Critical Phenomena in General Relativity

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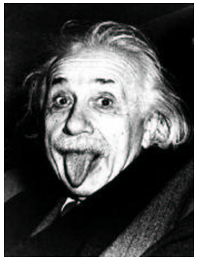
Outline

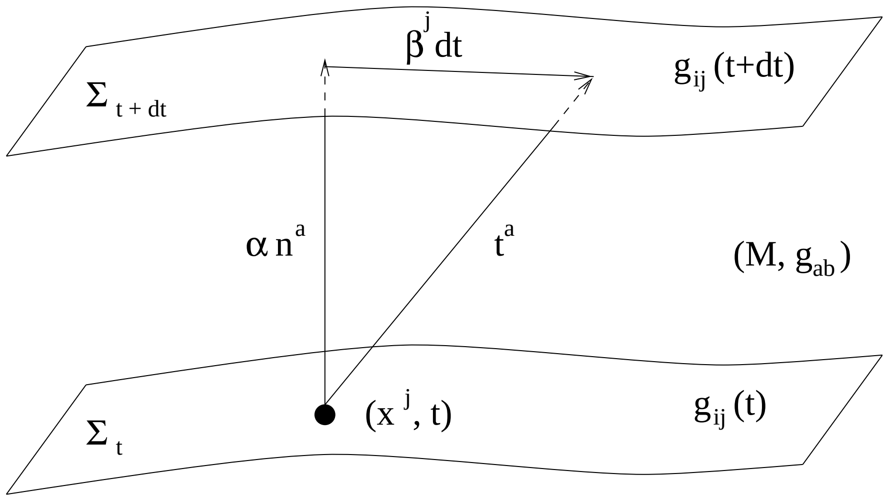
- Introduction to General Relativity (GR) and Numerical Relativity
- General Relativistic Hydrodynamics
- Neutron Stars
- Introduction to Critical Phenomena in GR
- Results
- Conclusion

Introduction to General Relativity

Newton	Einstein
	
Flat, Cartesian Geometry	Curved, Lorentzian Geometry
$dt\ , d\vec{\mathbf{r}}^2 = \text{constants}$	$ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu = \text{constant}$
Instantaneous Forces, Signals	Local-frame's speed limit = c
$\vec{F}_{\text{Gravity}} = -m\vec{\nabla}\phi$	No $\vec{F}_{\text{Gravity}}$, just free-fall!
$\vec{F}_{\text{Total}} = m\vec{a}$	$G_{\mu\nu} = 8\pi T_{\mu\nu} \quad \mu, \nu \in \{0, 1, 2, 3\}$
Simple, linear PDE's	Difficult, many highly-nonlinear PDE's, in general

The Arnowitt-Deser-Misner (ADM) 3+1 Formalism

$$ds^2 = \left(-\alpha^2 + \beta^i \beta_i\right) dt^2 + 2\beta_i dt dx^i + g_{ij} dx^i dx^j$$



- α, **lapse**, relates coord. time to proper time
- β^j, **shift**, how x^j translates between slices;
- n^a, time-like unit normal vector to Σ
- g_{ij}, spatial metric on Σ
- $t^a = \alpha n^a + \beta^a$, time-like tangent to coordinate's world line.

ADM *continued*

- ADM = constrained Hamiltonian formulation of GR;
- Foliating spacetime provides "time direction" through slices $\longrightarrow$ (Cauchy) Initial Value Problem;
- $g_{ij}(t, x^k)$ and extrinsic curvature, $K_{ij}(t, x^k)$, are the dynamical variables, where K_{ij} is a conjugate momentum to g_{ij} ($K_{ij} \equiv -\frac{1}{2}\pounds_n g_{ij}$)
- $g_{\mu\nu} = 10$ "fields", 4 Coordinate Conditions, 4 Constraints
- (10 - 4 - 4) = 2 Dynamical degrees of freedom in EQ's which follow 2nd-order hyperbolic PDE's or wave equations
  $\longrightarrow$ Gravitational Waves, aka Ripples in the Fabric of Spacetime.... !!

Fluid and Geometry Equations

Metric:

$$ds^2 = -\alpha\left(r,t\right)^2 dt^2 + a\left(r,t\right)^2 dr^2 + r^2\left(d\theta^2 + sin^2\theta d\phi^2\right)$$

Stress-Energy Tensor:

$$T_{ab} \;=\; \left(\rho_\circ + \rho_\circ\epsilon + P\right) u_a u_b + P g_{ab}$$

Equations of Motion: Local Conservation of Energy and Particle Number

$$\nabla_\mu T^\mu{}_\nu = 0 \quad , \quad \nabla_\mu\left(\rho_\circ u^\mu\right) = 0$$

Equation of State

$$P = (\Gamma - 1)\rho_\circ\epsilon$$

$$D = a\rho_\circ W \ , \quad S = (\rho + P)\, W^2 v \ , \quad \tau = S/v - D - P$$

$$v = \frac{a u^r}{\alpha u^t} \quad , \quad W^2 = \frac{1}{1 - v^2}$$

Slicing Condition :

$$\frac{\alpha'}{\alpha} = a^2 \left[4\pi r \left(S v + P \right) + \frac{1}{2r} \left(1 - 1/a^2 \right) \right]$$

Hamiltonian Constraint :

$$\frac{a'}{a} = a^2 \left[4\pi r \left(\tau + D \right) - \frac{1}{2r} \left(1 - 1/a^2 \right) \right]$$

g_{rr} **Evolution :**

$$\dot{a} = -4\pi r \alpha a^2 S$$

Fluid Equations of Motion:

$$\nabla_\mu T^\mu_\nu = 0 \quad , \quad \nabla_\mu J^\mu_\nu = 0$$

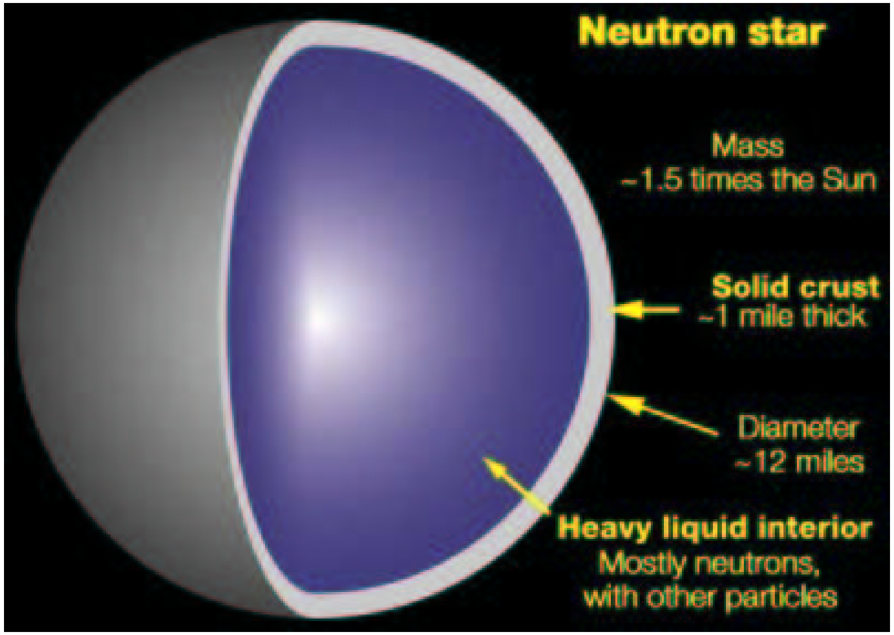
$$\boxed{\dot{\mathbf{q}} + \tfrac{1}{r^2}\left(r^2 \tfrac{\alpha}{a}\mathbf{f}\right)' = \boldsymbol{\psi}}$$

$$\mathbf{q} = \begin{bmatrix} D \\ S \\ \tau \end{bmatrix} , \ \mathbf{f} = \begin{bmatrix} Dv \\ Sv + P \\ v\,(\tau + P) \end{bmatrix} , \ \boldsymbol{\psi} = \begin{bmatrix} 0 \\ \Sigma \\ 0 \end{bmatrix} , \ \mathbf{w} = \begin{bmatrix} \rho_\circ \\ v \\ P \end{bmatrix}$$

$$\Sigma \equiv \Theta + \frac{2P\alpha}{ra}$$

$$\Theta \equiv \alpha a \left[(Sv - \tau - D) \left(8\pi r P + \frac{1}{2r}\left(1 - 1/a^2\right) \right) + \frac{P}{2r}\left(1 - 1/a^2\right) \right]$$

