Critical Phenomenon in Perturbed Neutron Star Models

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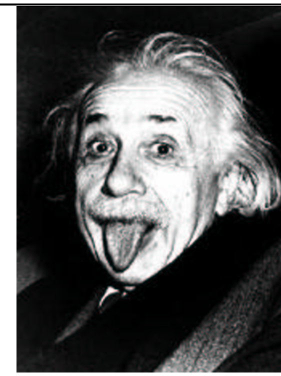
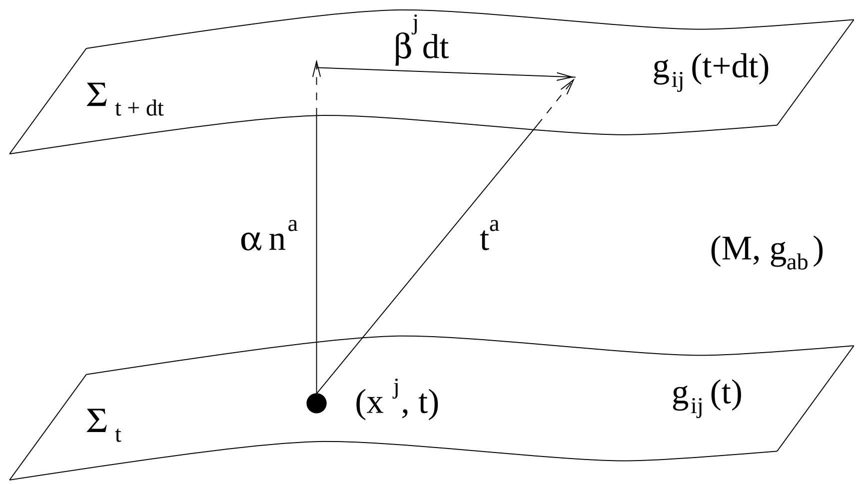
Outline

- Introduction to General Relativity (GR)
- General Relativistic Hydrodynamics and its Numerical Methods
- Neutron Stars
- Introduction to Critical Phenomena in GR
- Results
- Conclusion

Newton	Einstein
	
Flat, Cartesian Geometry	Curvey Geometry
Space & Time distinct	Space $\leftrightarrow$ Time
Δt , $\Delta \vec{\mathbf{r}}^2 =$ constants	$\Delta s^2 = \sum_{\mu,\nu} g_{\mu\nu} \Delta x^\mu \Delta x^\nu =$ constant
Instantaneous Forces, Signals	Universe's speed limit $= c$
$\vec{F}_{\text{Gravity}} = -m \vec{\nabla} \phi$	No $\vec{F}_{\text{Gravity}}$, just free-fall!
$\vec{F}_{\text{Gravity}} = m\vec{a}$	$G_{\mu\nu} = 8\pi T_{\mu\nu}$
Simple, linear PDE's	Difficult, many nonlinear PDE's, in general

The Arnowitt-Deser-Misner (ADM) 3+1 Formalism



$$ds^2 = \left(-\alpha^2 + \beta^i \beta_i\right) dt^2 + 2\beta_i dt dx^i + g_{ij} dx^i dx^j$$

- α, **lapse**, relates coord. time to proper time
- β^j, **shift**, how x^j translates between slices;
- n^a, time-like unit normal vector to Σ
- g_{ij}, spatial metric on Σ
- $t^a = \alpha n^a + \beta^a$, time-like tangent to coordinate's world line

ADM continued

- Spacetime $\rightarrow$ Space + Time (3+1);

- Provides "time direction" through slices $\rightarrow$ Initial Value Problem;

- $g_{ij}(t, x^k)$ and $K_{ij}(t, x^k)$ are the dynamical variables

  where K_{ij} is a conjugate momentum to g_{ij}

- $g_{\mu\nu} = 10$ "fields", 4 Coordinate Conditions, 4 Constraints

- 10 - 4 - 4 = 2 Dynamical degrees of freedom in EQ's
  $\rightarrow$ Gravitational Waves!!

Fluid and Geometry Equations

Metric:

$$ds^2 = -\alpha\,(r,t)^2\,dt^2 + a\,(r,t)^2\,dr^2 + r^2\left(d\theta^2 + sin^2\theta d\phi^2\right)$$

Stress-Energy Tensor:

$$T_{ab} \;=\; (\rho_\circ + \rho_\circ\epsilon + P)\,u_a u_b + P g_{ab}$$

Equation of State

$$P = (\Gamma - 1)\rho_\circ\epsilon$$

$$D = a\rho_\circ W \ , \quad S = (\rho + P)\, W^2 v \ , \quad \tau = S/v - D - P$$

$$v = \frac{au^r}{\alpha u^t} \quad , \quad W^2 = \frac{1}{1 - v^2}$$

Slicing Condition :

$$\frac{\alpha'}{\alpha} = a^2 \left[4\pi r\, (Sv + P) + \frac{1}{2r} \left(1 - 1/a^2 \right) \right]$$

Hamiltonian Constraint :

$$\frac{a'}{a} = a^2 \left[4\pi r\, (\tau + D) - \frac{1}{2r} \left(1 - 1/a^2 \right) \right]$$

g_{rr} **Evolution :**

$$\dot{a} = -4\pi r \alpha a^2 S$$

Equations of Motion:

$$\nabla_\mu T^\mu_\nu = 0 \quad , \quad \nabla_\mu J^\mu_\nu = 0$$

$$\boxed{\dot{\mathbf{q}} + \tfrac{1}{r^2}\left(r^2 \tfrac{\alpha}{a}\mathbf{f}\right)' = \boldsymbol{\psi}}$$

