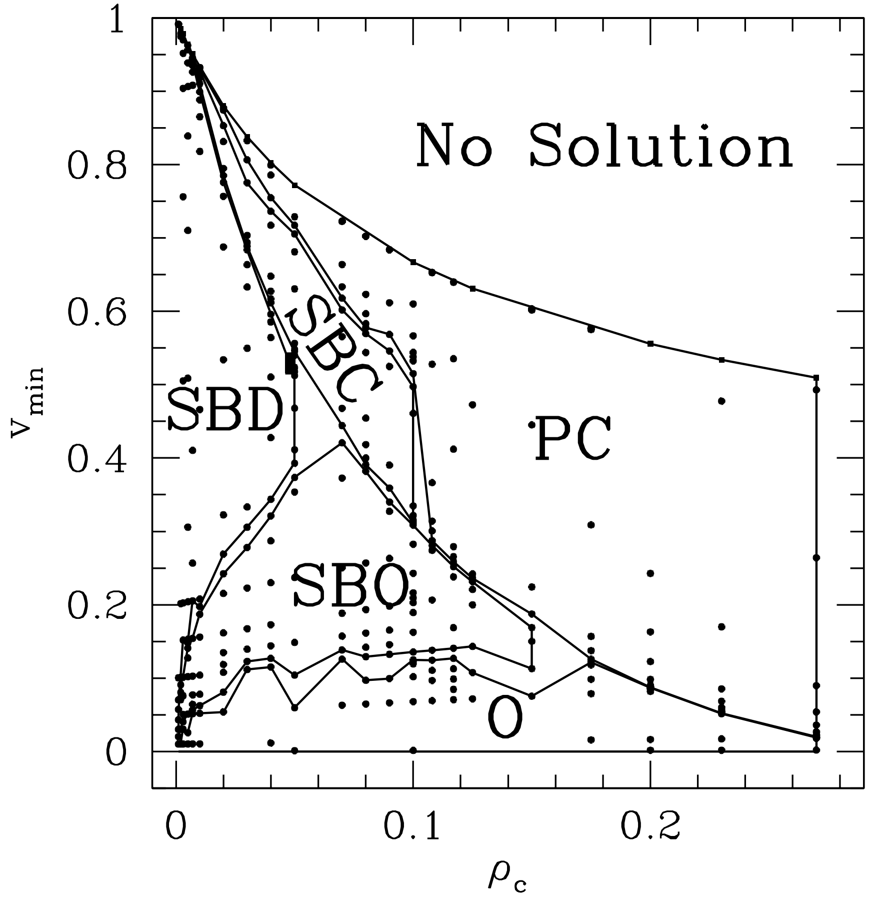
- **New formulation of fluid equations of motion:**

$$\Pi = \tau + S \quad , \quad \Phi = \tau - S$$

  - Formulation improves accuracy of $\tau \pm S$ since $\tau \rightarrow |S|$ as $|v| \rightarrow 1$

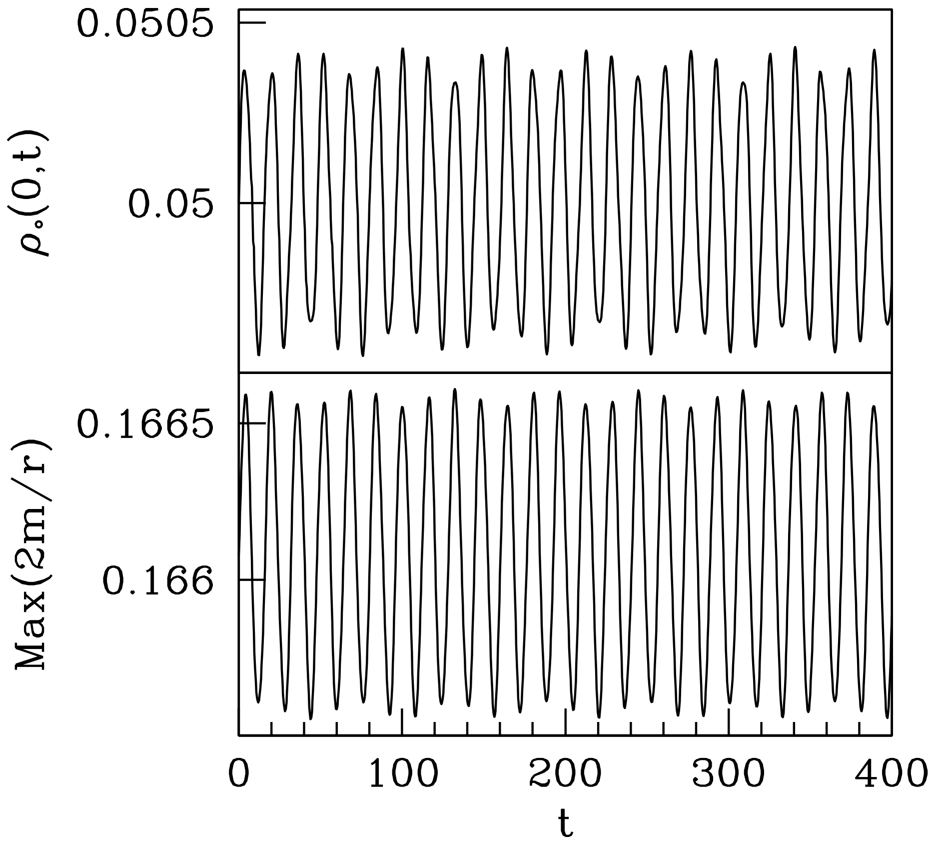
- **Smoothing about sonic point in Type II collapse:**
  - Instability sets in as expansion shock develops;
  - Dissipation subdues instability at discontinuity;
  - Smoothing $=$ Point-wise, nearest-neighbor averaging;

Parameter Space Survey



- Previous work:
  - S. Shapiro and Teukolsky (1980)
  - Gourgoulhon (1992)
  - Novak (2001)
- Parameterized by $v_{\rm min}$ and ρ_c
- Dynamical scenarios:
  - Normal **O**scillations (**O**)
  - **S**hock/**B**ounce/**O**scillations (**SBO**)
  - **S**hock/**B**ounce/**D**ispersal (**SBD**)
  - **S**hock/**B**ounce/**C**ollapse (**SBC**)
  - **P**rompt **C**ollapse (**PC**)

Normal Oscillations (O)



- All stars, small $v_{\rm min}$
- Perturbed stable solution
- Normal, radial oscillations

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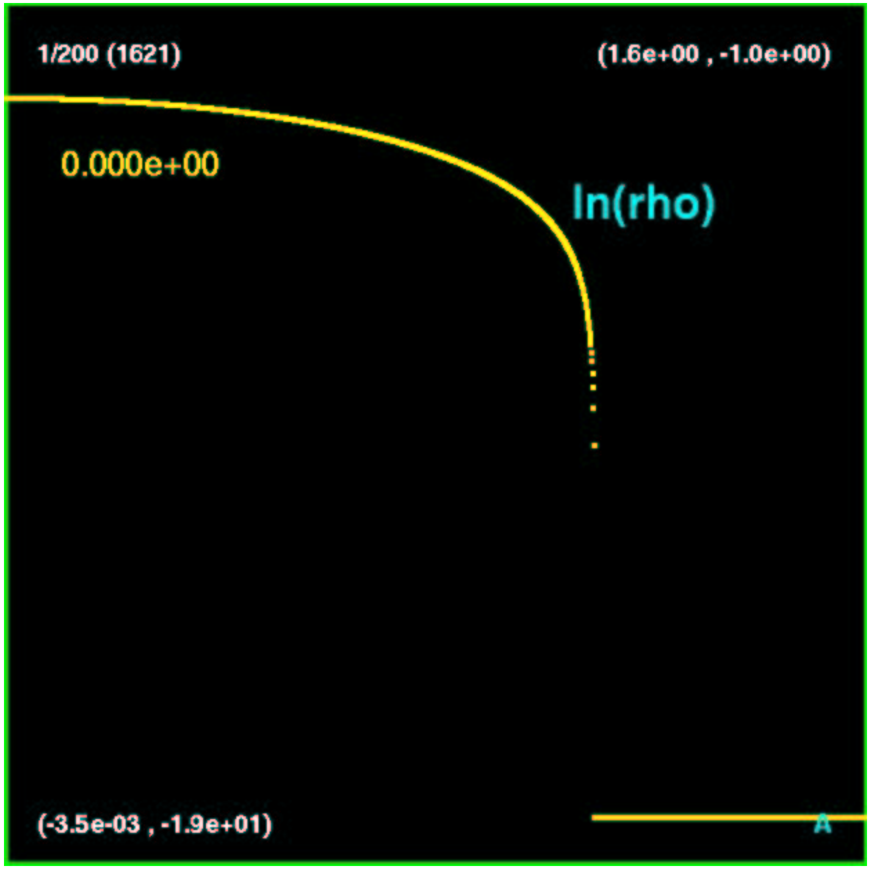