Neutron Stars

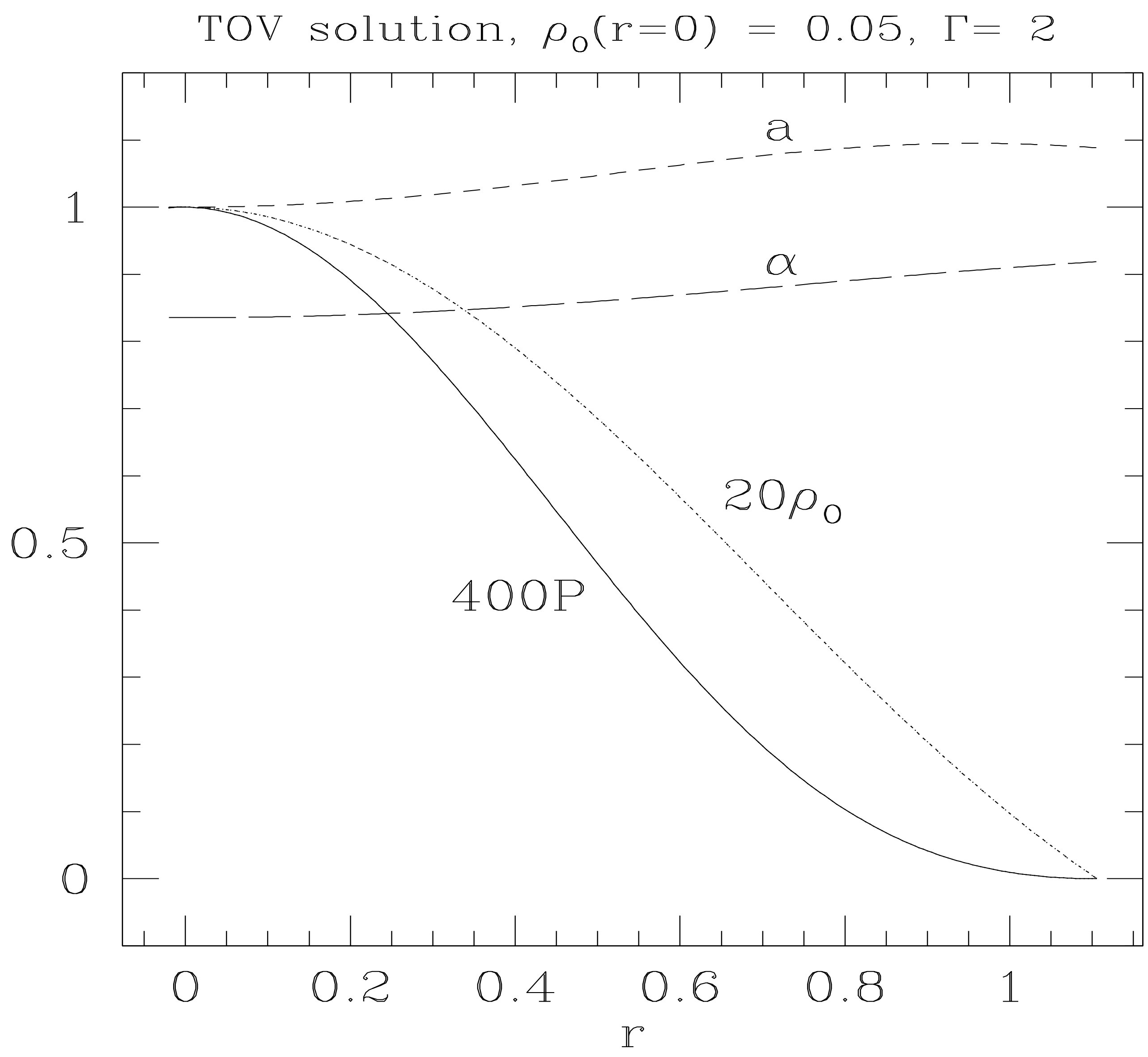


- NS = Big “Nucleus”, consists primarily of neutrons in condensed state;
- Most compact matter (non-BH) object;
- $\rho_{Avg} = 2 \times 10^{18}\ kg/m^3$, $\Rightarrow\ 1km^3$ of NS $= M_{\mathrm{Earth}}$

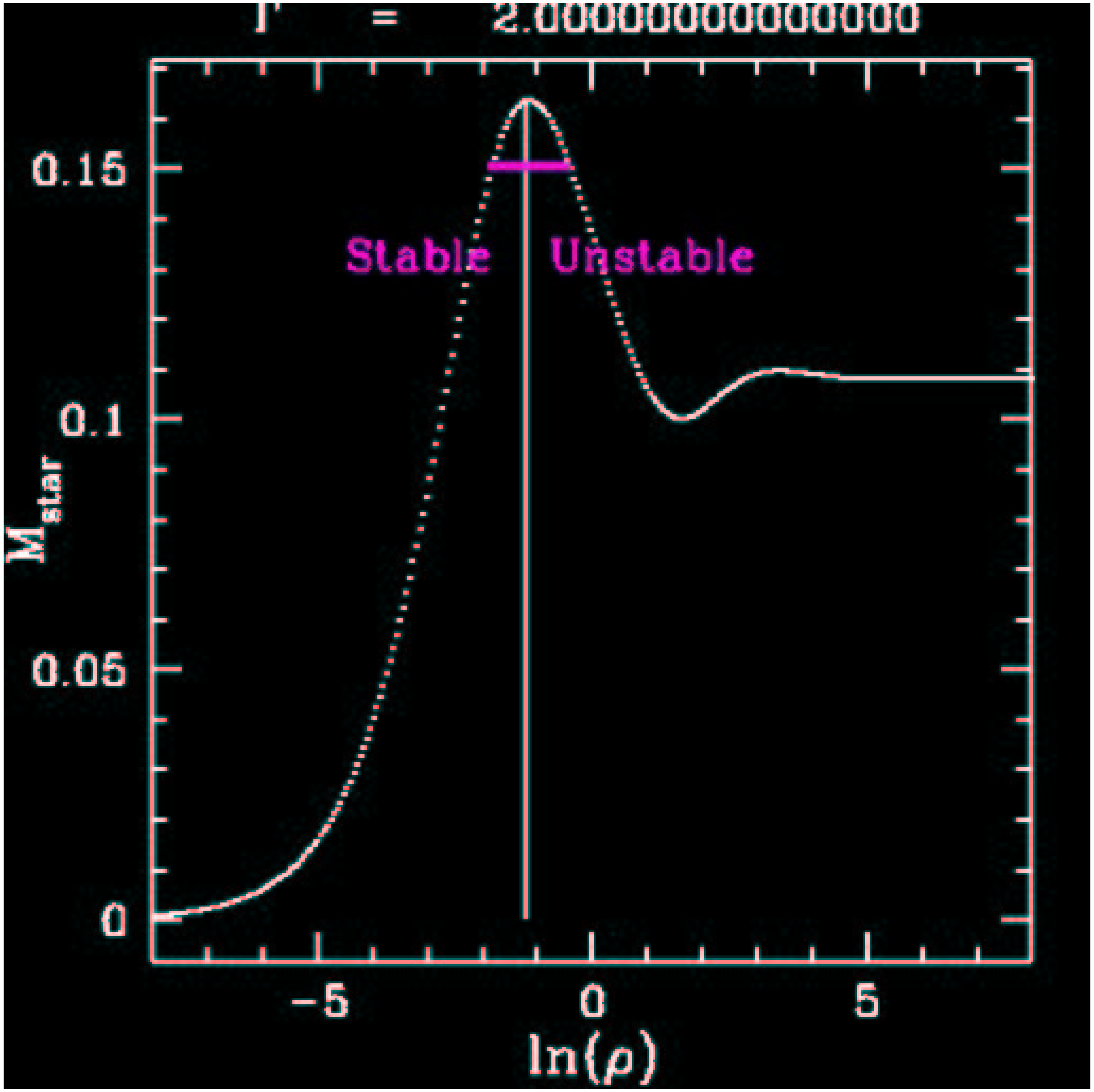
  $\Rightarrow$ Need GR to properly describe;
- Model as static, spherical solution to $G_{\mu\nu} = 8\pi T_{\mu\nu}$

  $\Rightarrow$ Tolman-Oppenheimer-Volkoff equations.

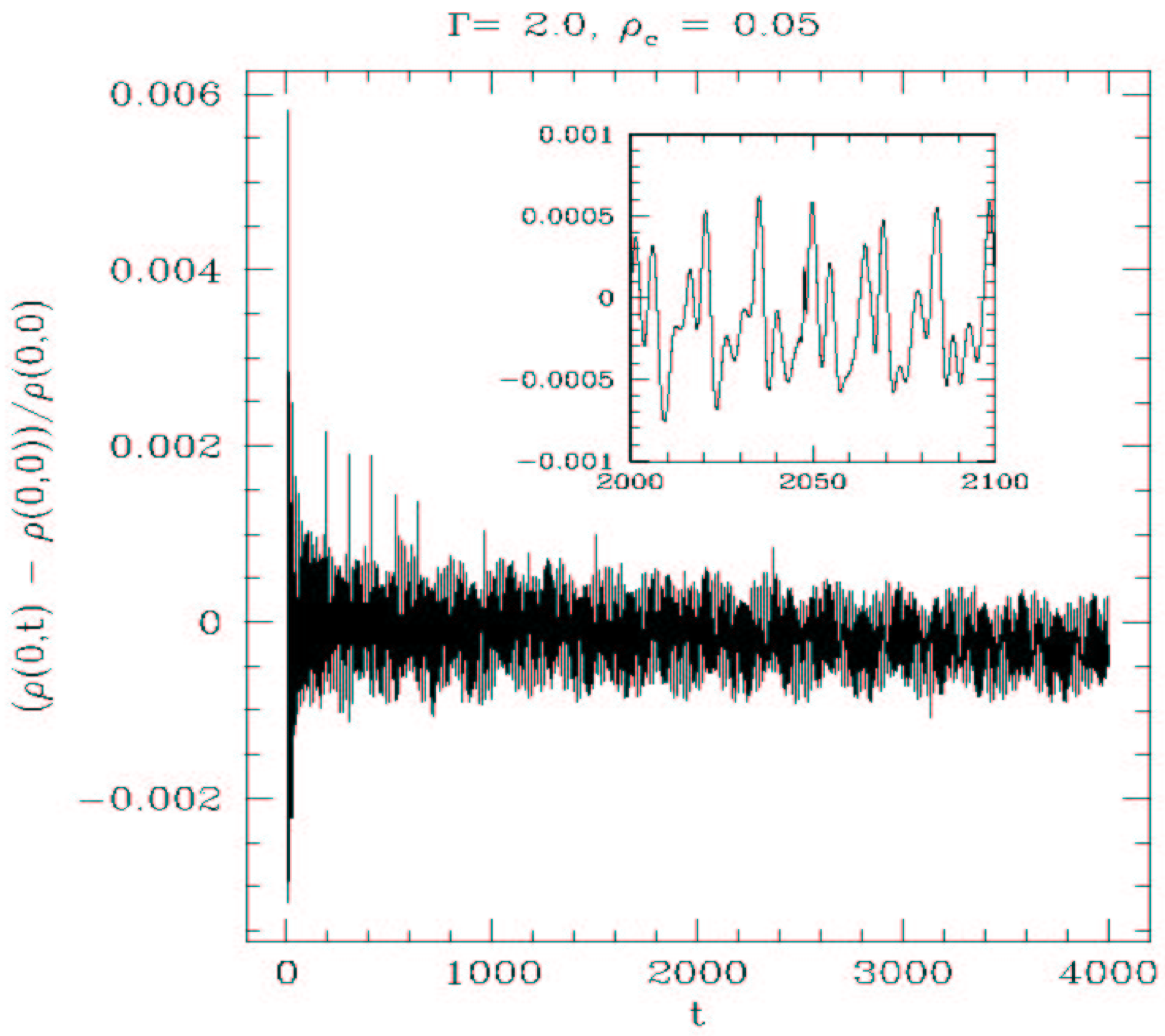
A TOV, or Static Star, Solution



Mass v. $\ln \rho_\circ(r=0)$



Evolution of a “Static” Star Solution



A Decade of Critical Phenomena

- M. W. Choptuik “Universality and Scaling in Gravitational Collapse of a Massless Scalar Field”, PRL **70**, 1, January 4, 1993.
- Crit. Phen. observed anywhere you have (BH)/(No BH);
- “Tuning” of initial data to Critical Solution
  $\longrightarrow$ eliminate the 1 unstable mode from solution;
- General feature of gravitational collapse, observed in many different matter models (even w/o matter!: Brill Gravitational Waves)
- Some Crit. Solutions are “Naked Singularities”!