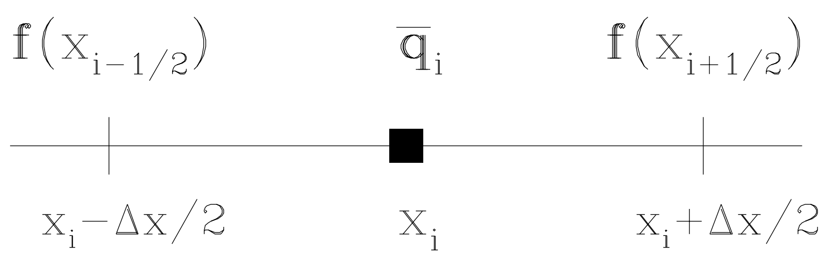
$$\mathbf{q} = \begin{bmatrix} D \\ S \\ \tau \end{bmatrix} \;,\; \mathbf{f} = \begin{bmatrix} Dv \\ Sv + P \\ v\,(\tau + P) \end{bmatrix} \;,\; \boldsymbol{\psi} = \begin{bmatrix} 0 \\ \Sigma \\ 0 \end{bmatrix} \;,\; \mathbf{w} = \begin{bmatrix} \rho_\circ \\ v \\ P \end{bmatrix}$$

$$\Sigma \equiv \Theta + \frac{2P\alpha}{ra}$$

$$\Theta \equiv \alpha a \left[(Sv - \tau - D)\left(8\pi r P + \frac{1}{2r}\left(1 - 1/a^2\right)\right) + \frac{P}{2r}\left(1 - 1/a^2\right)\right]$$

Finite Volume Method

- Naturally Conservative and ensures weak solutions are found;
- Can handle discontinuities (shocks);
- Accurately handle highly-relativistic flows;



Take $\partial_t \mathbf{q} + \partial_x \mathbf{f} = 0$

- $\int_{-\infty}^{\infty} q(x,t)dx =$ a conserved quantity
- $f(x,t) = f(q(x,t))$ flux of q at x, t
- Cells $\mathcal{C}_i$ centered at x_i with boundaries at $x_i - \Delta x/2$, $x_i + \Delta x/2$.

$$\bar{q}_i(t) = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)dx$$

$$\frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)dx = f\left(q(x_{i-1/2},t)\right) - f\left(q(x_{i+1/2},t)\right)$$

$$\Delta x\left(\bar{q}_i(t_2) - \bar{q}_i(t_1)\right) = \int_{t_1}^{t_2} f\left(q(x_{i-1/2},t)\right)dt - \int_{t_1}^{t_2} f\left(q(x_{i+1/2},t)\right)dt$$

- Define Numerical Flux, F, as "time average" of f:

$$F_i^n = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f\left(q(x_i,t)\right)dt$$

$$\bar{q}_i^{n+1} = \bar{q}_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

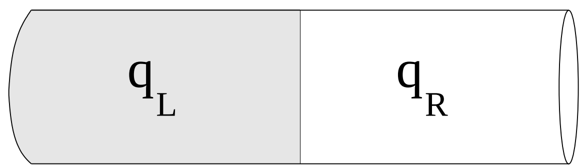
where $t^n = (n-1)\Delta t$;

Riemann Problem

- Find $\mathbf{q}(x,t)$ given

$$\partial_t \mathbf{q} + \partial_x \mathbf{f} = 0$$

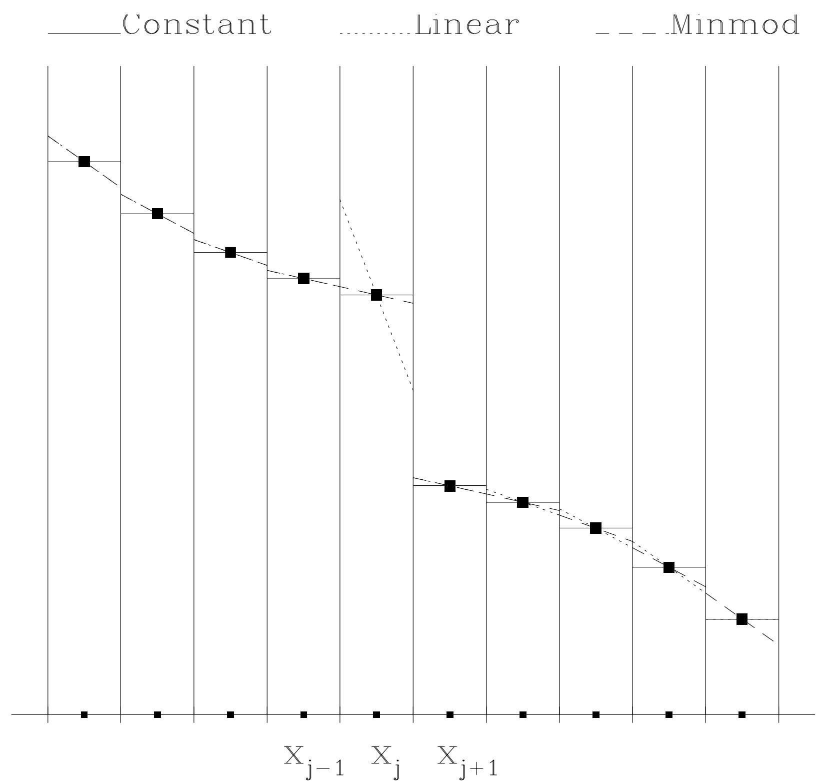
$$\mathbf{q}(x, t=0) \;=\; \begin{cases} \mathbf{q}_L & \text{for } x < 0 \\ \mathbf{q}_R & \text{for } x > 0 \end{cases}$$



- Solution via characteristics of quasi-linear form

$$\mathbf{q}_t + \mathbf{A}\mathbf{q}_x = 0 \quad , \quad \mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{q}}$$

and via Rankine-Hugoniot jump conditions,
which tell how discontinuities propagate.