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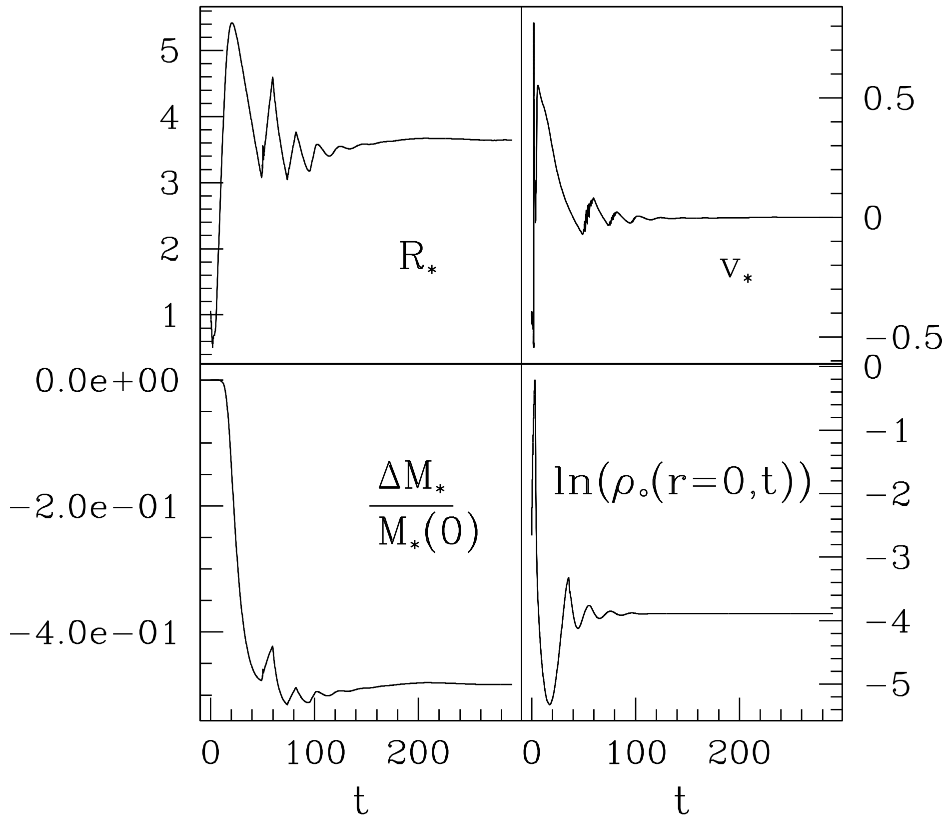
- Normal, radial oscillations
- Movies:

  $\ln\left(\rho_{\circ}(r,t)\right), \quad \rho_{\circ}(r,t), \quad v(r,t)$

Shock/Bounce/Oscillations (SBO)



- Moderately compact stars, intermediate $v_{\min}$
- Bounce, Core's Rebound $\longrightarrow$ Mass Ejection
- Highly-damped oscillations about sparser star

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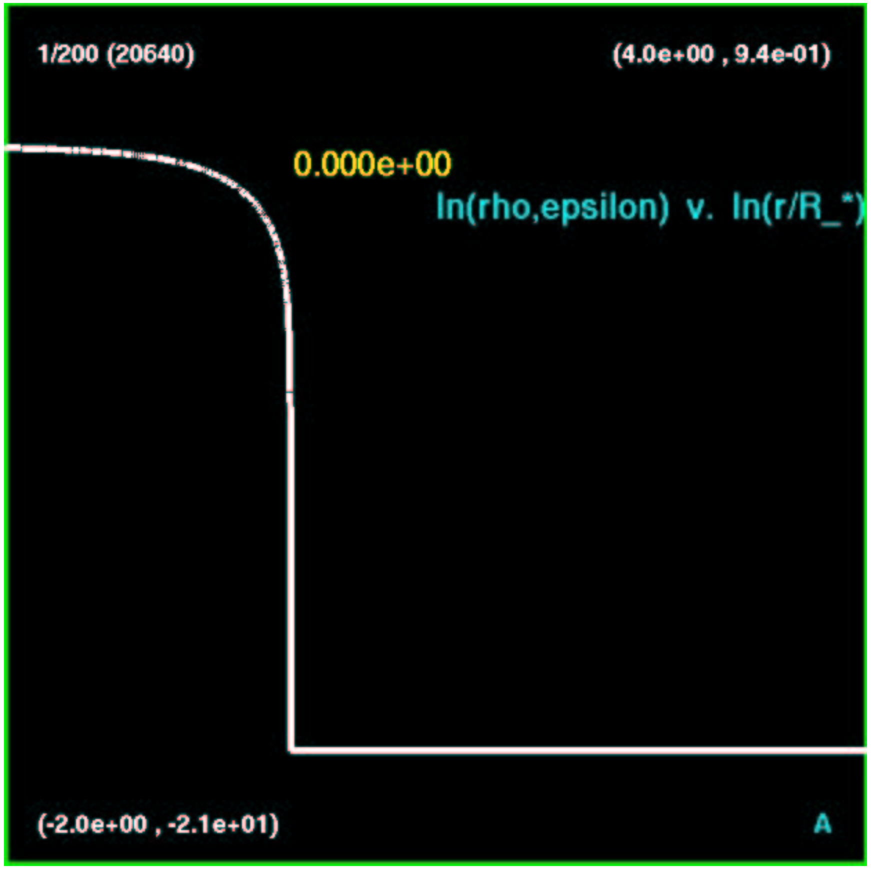

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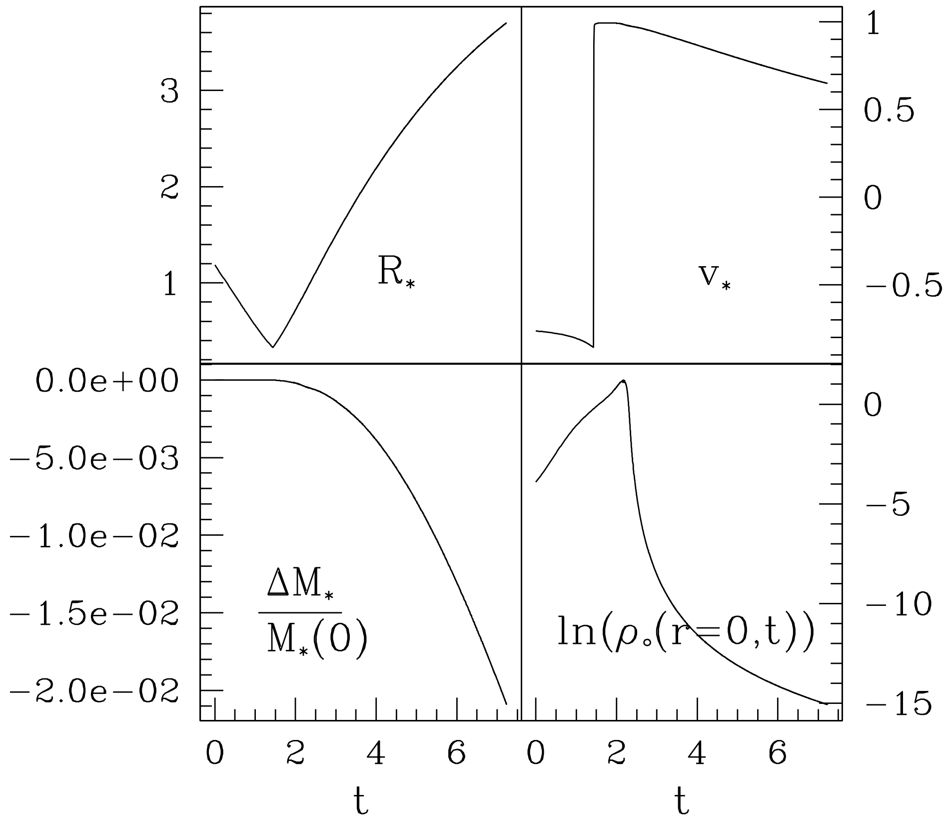
- Movies:

$$\ln\left(\rho_\circ\right) \;\&\; \ln\left(\epsilon\right) \quad \text{vs.} \quad \{\ln(r/R_*), t\} \text{ ,}$$

$$v \quad \text{vs.} \quad \{\ln\left(r/R_*\right), t\}$$

Shock/Bounce/Dispersal (SBD)