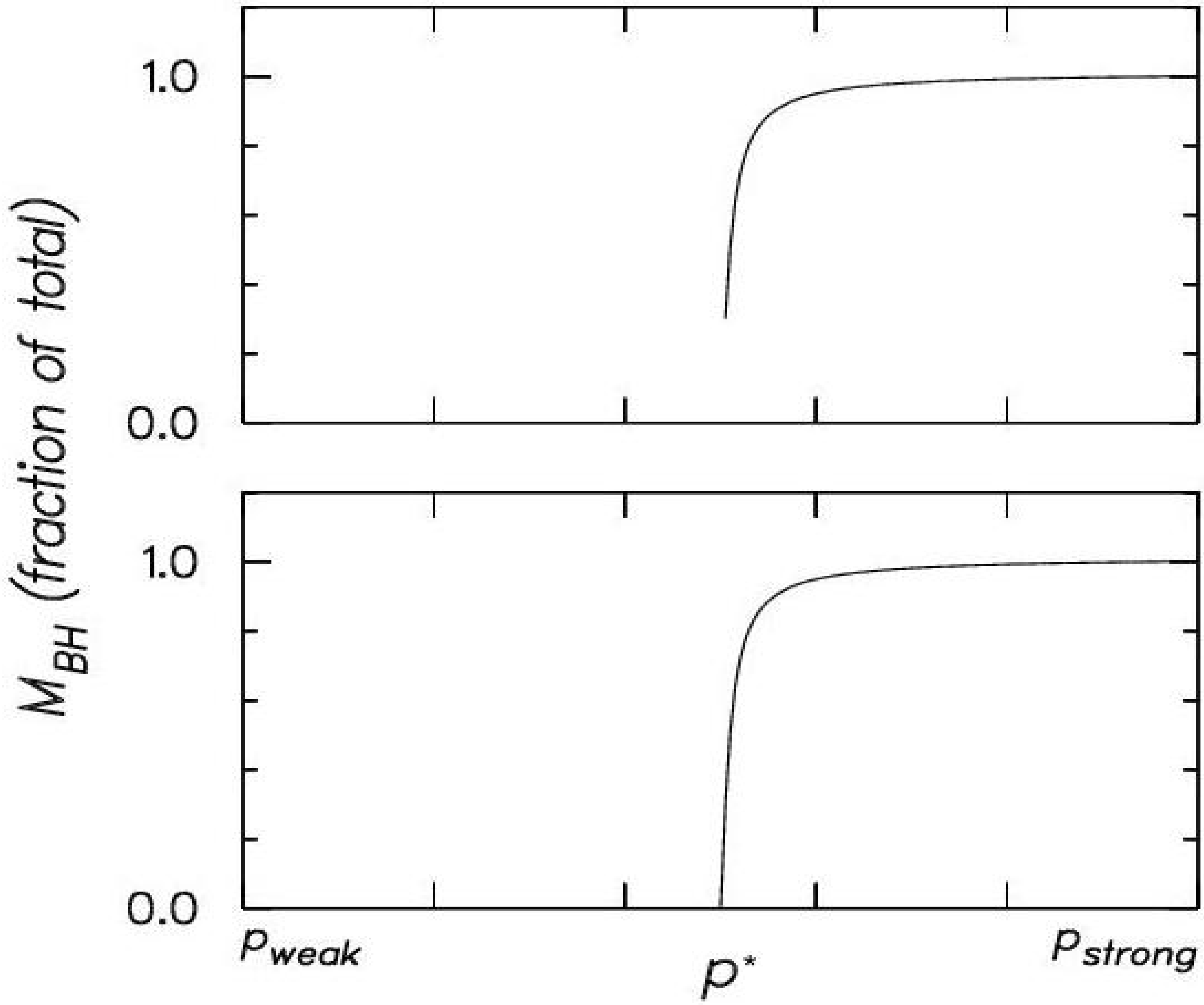
Systems Found to Exhibit Critical Phenomena

Matter	Type	Collapse simulations	Critical solution	Perturbations of crit. soln.
Perfect fluid $p = k\rho$	II	[69, 142]	CSS [69, 138, 142]	[138, 128, 93, 97]
Real scalar field:				
– massless, min. coupled	II	[47, 48, 49]	DSS [89]	[90, 139]
– massive	I	[32]	oscillating [165]	
	II	[49]	DSS [104, 99]	[104, 99]
– conformally coupled	II	[48]	DSS	
– 4+1	II	[16]		
– 5+1	II	[77]		
Massive complex scalar field	I (II)	[110]	[165]	[110]
Massless scalar electrodynamics	II	[117]	DSS [99]	[99]
2-d sigma model				
– complex scalar ($\kappa = 0$)	II	[50]	DSS [90]	[90]
– axion-dilaton ($\kappa = 1$)	II	[101]	CSS [67, 101]	[101]
– scalar-Brans-Dicke ($\kappa > 0$)	II	[136, 133]	CSS, DSS	
– general κ including $\kappa < 0$	II		CSS, DSS [115]	[115]
$SU(2)$ Yang-Mills	I	[53]	static [12]	[131]
	II	[53]	DSS [92]	[92]
	"III"	[55]	colored BH [17, 173]	[168, 172]
$SU(2)$ Skyrme model	I	[19]	static [19]	[19]
	II	[22]	static [22]	
$SO(3)$ Mexican hat	II	[134]	DSS	
Vlasov	I?	[160, 148]	[141]	

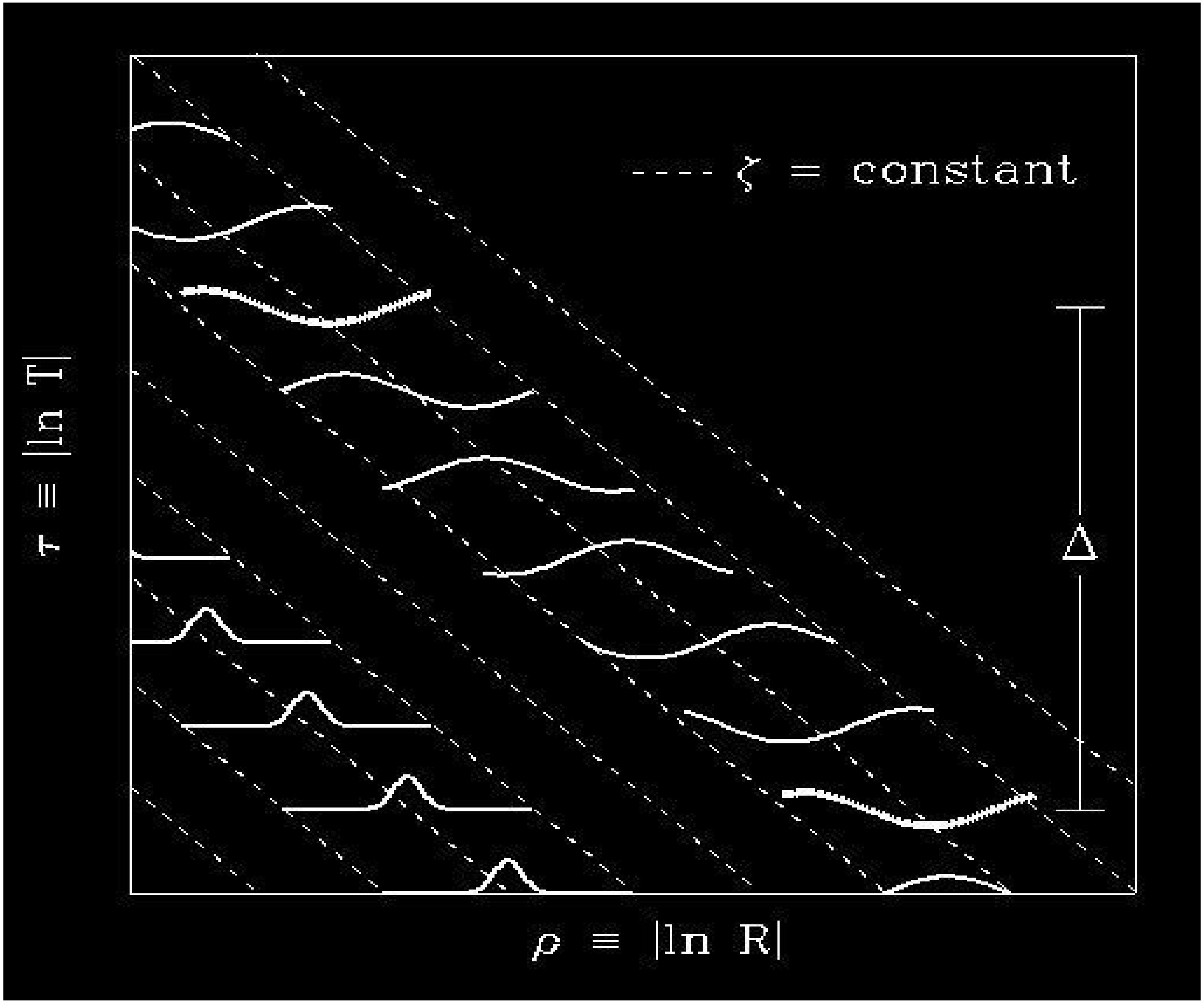
Types of Crit. Phen.