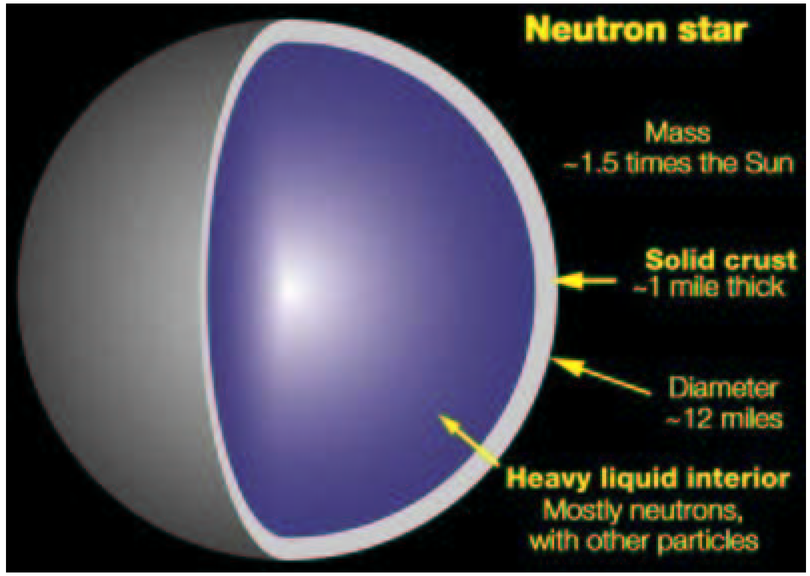


At $x = x_{j+1/2}$	Const.	Linear	Minmod
$\tilde{\mathbf{q}}_L =$	$\bar{\mathbf{q}}_j$	$\bar{\mathbf{q}}_j + \mathbf{s}_{j+1/2}\Delta x/2$	$\bar{\mathbf{q}}_j + \mathsf{M}(\mathbf{s}_{j+1/2}, \mathbf{s}_{j-1/2})\Delta x/2$
$\tilde{\mathbf{q}}_R =$	$\bar{\mathbf{q}}_{j+1}$	$\bar{\mathbf{q}}_{j+1} - \mathbf{s}_{j+3/2}\Delta x/2$	$\bar{\mathbf{q}}_{j+1} - \mathsf{M}(\mathbf{s}_{j+3/2}, \mathbf{s}_{j+1/2})\Delta x/2$

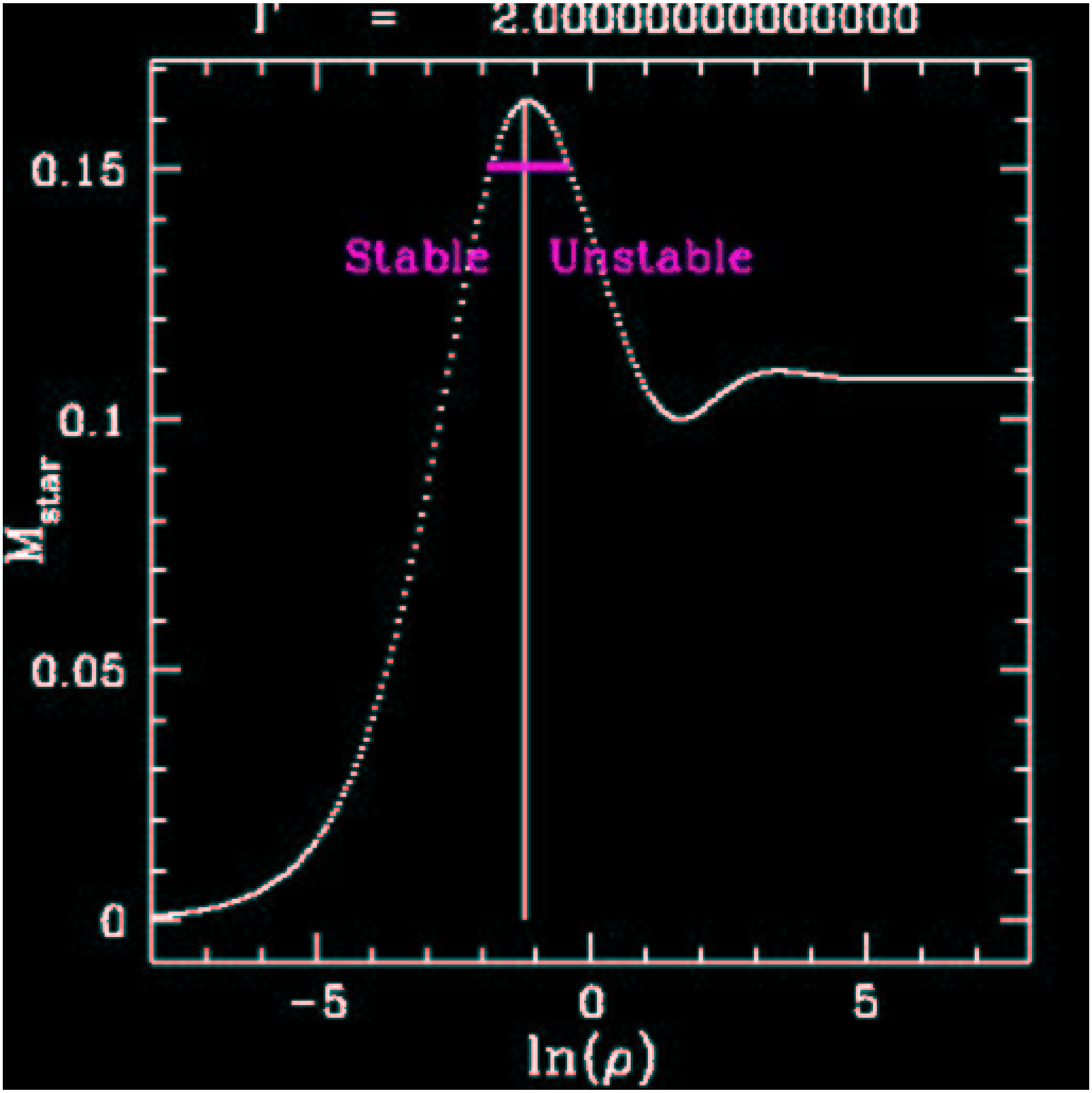
$$\mathbf{s}_{j+1/2} \equiv \left(\bar{\mathbf{q}}_{j+1} - \bar{\mathbf{q}}_j\right)/\Delta x$$