- Sparse stars, small—to—large $v_{\rm min}$
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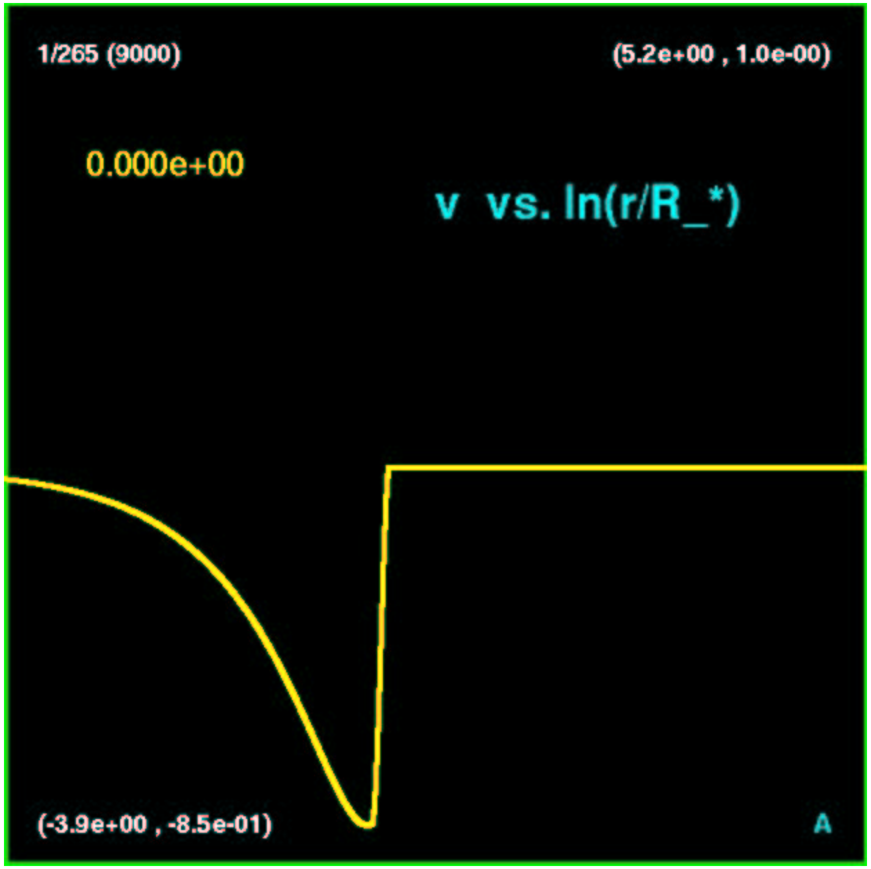

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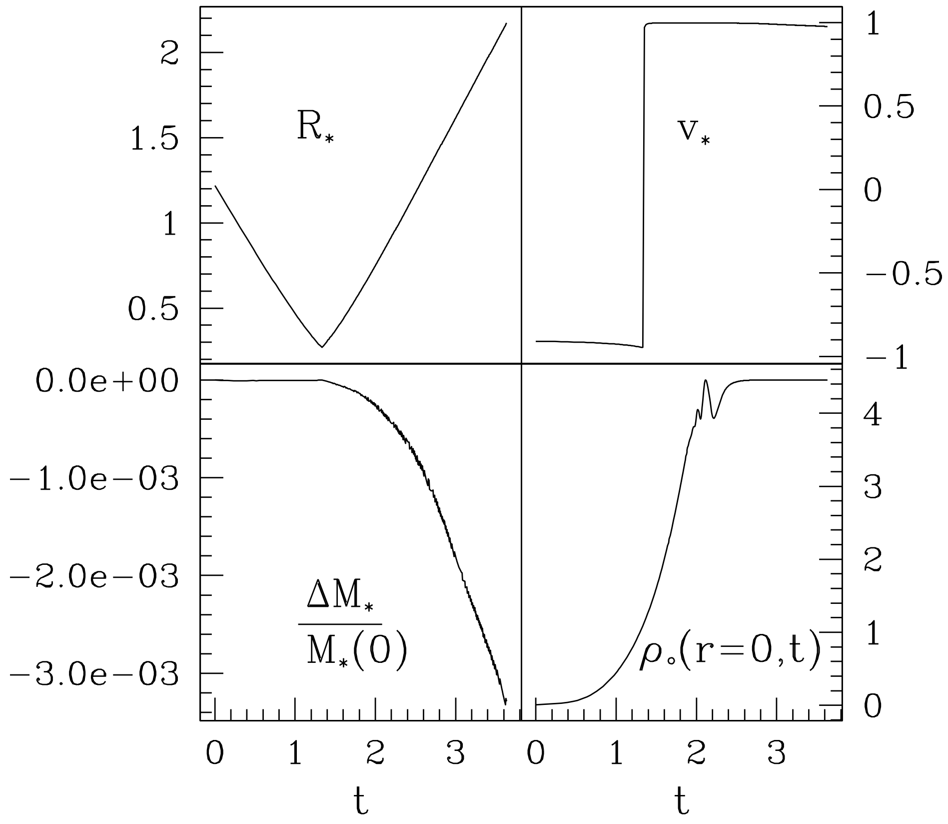
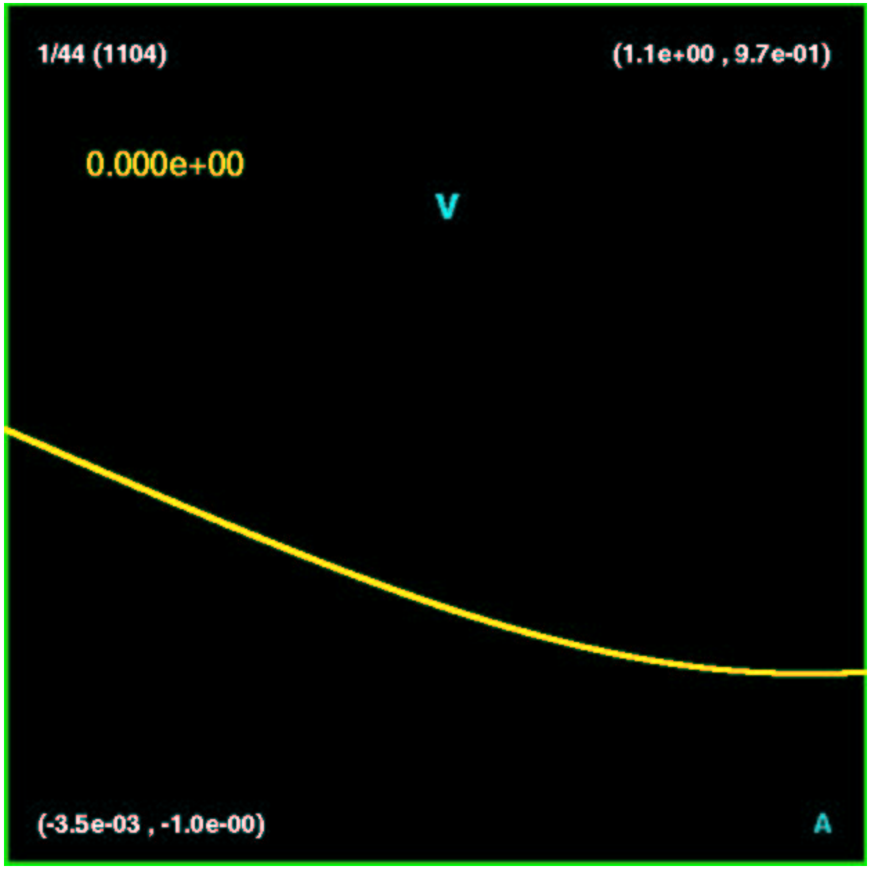


- Sparse—to—semi-dense stars, medium—to—large $v_{\rm min}$
- Bounce → Mass Ejection
- Black hole formation, $M_{\rm BH} < M_*$

Shock/Bounce/Collapse (SBC)



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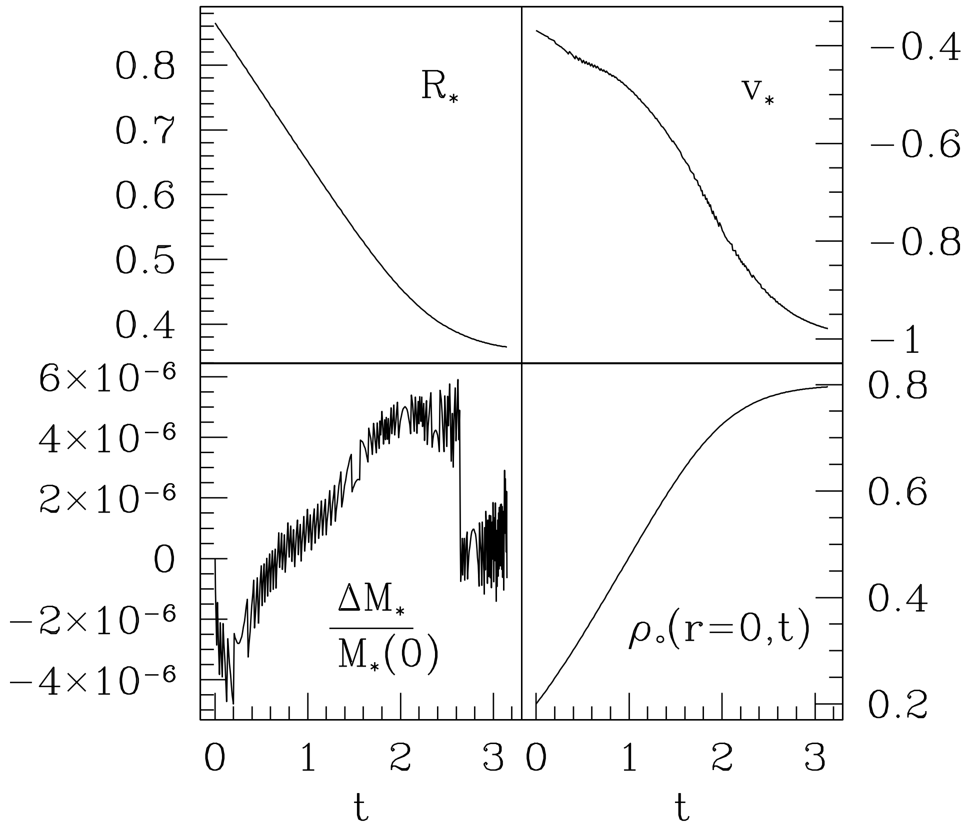
- Movies:

$$a(r,t)\ \ ,\ \alpha(r,t)\ \ ,\ \rho_\circ(r,t)\ \ ,\ v(r,t)$$

$$r \in [0, R_*]$$

Prompt Collapse (PC)