Type I	Type II
Discontinuous “Phase” Transition $M_{BH} \to M^* > 0$	Continuous “Phase” Transition $M_{BH} \to 0$
Static or Oscillatory	Cont. or Discretely Self-similar
$t_{\rm hang} \propto \left\|p - p^*\right\|^{-\gamma}$	$M_{\rm BH} \propto \left\|p - p^*\right\|^{\gamma}, T_{\rm max} \propto \left\|p - p^*\right\|^{2\gamma}$

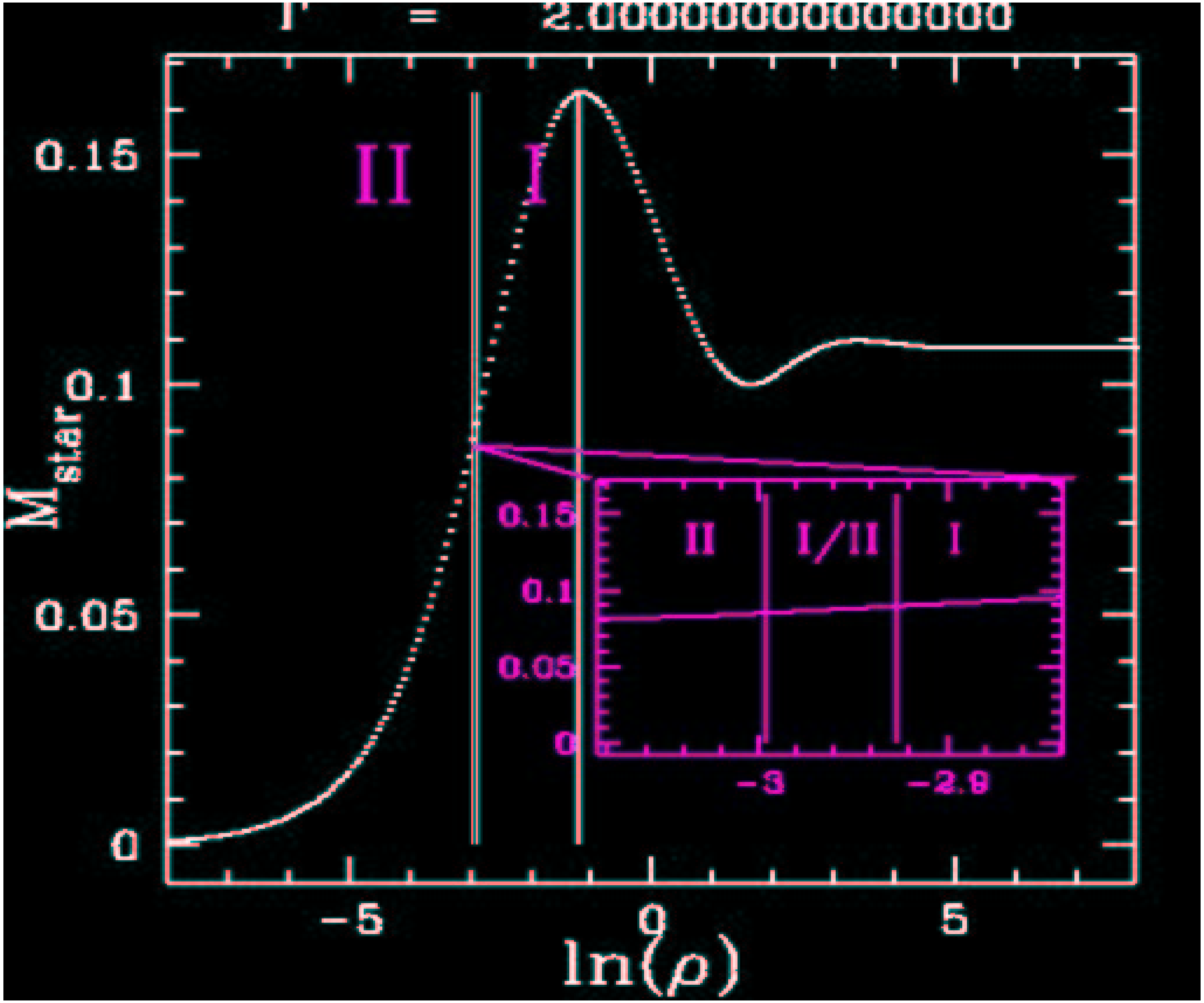
Black Hole Mass Scaling



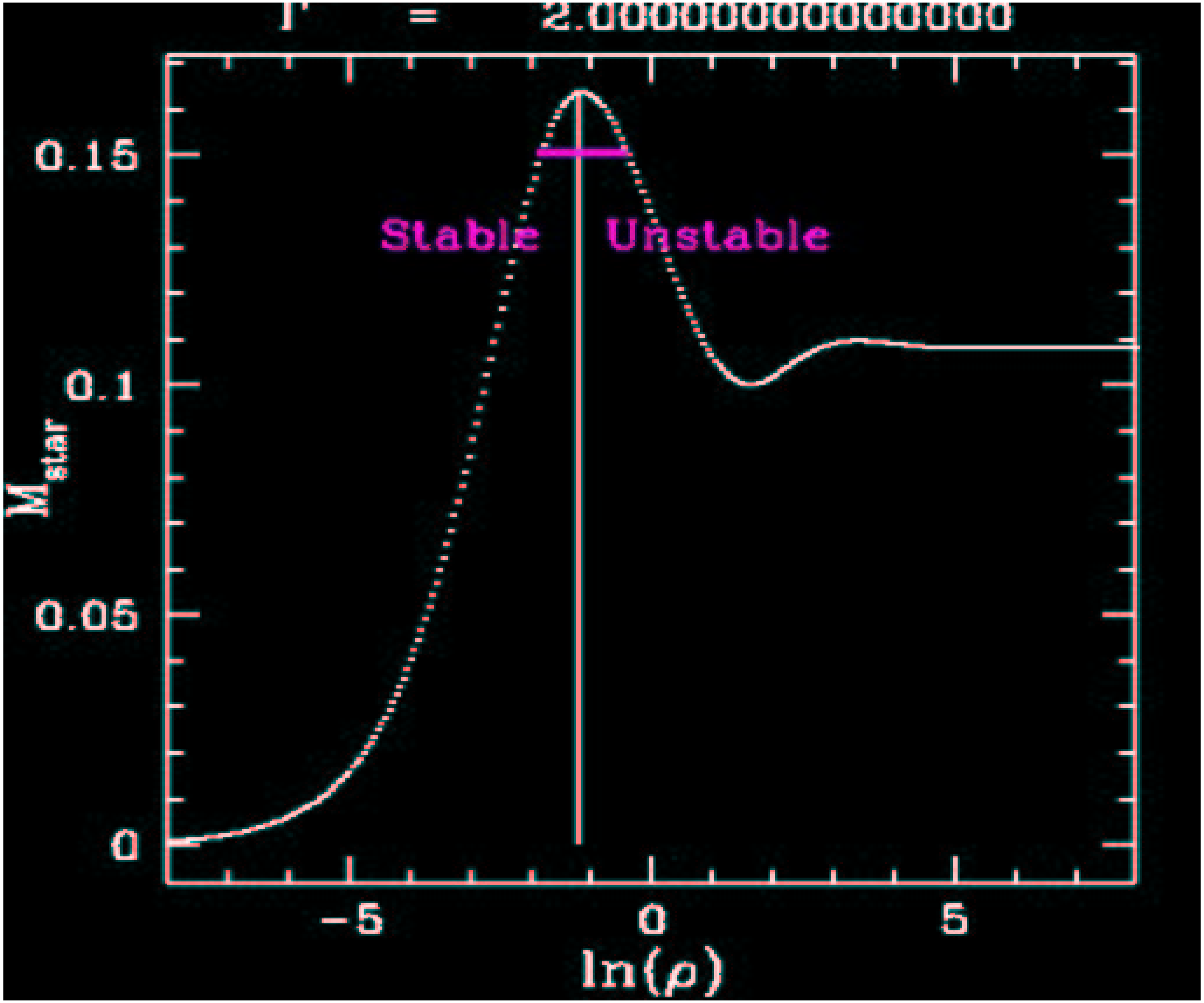
Cont. and Discrete Self-Similarity of Type-II