$$\mathsf{M}(a,b) = \mathsf{Minmod}(a,b) = \begin{cases} 0 & \text{if} \quad ab < 0 \\ a & \text{if} \quad |a| < |b| \quad \text{and} \quad ab > 0 \\ b & \text{if} \quad |b| < |a| \quad \text{and} \quad ab > 0 \end{cases}$$

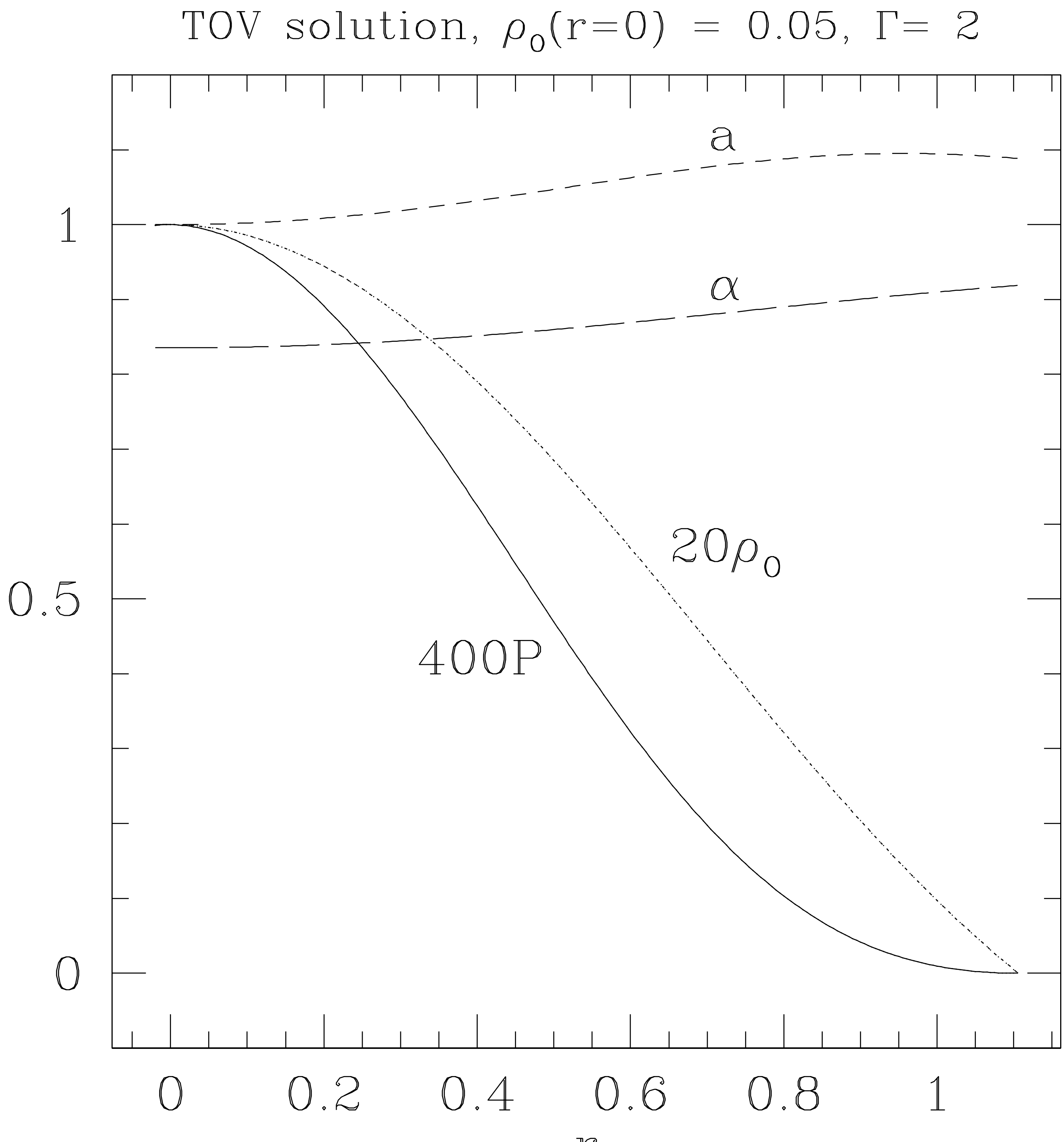
Neutron Stars



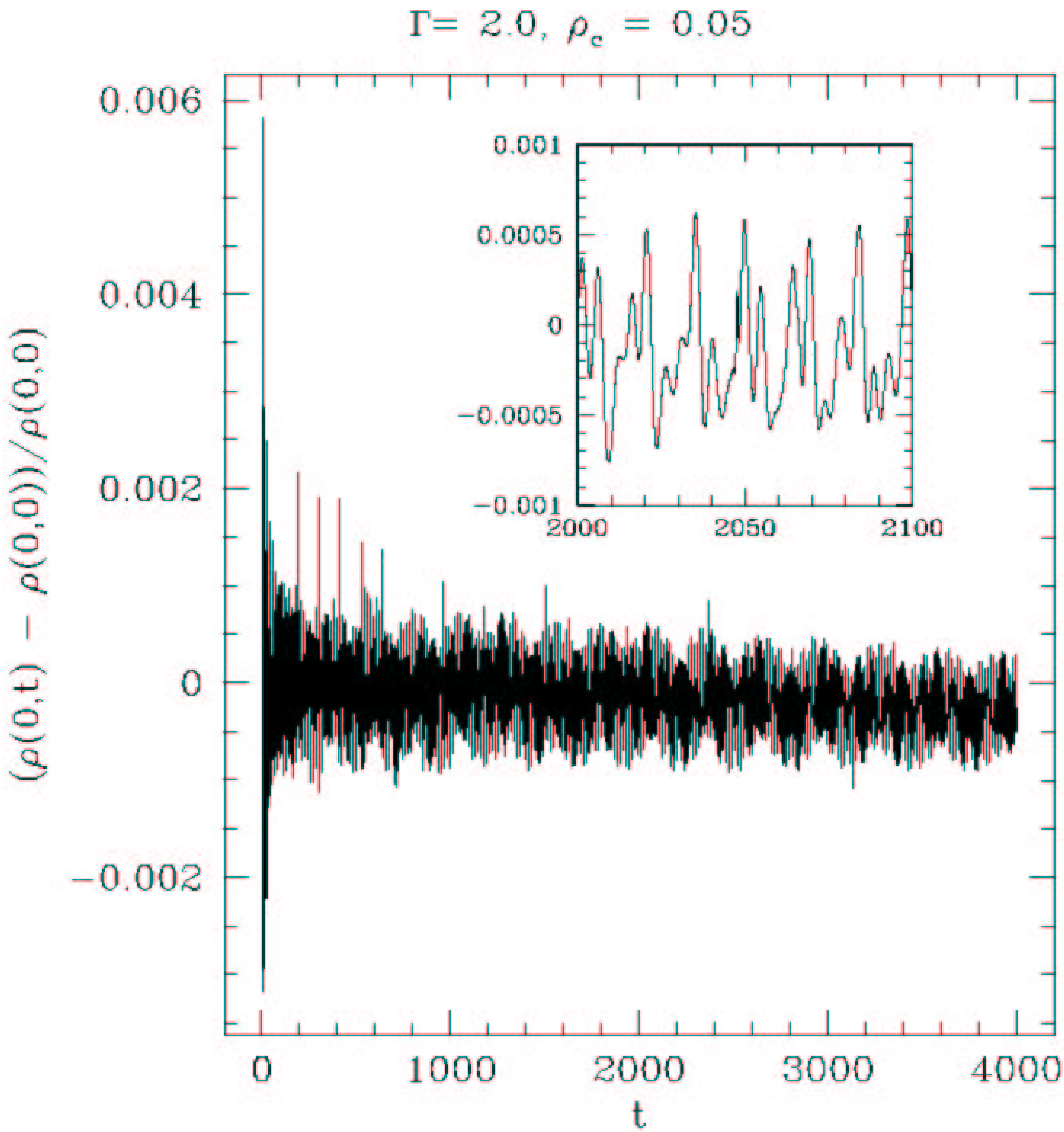
- NS = Big "Nucleus", Normal matter mostly vacuum;
- Most compact matter (non-BH) object;
- $\rho_{Avg} = 2 \times 10^{18}\ kg/m^3$, $\Rightarrow\ 1km^3$ of NS $= M_{\text{Earth}}$
  $\Rightarrow$ Need GR to properly describe;
- Model as static, spherical solution to $G_{\mu\nu} = 8\pi T_{\mu\nu}$
  $\Rightarrow$ Tolman-Oppenheimer-Volkoff equations.