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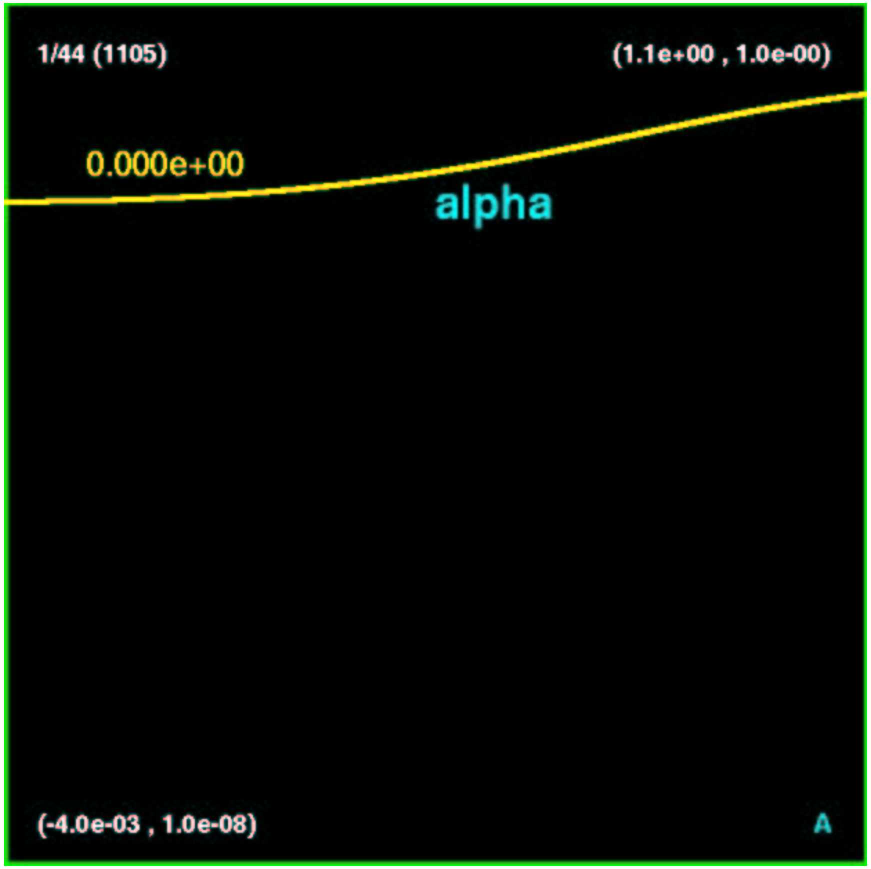

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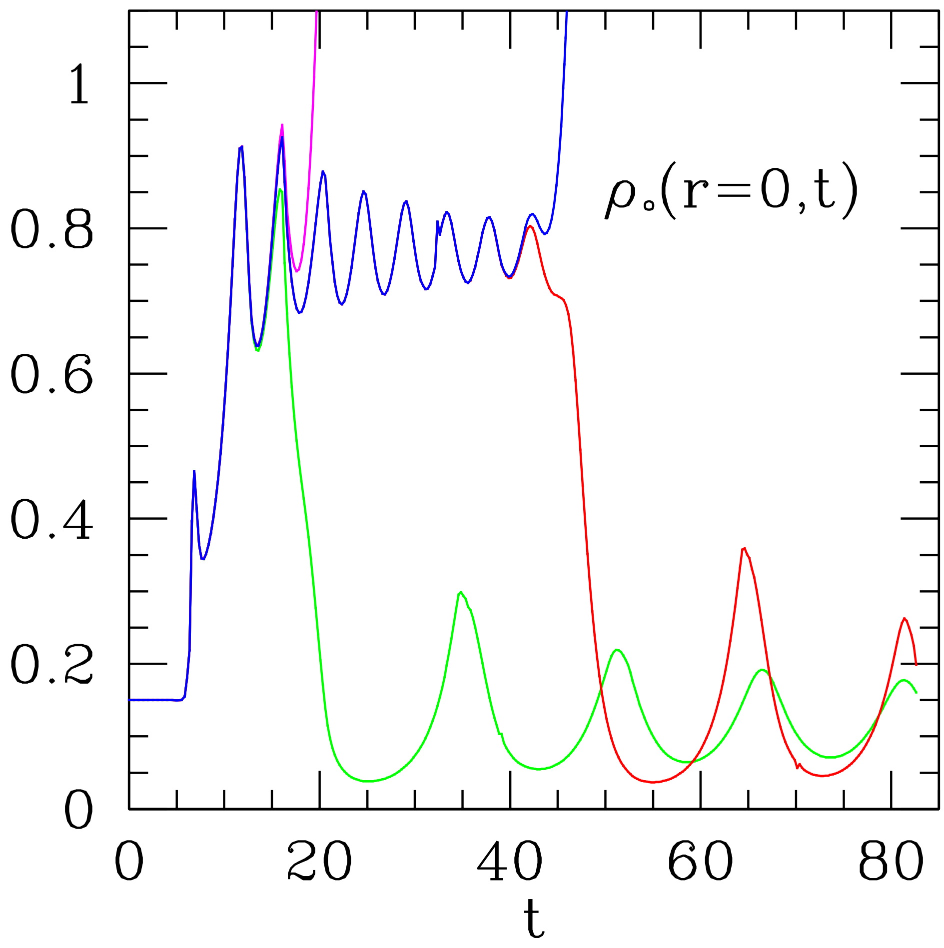
Type I Critical Phenomena

$\rho_c = 0.15$



- Hawley & Choptuik (2000): Boson Stars
- Vary p:
  $\phi(r,0) = p\,\mathtt{exp}\left(-\left[r - r_\circ\right]^2 / \Delta^2\right)$
- Large p $\rightarrow$ BLACK HOLE
- Small p $\rightarrow$ NO BLACK HOLE (e.g. perturbed star)
- Tuning away the only unstable mode

$$\Rightarrow T_\circ \propto \frac{1}{\omega_{Ly}} \ln|p - p^*|$$

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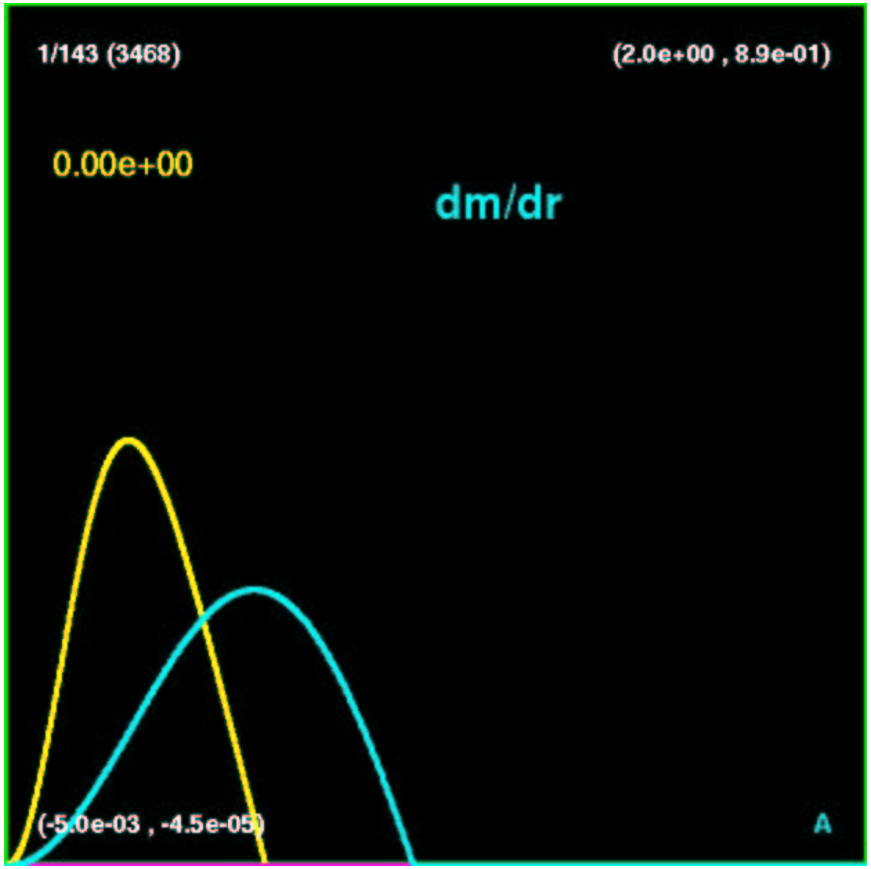