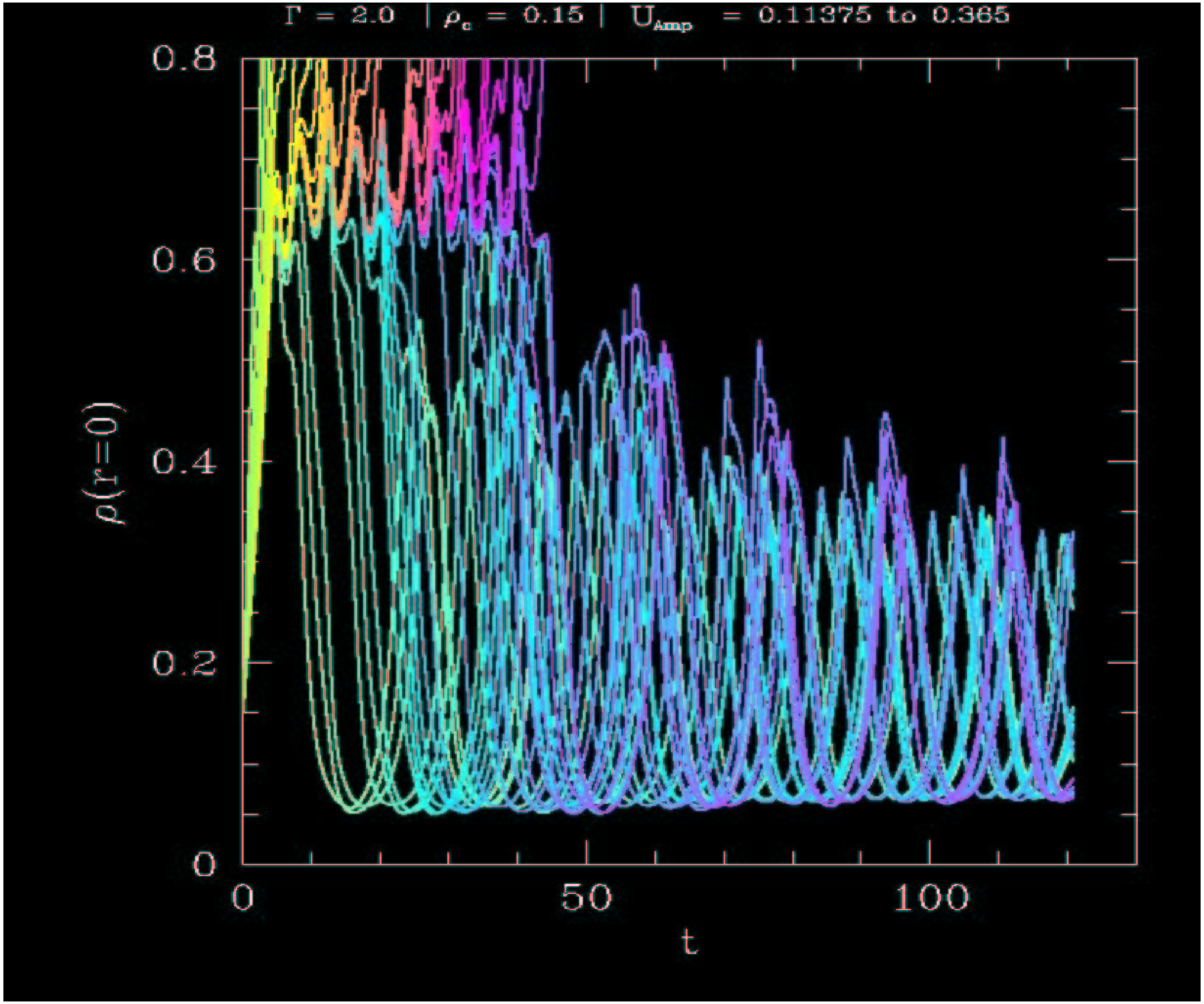
Types in TOV Solutions



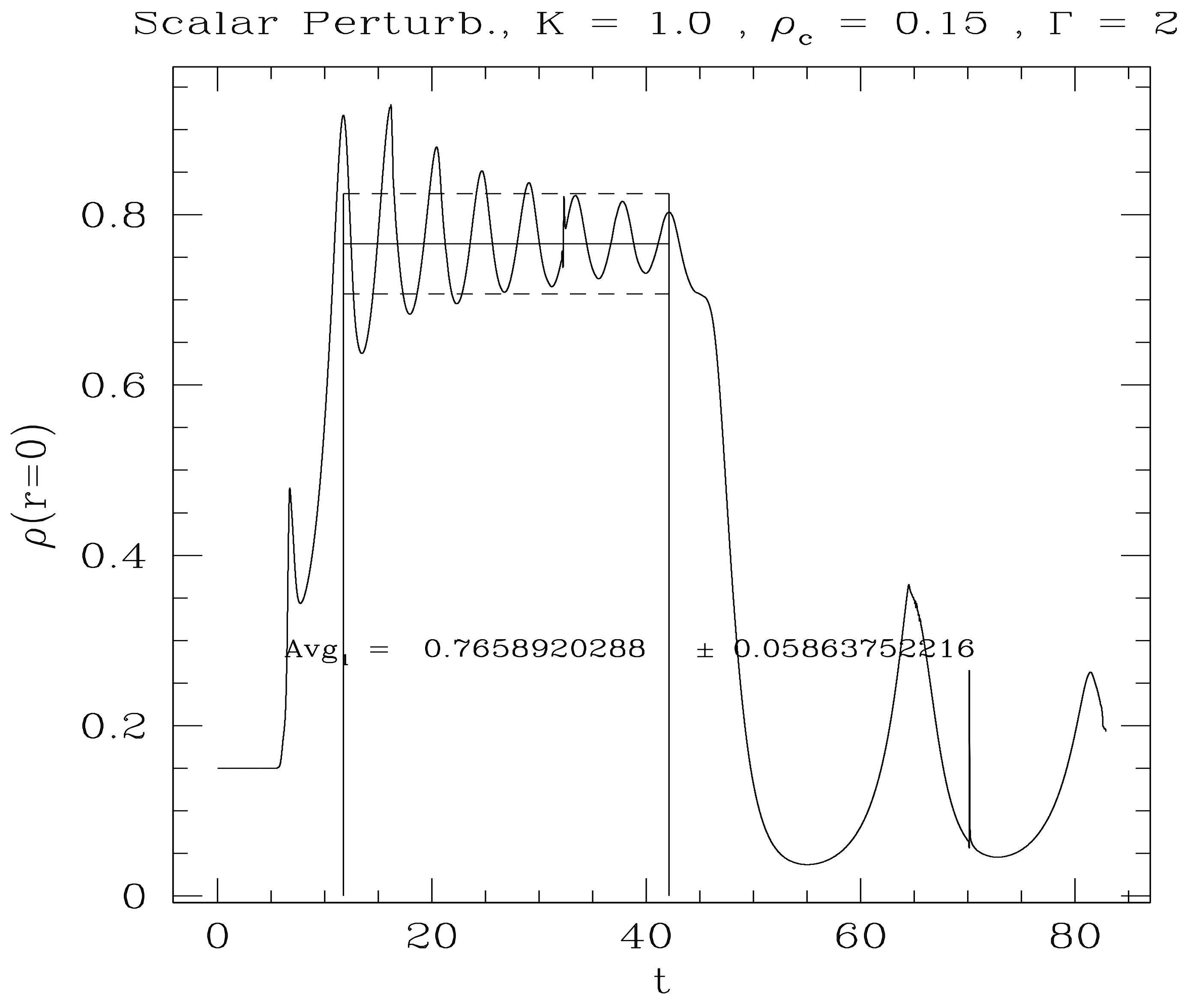
Type-I Transition



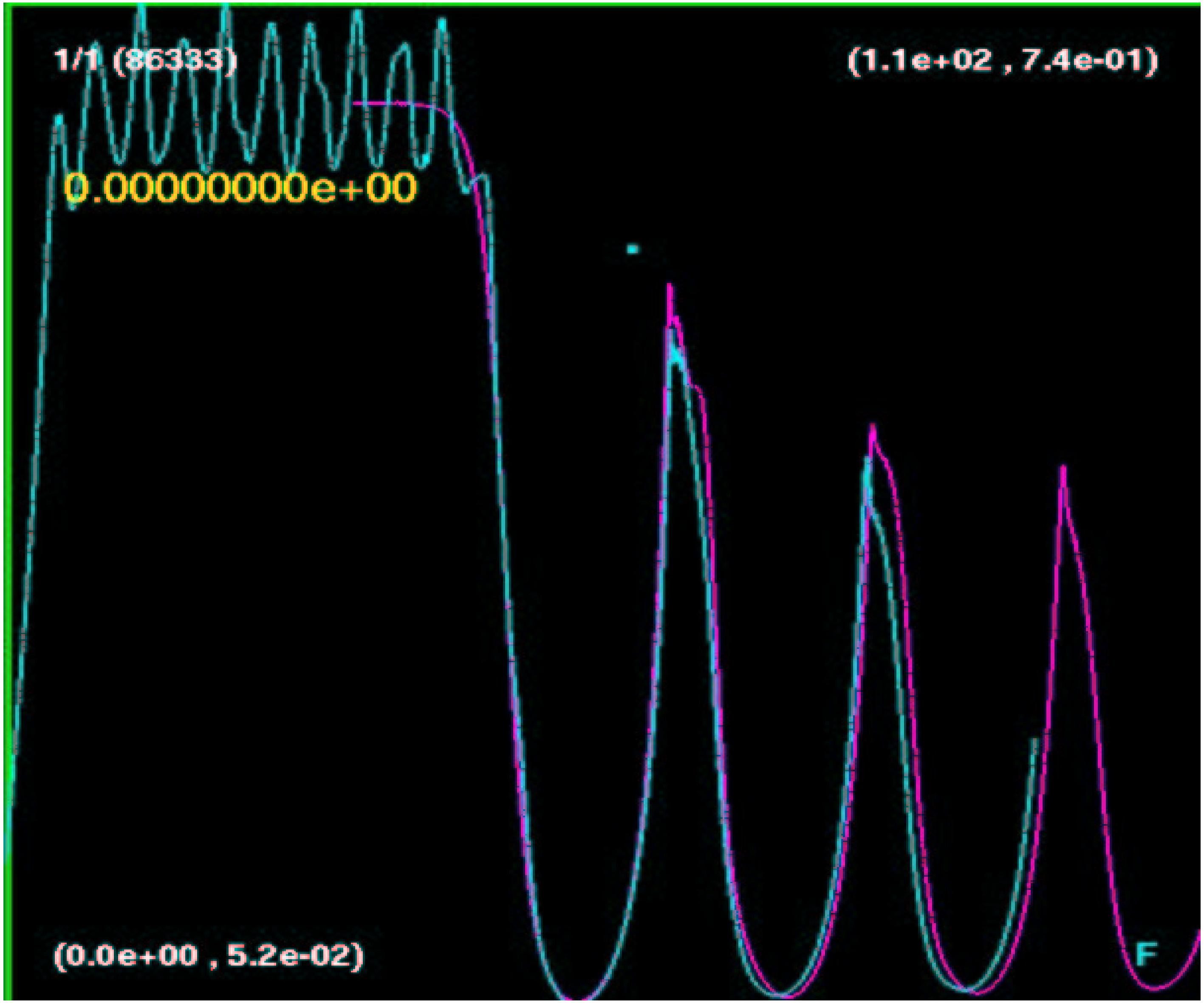
Tuning Type-I Solution



Typical Type-I Critical Solution