Γ = 2.0000000000000
0.15
0.1
0.05
0
Stable
Unstable
M_star
-5
0
5
ln(ρ)



TOV solution, $\rho_0(r=0)$ = 0.05, Γ= 2
a
α
$20\rho_0$
400P
1
0.5
0
0
0.2
0.4
0.6
0.8
1
r



Γ= 2.0, ρ_c = 0.05
0.006
0.004
0.002
0
−0.002
(ρ(0,t) − ρ(0,0))/ρ(0,0)
0
1000
2000
3000
4000
t
0.001
0.0005
0
−0.0005
−0.001
2000
2050
2100

A Decade of Critical Phenomena

- M. W. Choptuik "Universality and Scaling in Gravitational Collapse of a Massless Scalar Field", PRL **70**, 1, January 4, 1993.

- Crit. Phen. observed anywhere you have (BH)/(No BH);

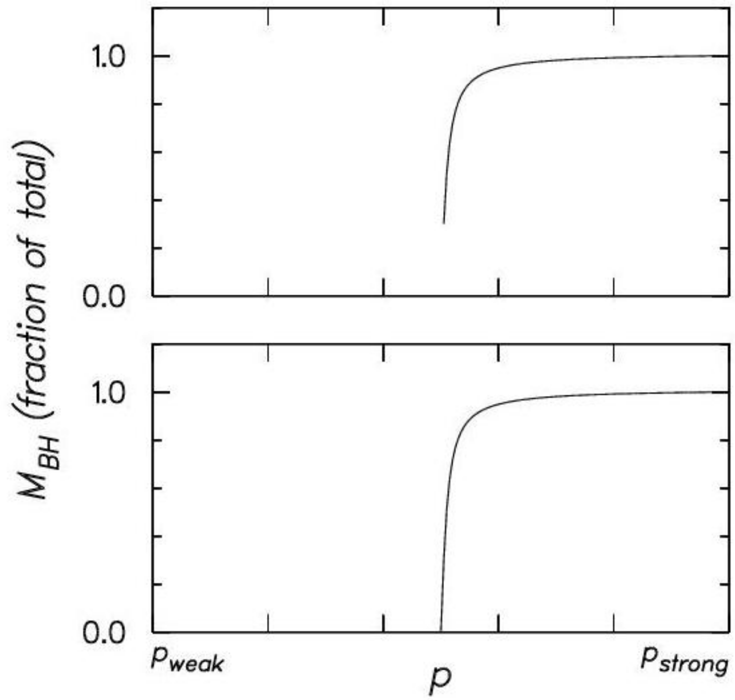
- "Tuning" of initial data to Critical Solution
  $\longrightarrow$ eliminate the 1 unstable mode from solution;

- General feature of gravitational collapse, observed in many different matter models (even w/o matter!)

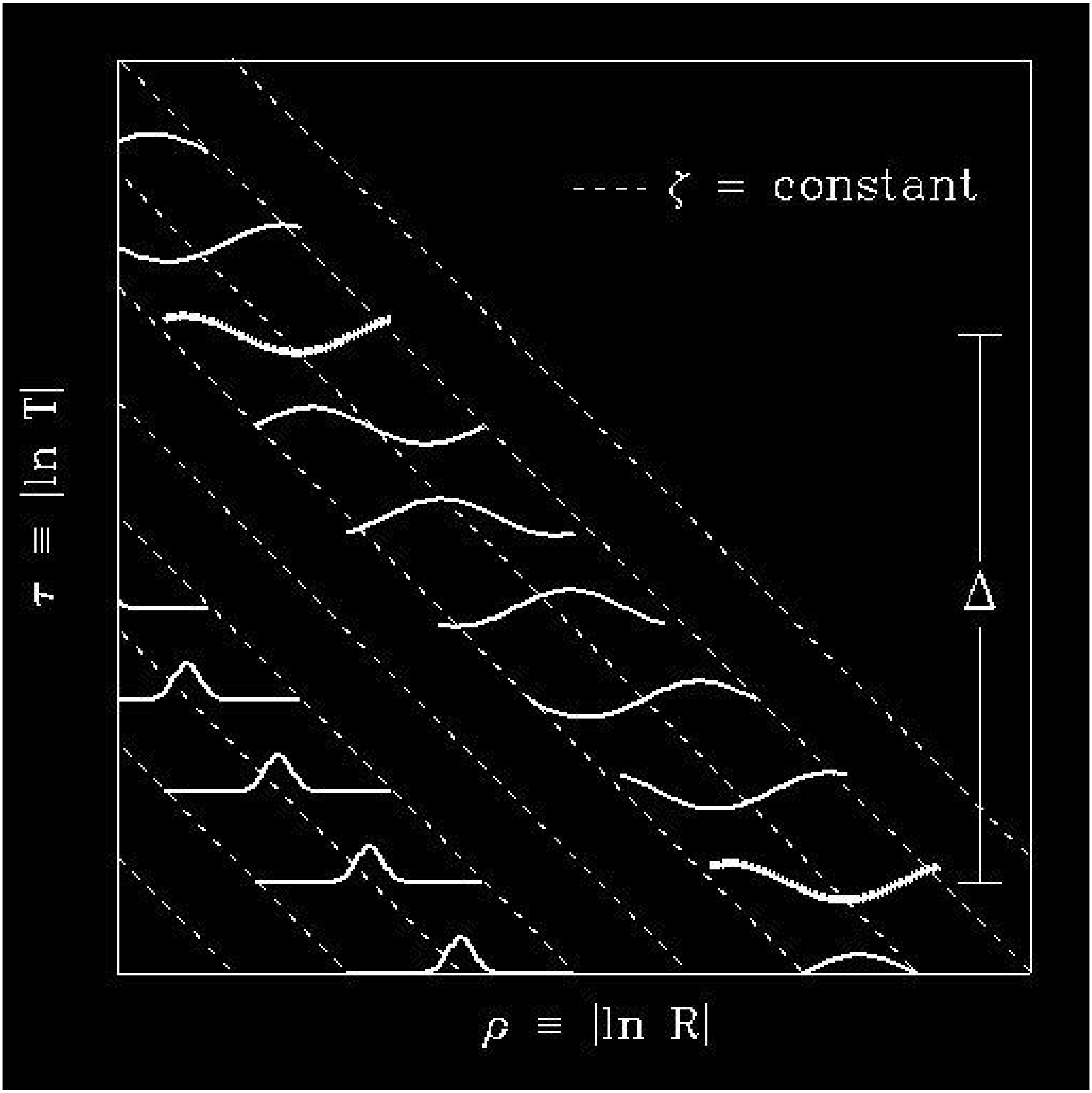
- Some Crit. Solutions are "Naked Singularities"!