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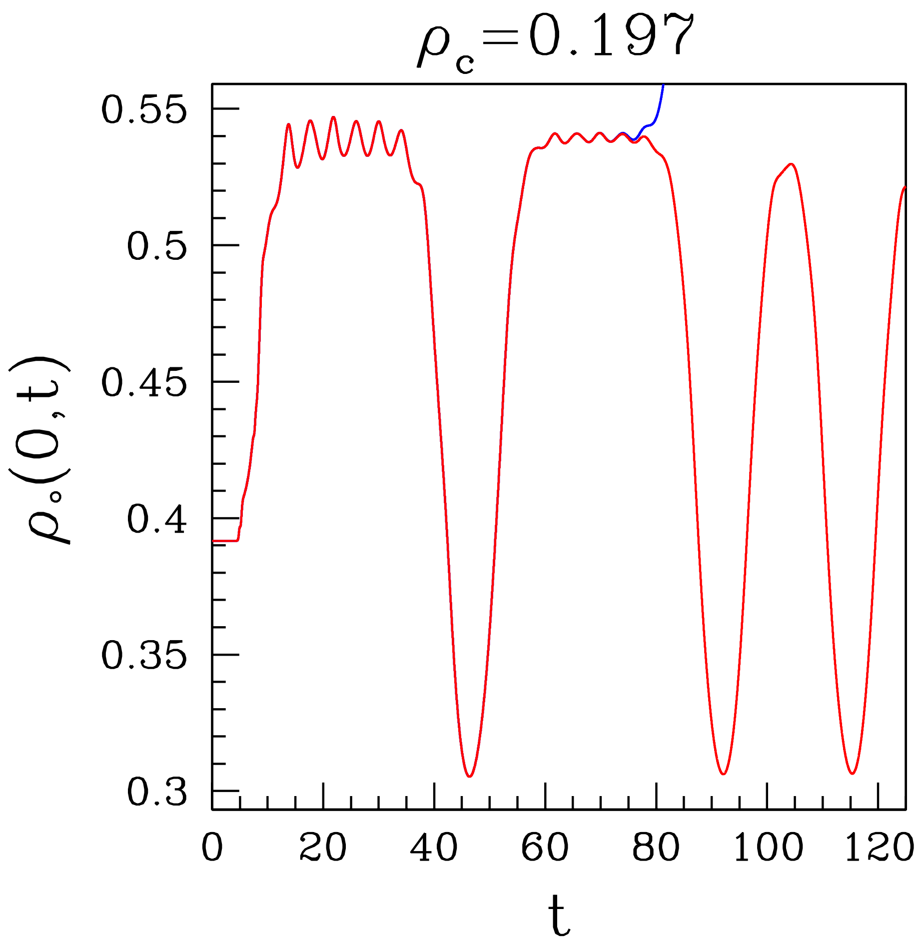
- Movies:

dm/dr , $\ln(dm/dr)$ (wide view) , $\ln(dm/dr)$ (closeup)

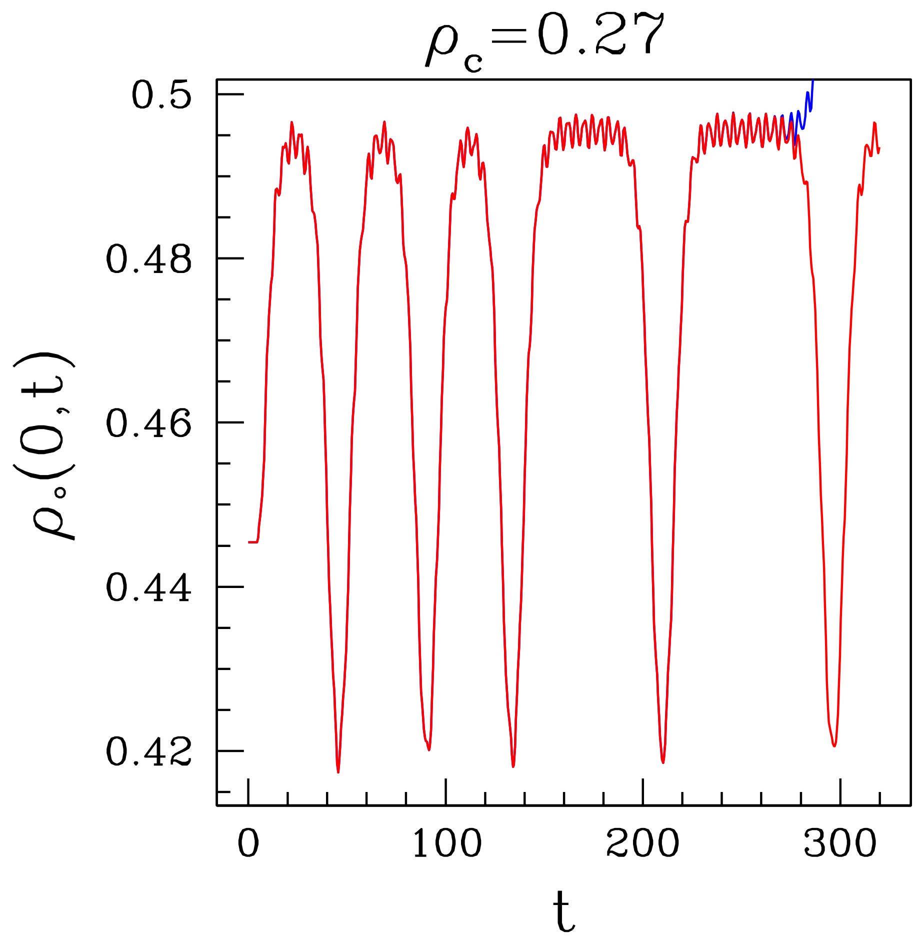
Type I: Anomalous Case $\rho_c = 0.197$



- Movie:

dm/dr

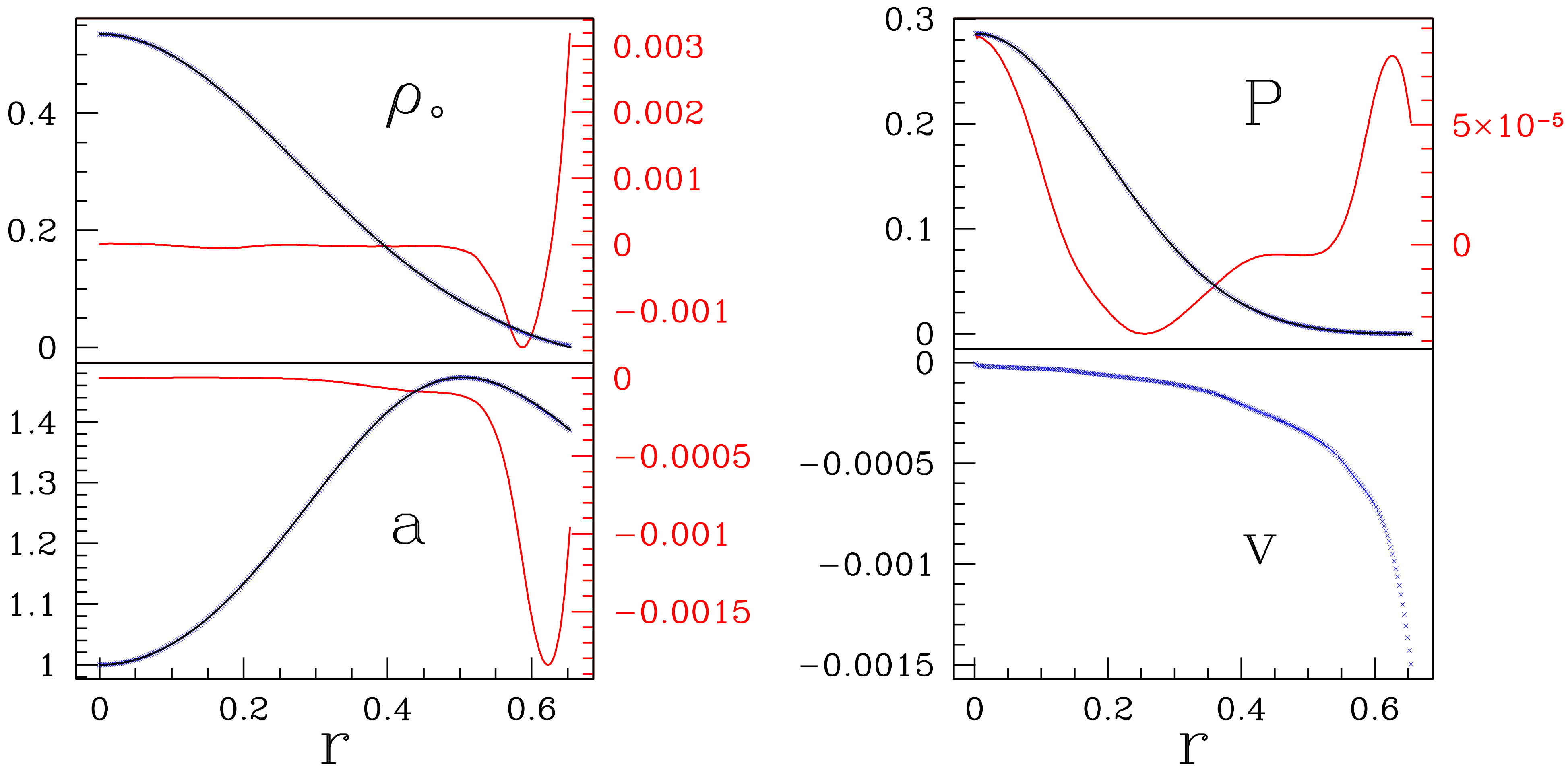
Type I: Anomalous Case $\rho_c = 0.27$



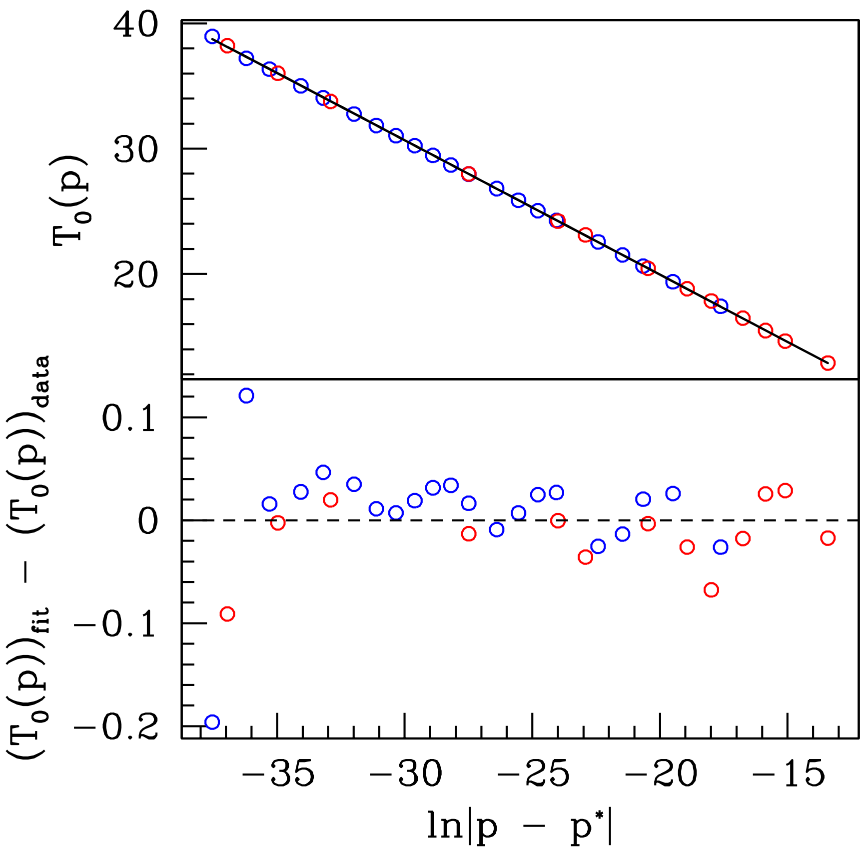
- Movies:

  dm/dr , $\ln(dm/dr)$

Critical Solution $\overset{?}{=}$ Unstable TOV $(\rho_c = 0.197)$



Scaling Behavior



- Expected scaling relationship:

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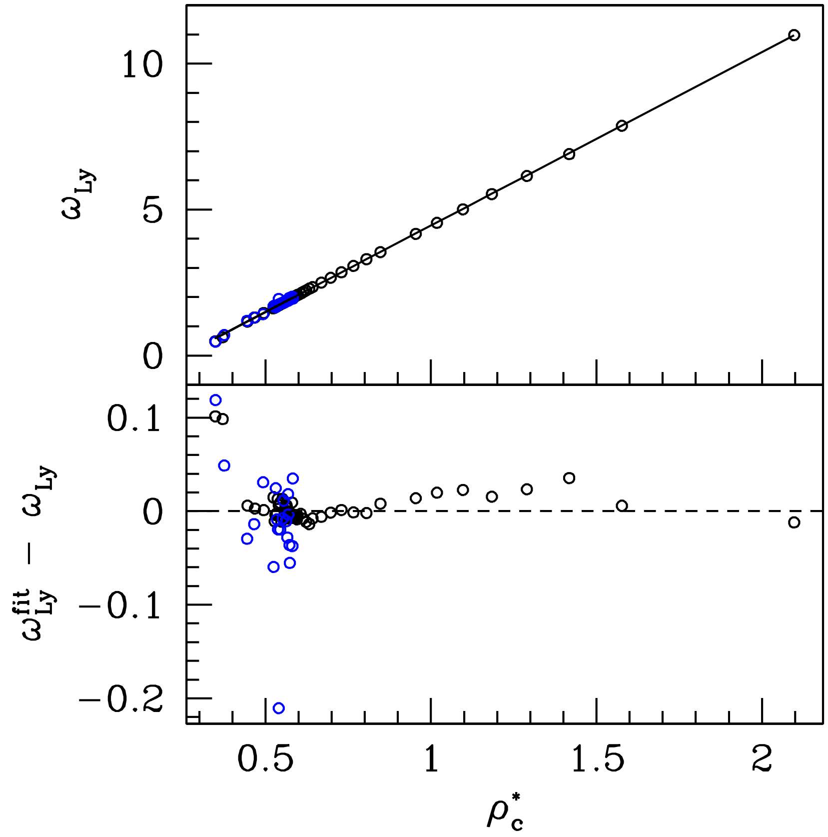

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- $\omega_{Ly} \propto \rho_c^*$

Type II Critical Phenomena: Motivation

- J. Novak (2001):
  - "Ideal-gas" EOS: $P = (\Gamma - 1)\,\rho_\circ \epsilon$, $\Gamma = 2$
  - Tuning star's init. vel. $\rightarrow$ Type II critical behavior;
  - $M_{BH} \propto |p - p^*|^\gamma$ with $\gamma \simeq 0.52$
- Neilsen and Choptuik (2000), Brady et al. (2002)
  - Studied ultra-relativistic fluid collapse;
  - A limit of "ideal-gas" case where $\rho \equiv (1 + \epsilon)\,\rho_\circ \simeq \rho_\circ \epsilon$
  - $P = (\Gamma - 1)\,\rho$, only EOS to admit CSS soln's;
  - For $\Gamma = 2$, $\gamma \simeq 0.95 \pm 0.02$
- Neilsen and Choptuik (2000)
  - For $\Gamma = 1.4$: Ideal-gas Type II Sol'n. $=$ Ultra-rel. Type II Sol'n.