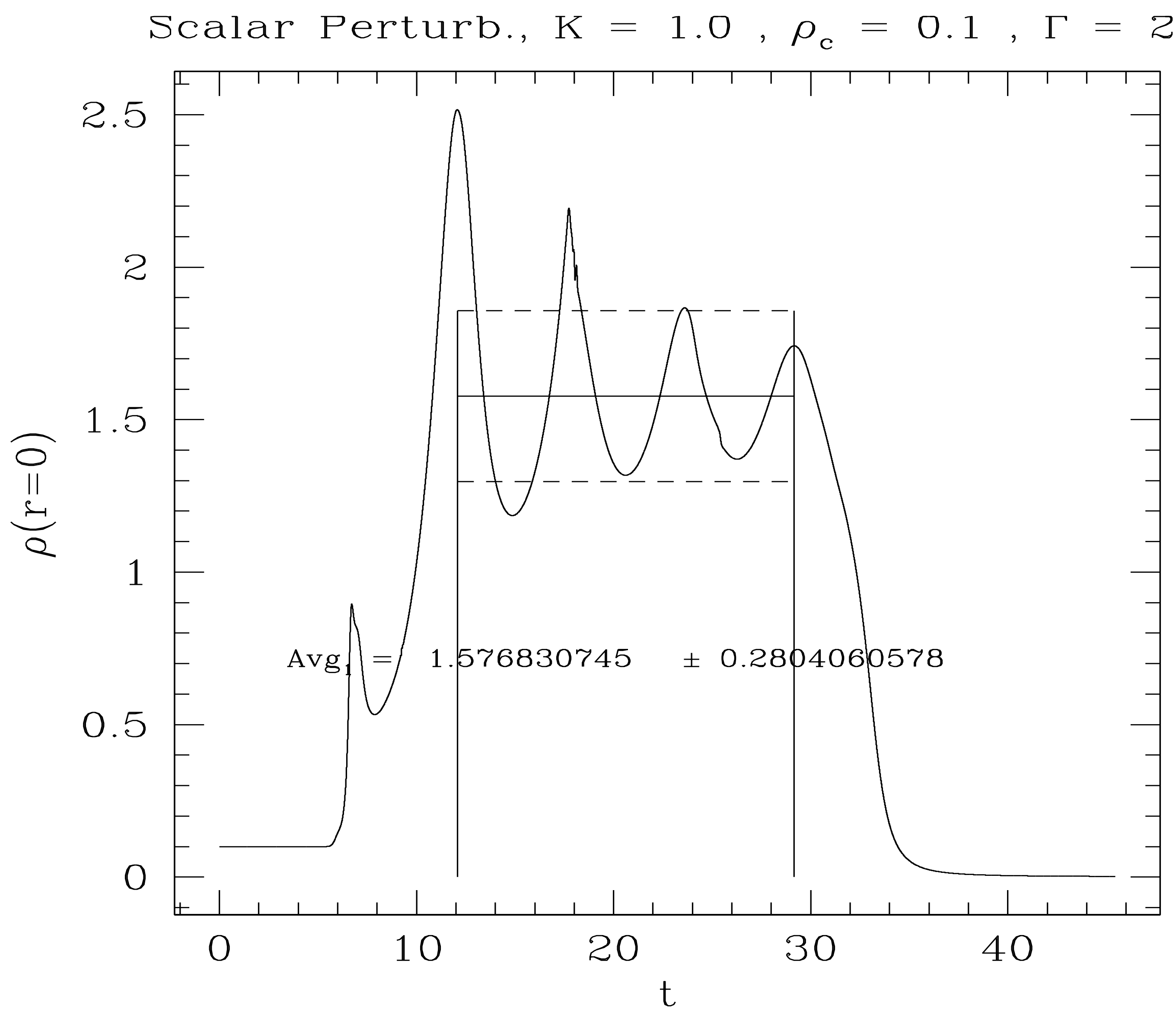


Matching with Unstable Solution

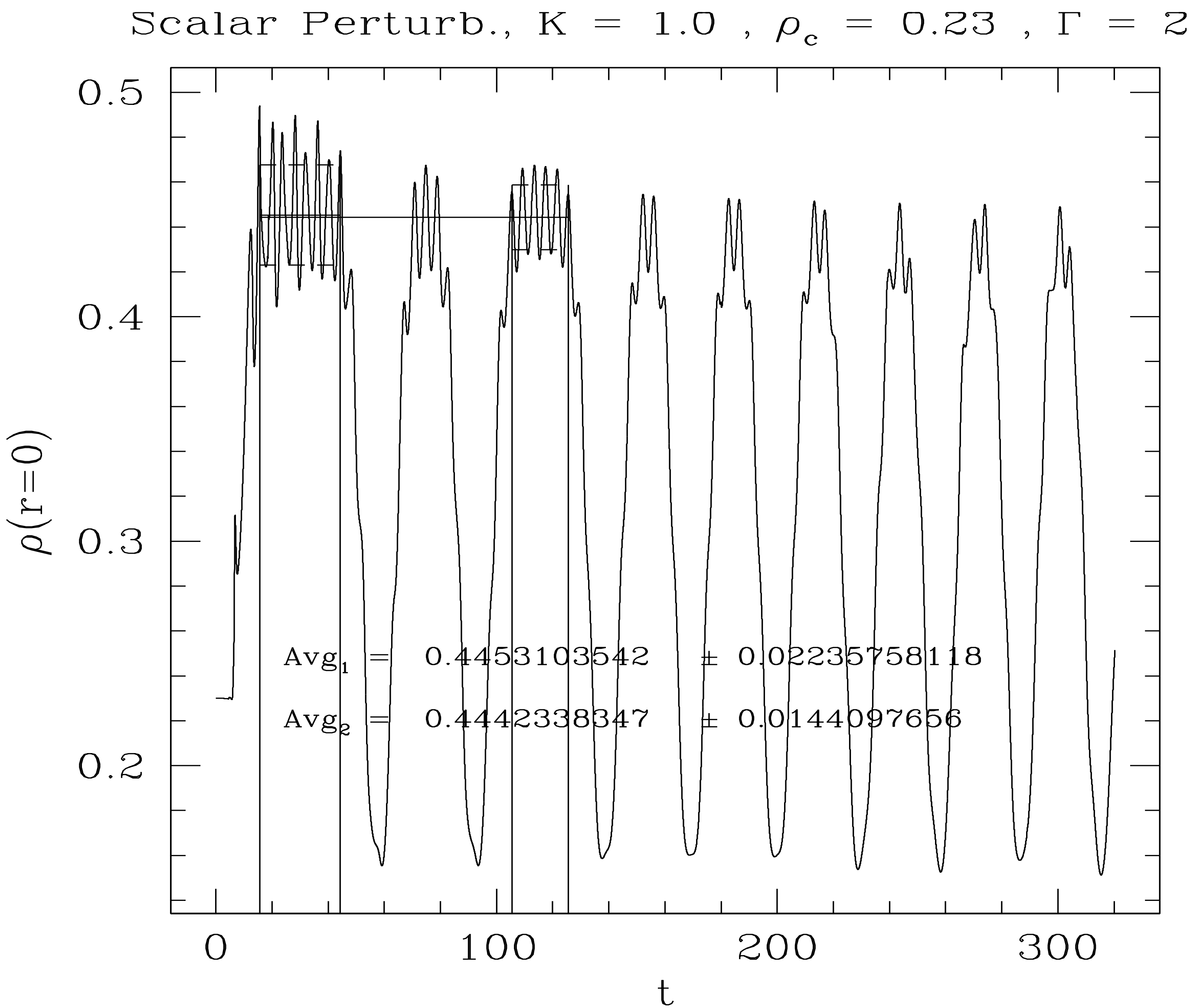


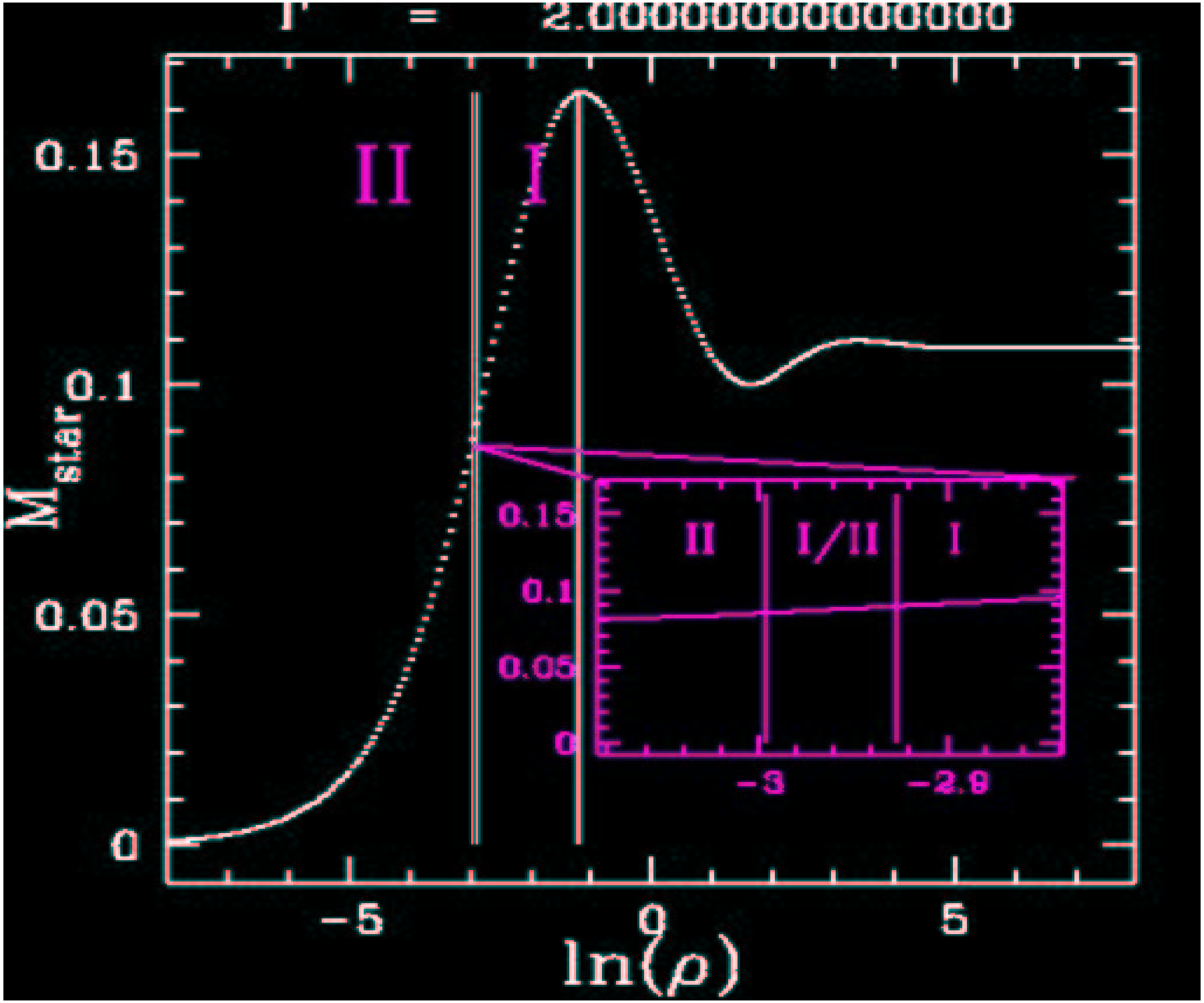
Movie #1 : `NearCrit.mpg`

Far from Turn-over Point, Dispersal/BH