Table 1: Critical collapse in spherical symmetry

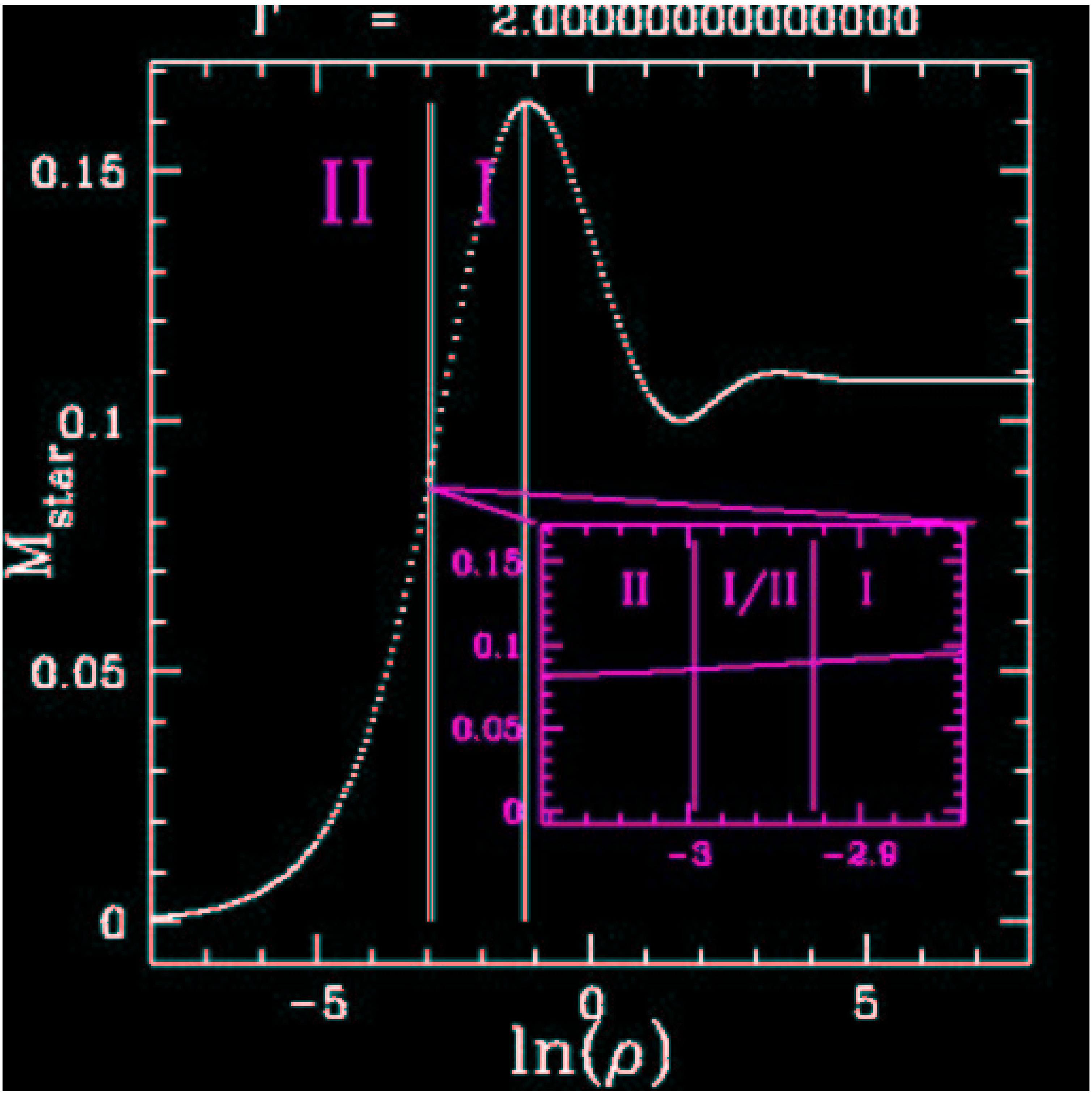
Matter	Type	Collapse simulations	Critical solution	Perturbations of crit. soln.
Perfect fluid $p = k\rho$	II	[69, 142]	CSS [69, 138, 142]	[138, 128, 93, 97]
Real scalar field:				
– massless, min. coupled	II	[47, 48, 49]	DSS [89]	[90, 139]
– massive	I	[32]	oscillating [165]	
	II	[49]	DSS [104, 99]	[104, 99]
– conformally coupled	II	[48]	DSS	
– 4+1	II	[16]		
– 5+1	II	[77]		
Massive complex scalar field	I (II)	[110]	[165]	[110]
Massless scalar electrodynamics	II	[117]	DSS [99]	[99]
2-d sigma model				
– complex scalar ($\kappa = 0$)	II	[50]	DSS [90]	[90]
– axion-dilaton ($\kappa = 1$)	II	[101]	CSS [67, 101]	[101]
– scalar-Brans-Dicke ($\kappa > 0$)	II	[136, 133]	CSS, DSS	
– general κ including $\kappa < 0$	II		CSS, DSS [115]	[115]
$SU(2)$ Yang-Mills	I	[53]	static [12]	[131]
	II	[53]	DSS [92]	[92]
	"III"	[55]	colored BH [17, 173]	[168, 172]
$SU(2)$ Skyrme model	I	[19]	static [19]	[19]
	II	[22]	static [22]	
$SO(3)$ Mexican hat	II	[134]	DSS	
Vlasov	I?	[160, 148]	[141]	