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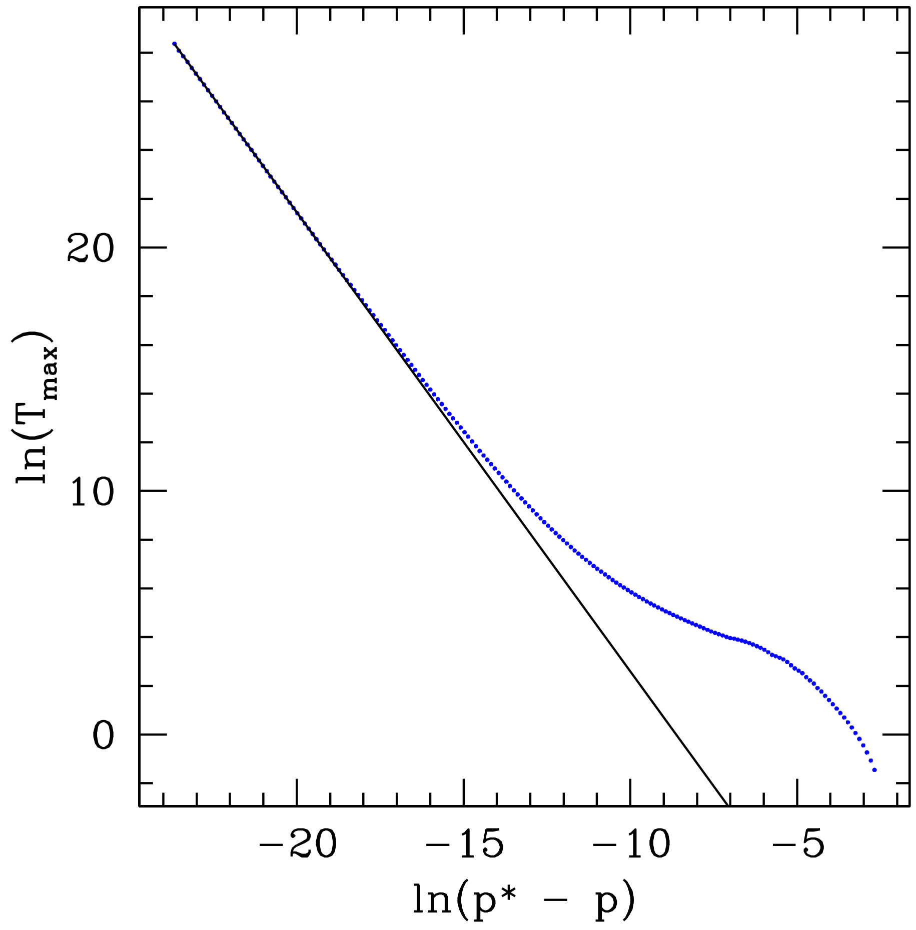
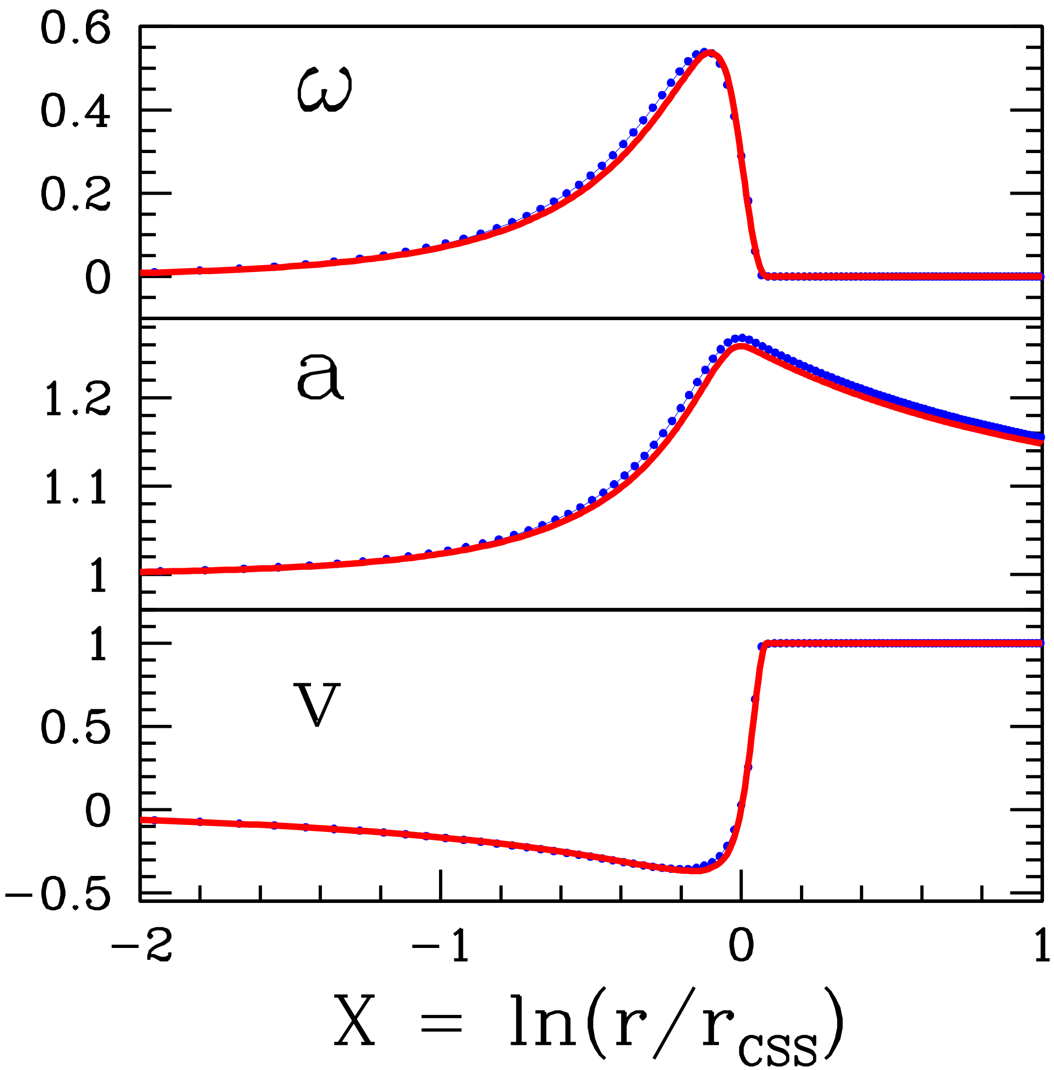
Critical Regime of Parameter Space



- $T_{\max} \equiv \text{Global Max.}(T^a{}_a)$
- $T^a{}_a = 3P - (\rho_\circ + \rho_\circ \epsilon)$
- Anticipated subcritical scaling behavior:
  $T_{\max} \propto |p - p^*|^{-2\gamma} \quad \gamma = 1/\omega_{Ly}$
- Novak tuned to $\ln|p^* - p| \simeq -7$

CSS Solutions of Ideal-gas and Ultra-rel.

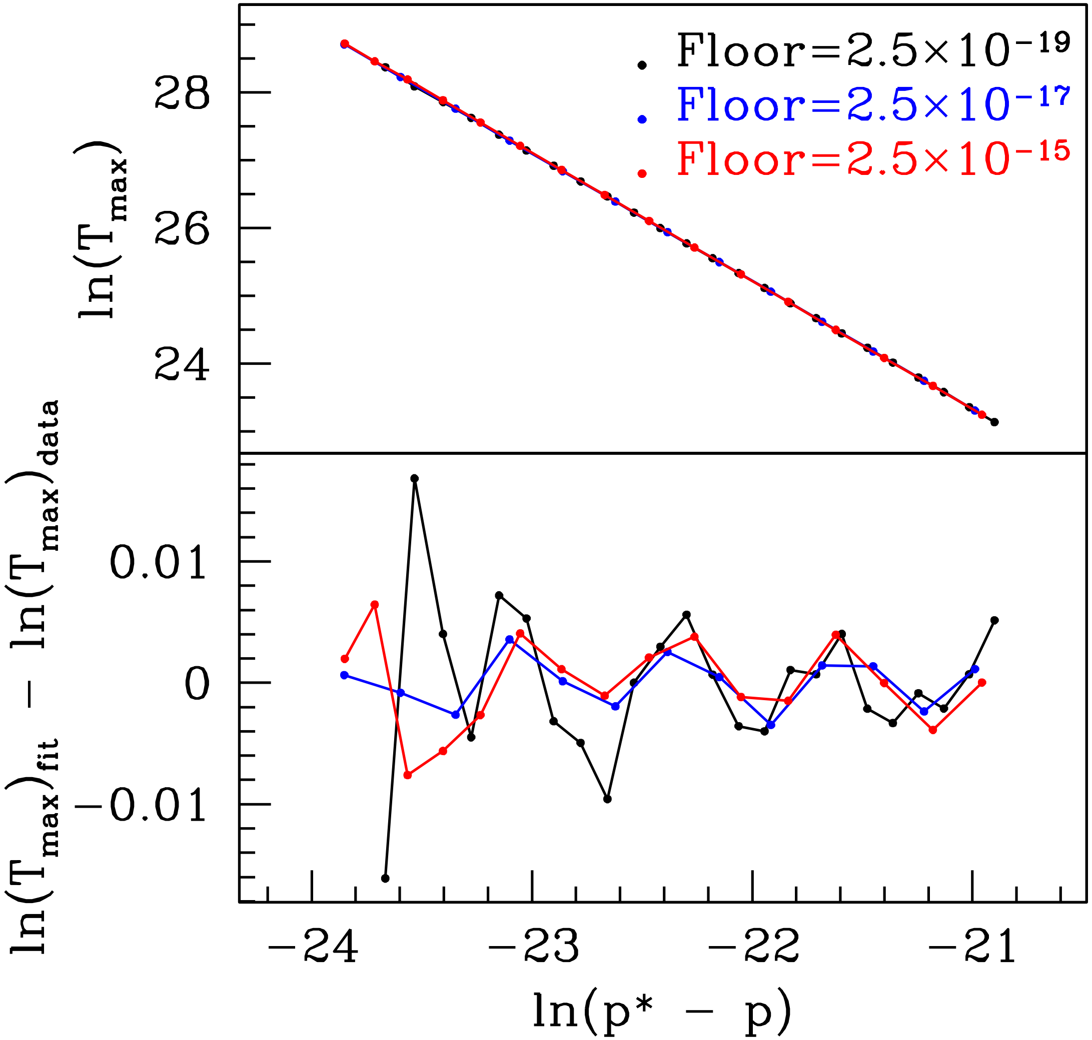


- Comparison of dimensionless quantities:
  - $\omega \equiv 4\pi r^2 a^2 \rho$
  - $a = \sqrt{g_{rr}}$
  - $v = \frac{a u^r}{\alpha u^t}$ = Eulerian Velocity

    (u^μ = Fluid's 4-velocity)
- Star: $\rho_c = 0.05$
- Ultra-relativistic fluid:

  Initial profile = Gaussian

Scaling of $T_{\max}$: Dependence on Fluid's Floor



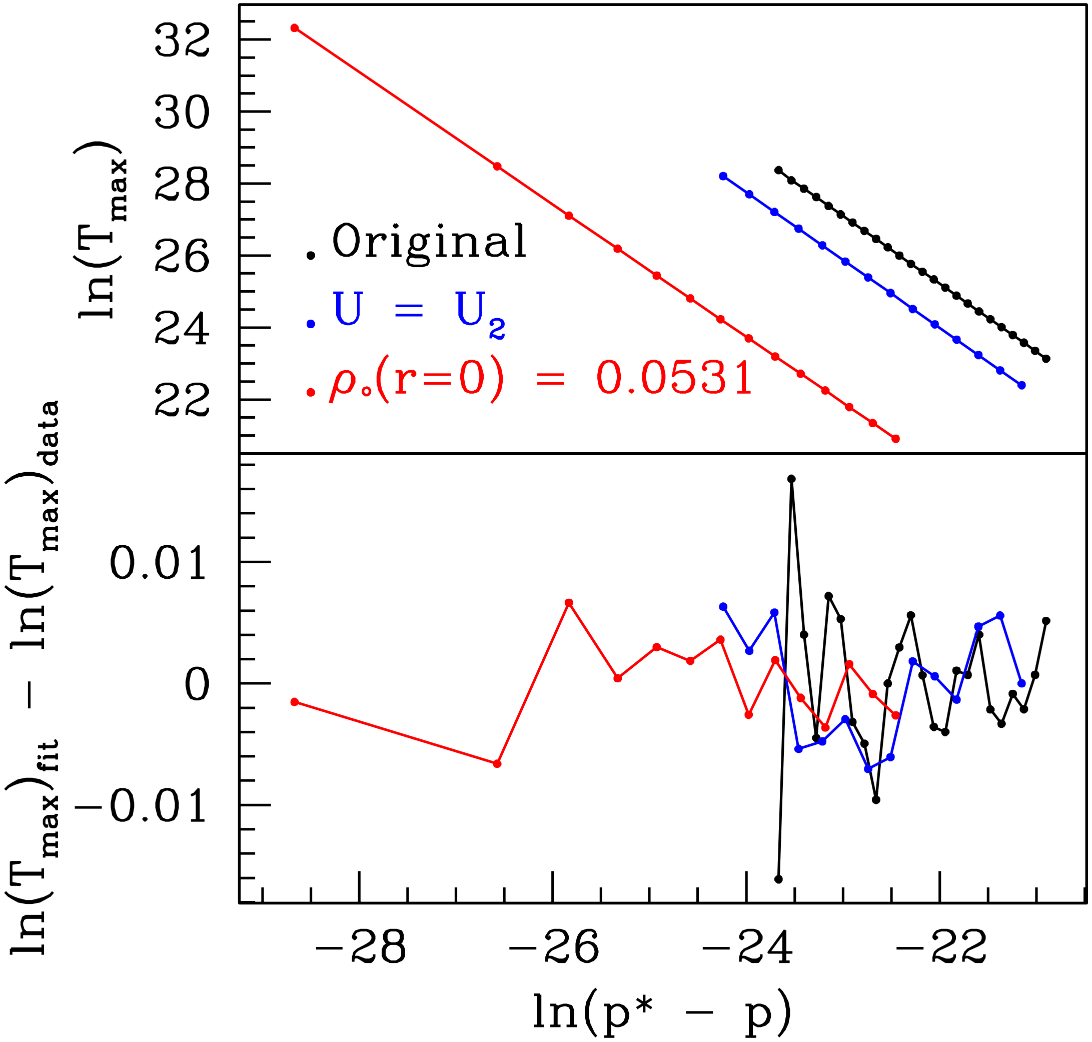
γ	p^*
0.9427	0.46875367383
0.9436	0.46875350285
0.9470	0.4687516089

- Floor used to prevent

  $v \geq 1 \quad , \quad P, \rho_\circ < 0$

- No significant effect;

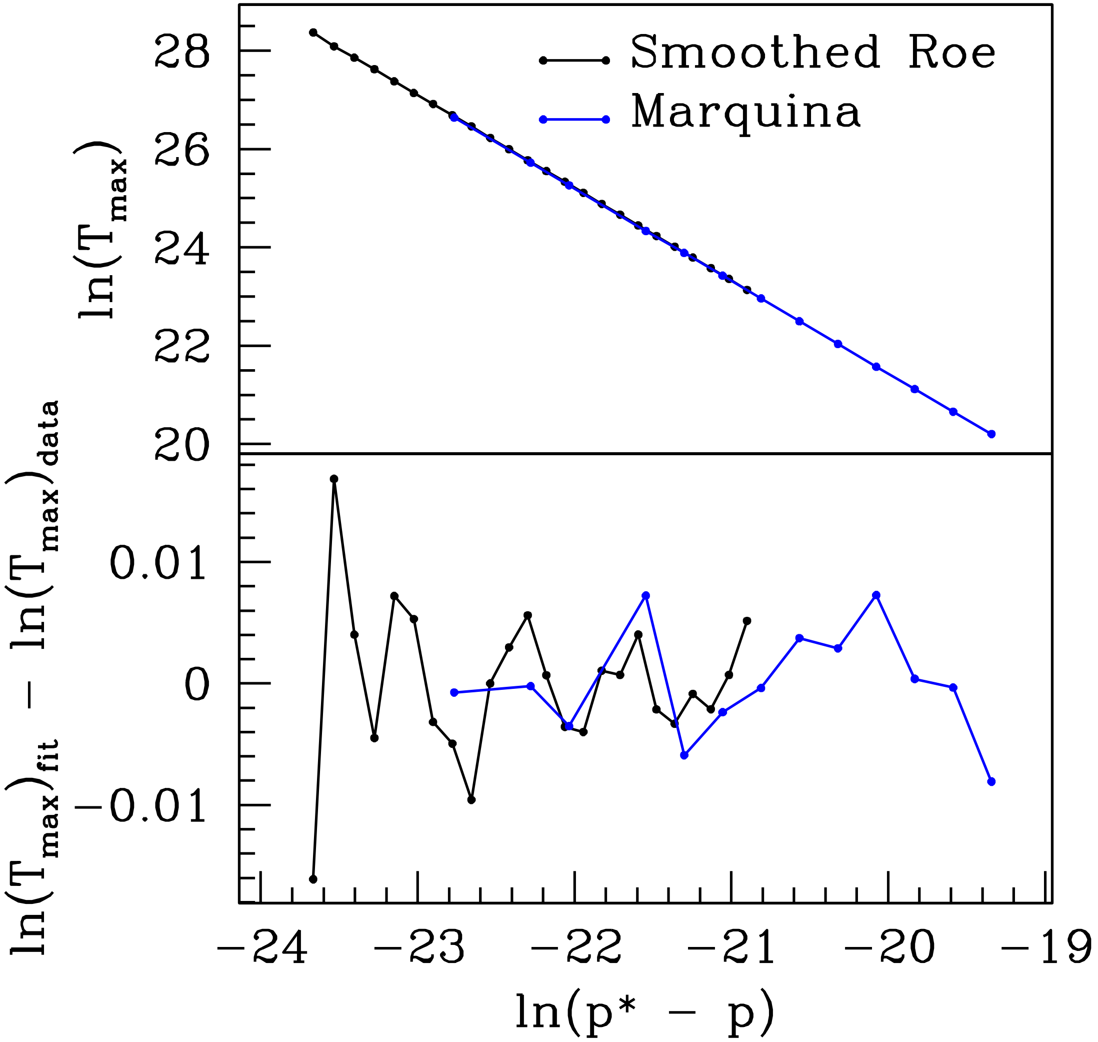
Scaling of $T_{\max}$: Different "Families"



γ	p^*
0.9427	0.46875367383
0.9423	0.42990315097
0.9187	0.4482047429836

- Suggests scaling is fairly independent of:
  - Functional form of perturbation;
  - Initial star configuration;

Scaling of $T_{\max}$: Different Flux Functions



γ	p^*
0.9427	0.46875367383
0.9399	0.46876822118

- Suggests scaling is independent of flux formula;
- Able to tune further with "Smoothed" Roe solver;

Comparison of Scaling Parameters

Noble and Choptuik	Ideal gas	$\gamma = 0.94 \pm 0.01$
Noble and Choptuik	Ultra-relativistic fluid	$\gamma = 0.9747$
Neilsen and Choptuik (2000) and Brady et al. (2002)	Ultra-relativistic fluid	$\gamma = 0.95 \pm 0.02$
Novak (2001)	Ideal gas	$\gamma \simeq 0.52$

Conclusion

- Parameter Space Survey:
  - Illuminated possible dynamical scenarios
  - Provided a backdrop for critical phenomena studies
- Type I Behavior:
  - Critical solutions $\simeq$ perturbed unstable TOV solutions
  - Found anticipated scaling behavior $T_\circ \propto \frac{1}{\omega_{Ly}} \ln |p - p^*|$
  - $\omega_{Ly} \propto \rho_c^*$
- Type II Behavior:
  - Ideal gas critical solution $\simeq$ ultra-relativistic critical solution
  - $\gamma_{\mathrm{ideal}} \simeq \gamma_{\mathrm{ultra-rel}}$

Future Work

- Type I Phenomena:
  - Compare results to ω_{Ly} of unstable TOV growing modes
  - Axially-symmetric collapse, effect of rotation
  - How $\omega_{Ly}(\rho_c^*)$ varies with Γ
  - Dependence on EOS

- Type II Phenomena:
  - Realistic equation of state
  - Axially-symmetric critical behavior
  - Develop general adaptive mesh refinement methods for relativistic fluids

Supporting Agencies

- NSERC = National Sciences and Engineering Research Council of Canada
- CIAR = Canadian Institute for Advance Research
- CFI = Canada Foundation for Innovation
- BCKDF = British Columbia Knowledge Development Fund

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