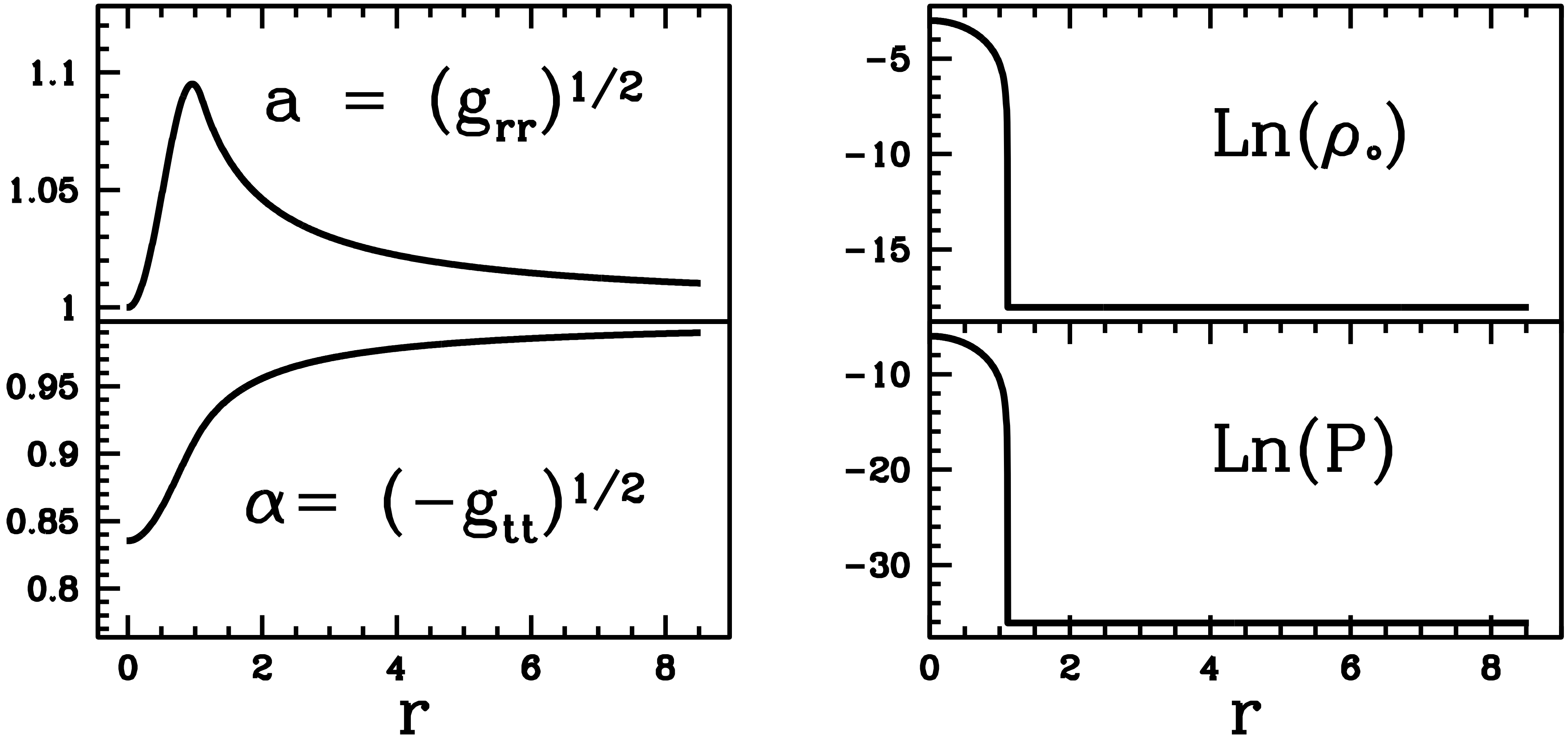


Crit. Sol'n Nearly Stable Near Turn-over Poin

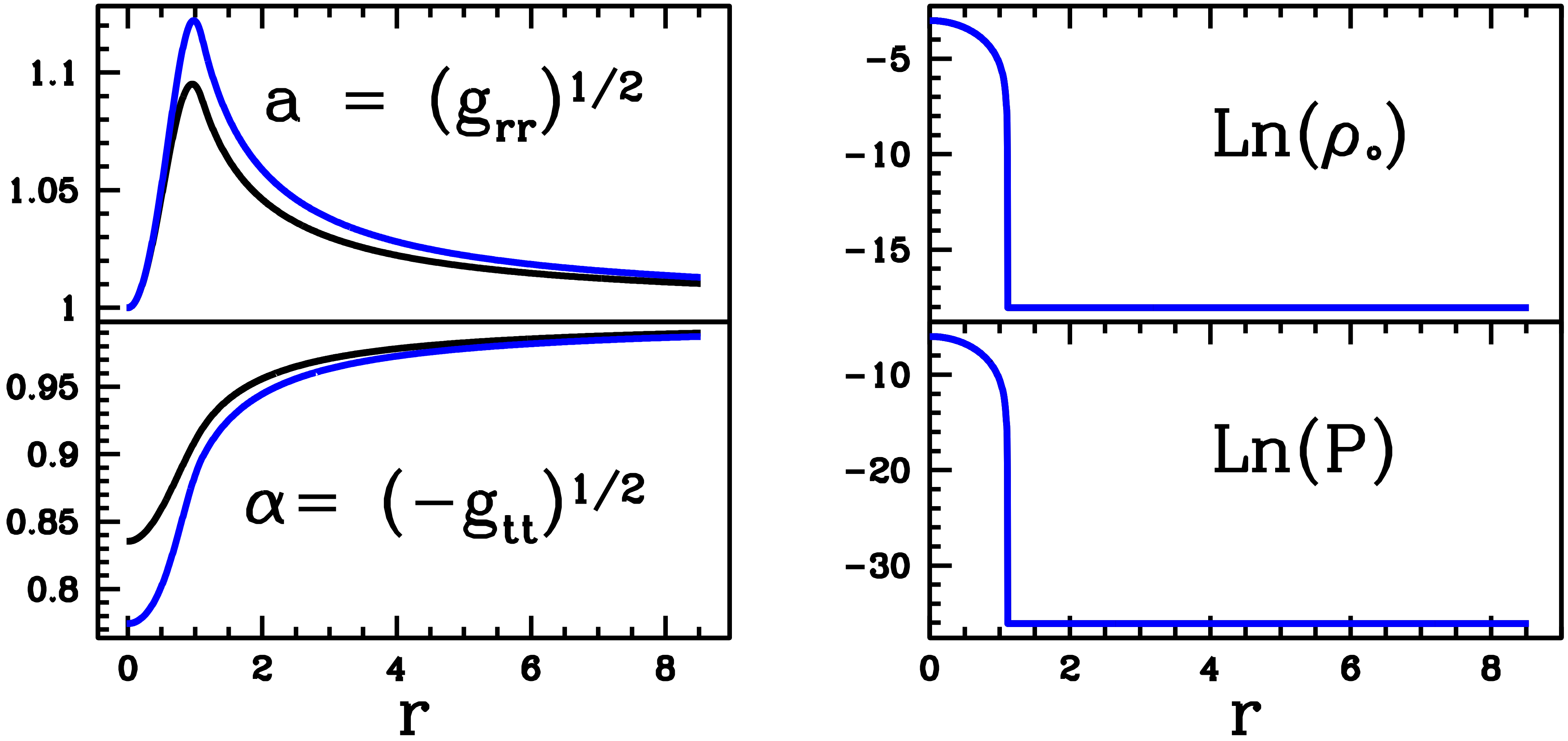



Γ = 2.00000000000000
0.15
0.1
0.05
0
M_{star}
II
I
-5
0
5
ln(ρ)
0.15
0.1
0.05
0
II
I/II
I
-3
-2.9

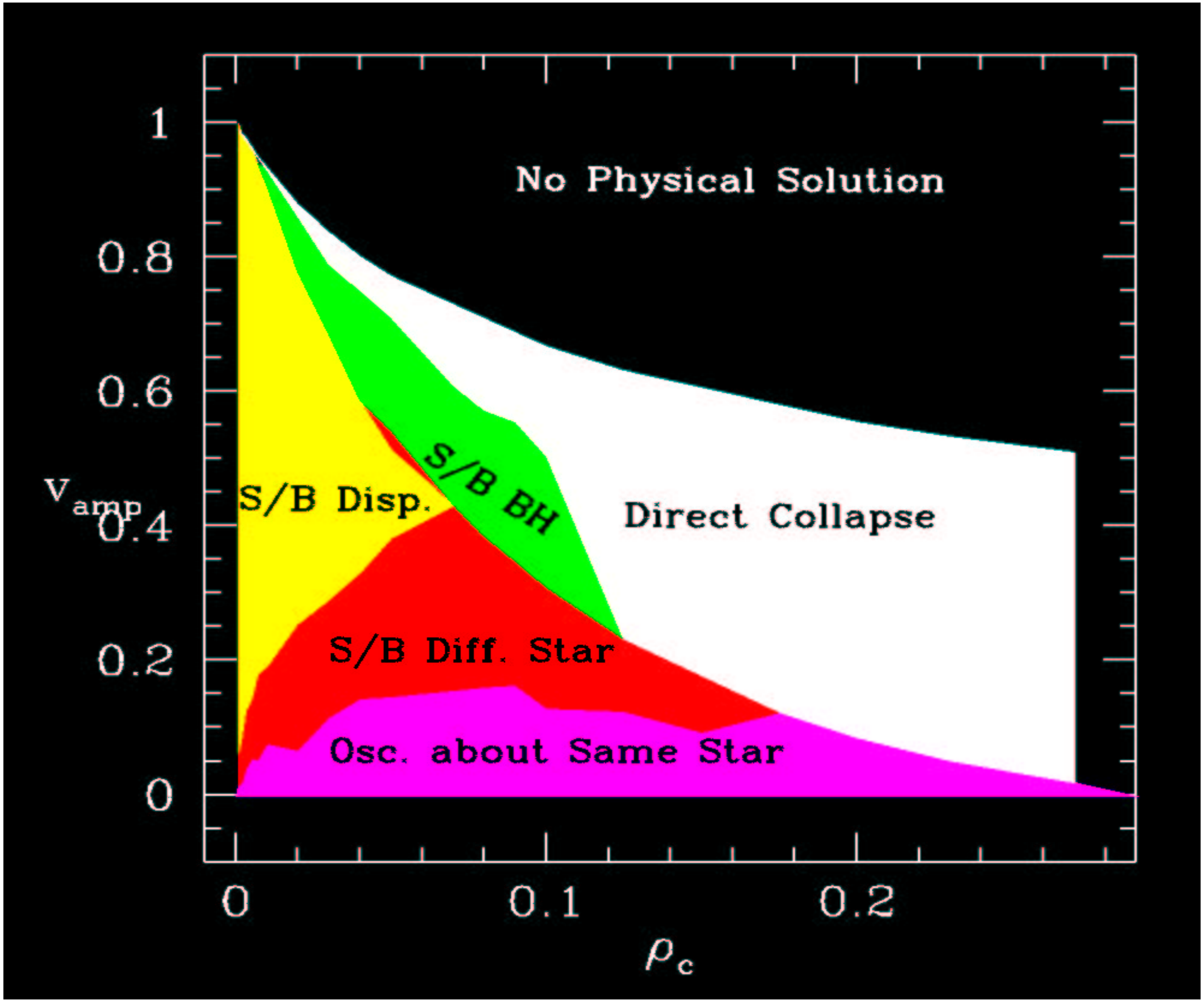
Initial Data : TOV Solution



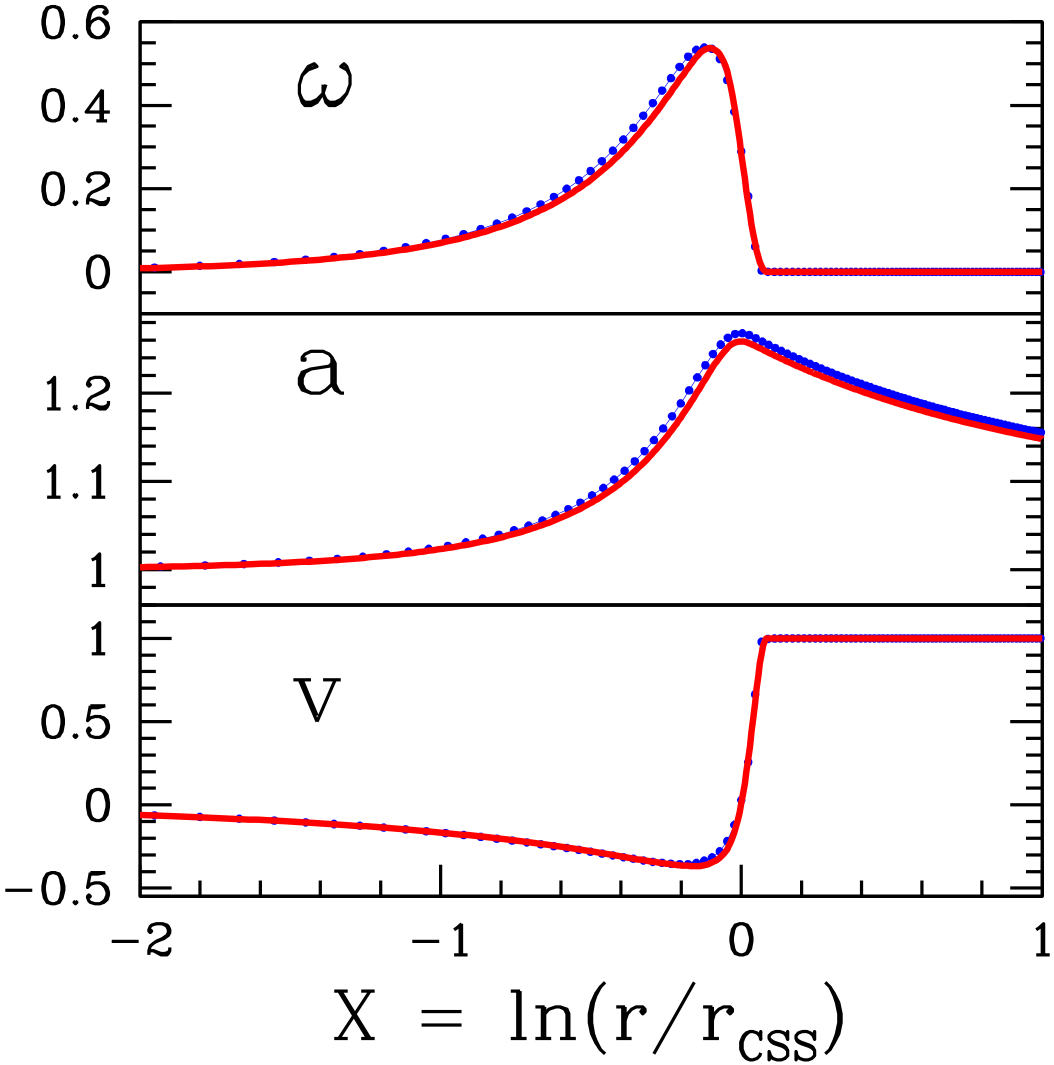
Initial Data : TOV + In-going Velocity



Parameter Space of Final States



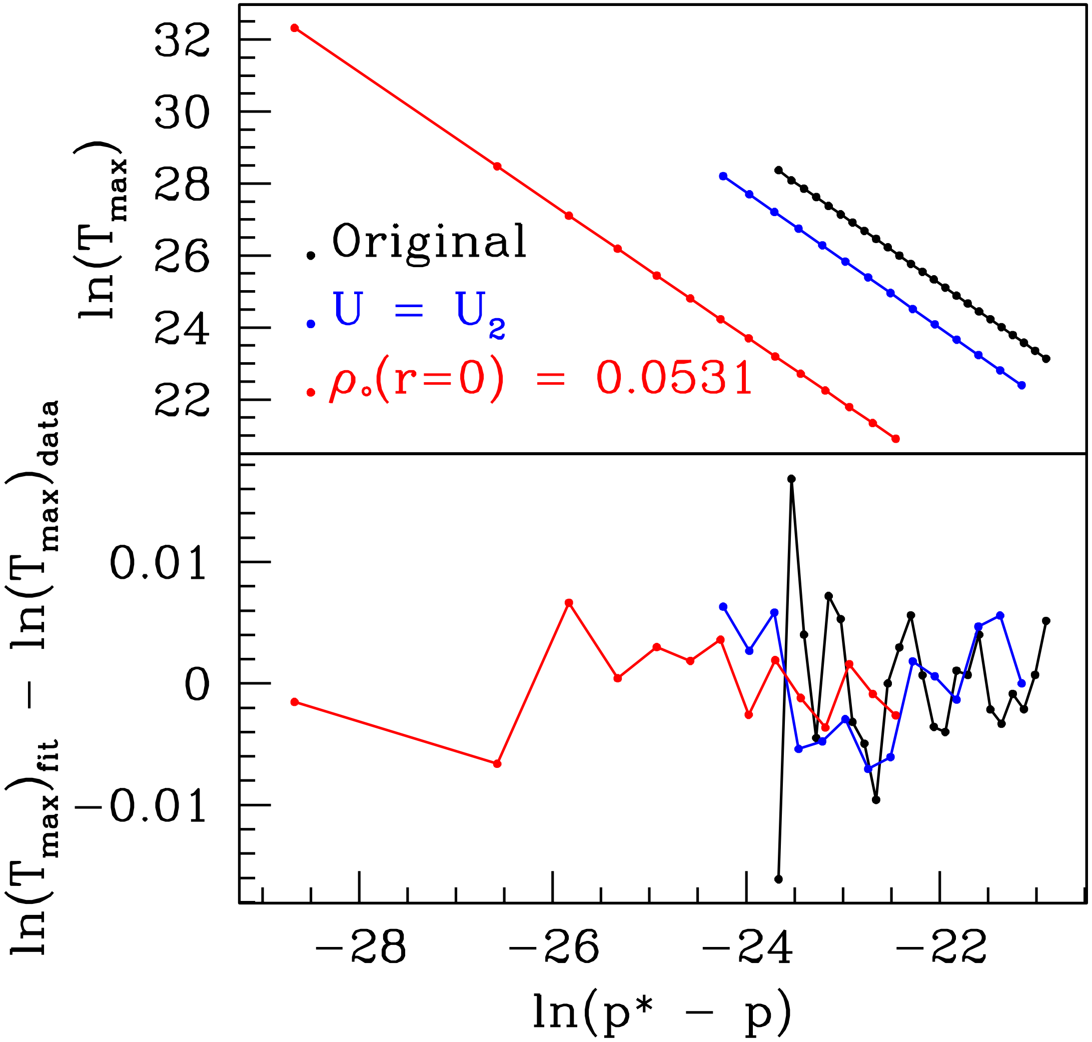
CSS Solutions of Ideal-gas and Ultra-rel.



- Comparison of dimensionless quantities:
  - $\omega \equiv 4\pi r^2 a^2 \rho$
  - $a = \sqrt{g_{rr}}$
  - $v = \frac{au^r}{\alpha u^t}$ = Eulerian Velocity

    (u^μ = Fluid's 4-velocity)
- Star parameters at $t = 0$:
  - $\rho_\circ \left(r = 0\right) = 0.05$
  - $P = \rho_\circ^2 \ , \ \epsilon = P/\rho_\circ$

Scaling of $T_{\max}$: Different "Families"



γ	p^*
0.94272	0.46875367383
0.94234392	0.42990315097
0.918693	0.4482047429836

- Suggests scaling is fairly independent of:
  - Functional form of perturbation;
  - Initial star configuration;

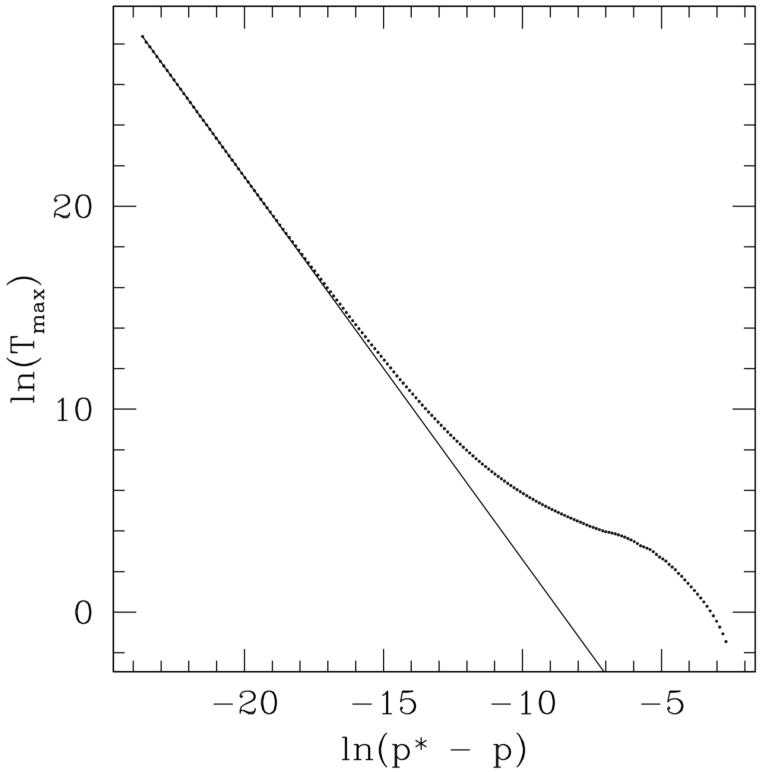
Scaling Parameters

Test Type	ρ_c	Floor	Δr	U	γ	p^*
-	0.05	2.5×10^{-19}	$4h$	U_1	0.94272	0.46875367383
Floor	0.05	2.5×10^{-17}	$4h$	U_1	0.94358	0.46875350285
Floor	0.05	2.5×10^{-15}	$4h$	U_1	0.9469707	0.4687516089
Family	0.05	2.5×10^{-19}	$4h$	U_2	0.94234392	0.42990315097
Family	0.0531	2.5×10^{-19}	$4h$	U_1	0.918693	0.4482047429836

- Our average : $\gamma \;=\; 0.94 \pm 0.01$
- Brady et al. (2002) (averaged over diff. methods): $\gamma = 0.95 \pm 0.02$

Type-II Conclusions

- (Ideal-gas Type-II Sol'n.) $\simeq$ (Ultra-rel. Type-II Sol'n.) for $\Gamma = 2$
- $\gamma_{\rm ideal} \simeq \gamma_{\rm ultra-rel.}$
- Novak (2001) did not sufficiently tune toward p^*



Conclusions

- Type-I and Type-II Phen. observed in TOV solutions;
- Some evidence for overlap in param. space of Type-I/Type-II;
- Type-I Crit. Sol'n = Unstable Sol'n w/ same mass;
- Type-II Crit. Sol'n = Crit. Sol'n of Self-similar EOS fluid;