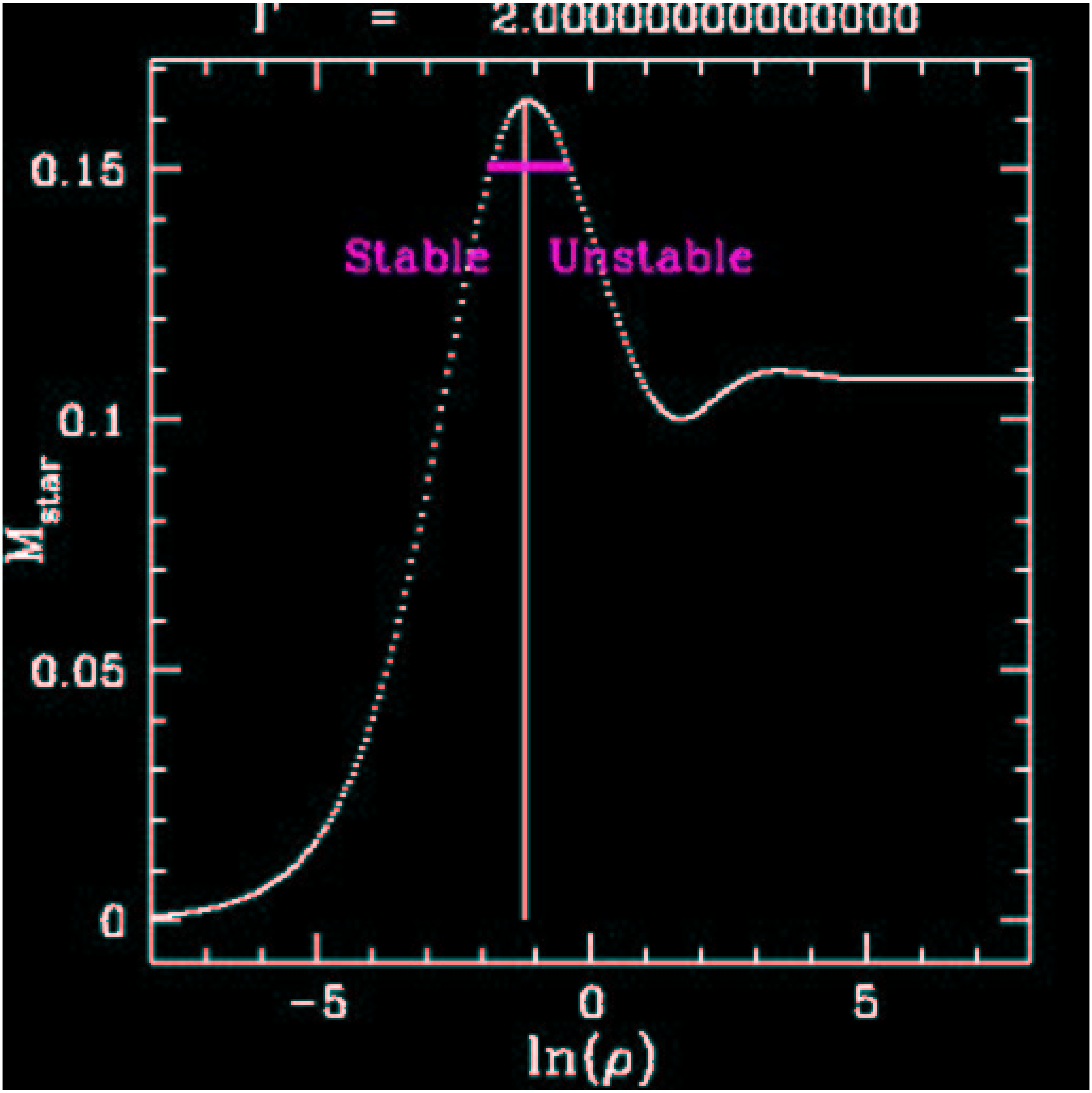
Type I	Type II
Discontinuous "Phase" Transition $M_{BH} \to M^* > 0$	Continuous "Phase" Transition $M_{BH} \to 0$
Static or Oscillatory	Cont. or Discretely Self-similar
$t_{\mathrm{hang}} \propto \lvert p - p^* \rvert^{-\gamma}$	$M_{\mathrm{BH}} \propto \lvert p - p^* \rvert^{\gamma}, T_{\mathrm{max}} \propto \lvert p - p^* \rvert^{2\gamma}$



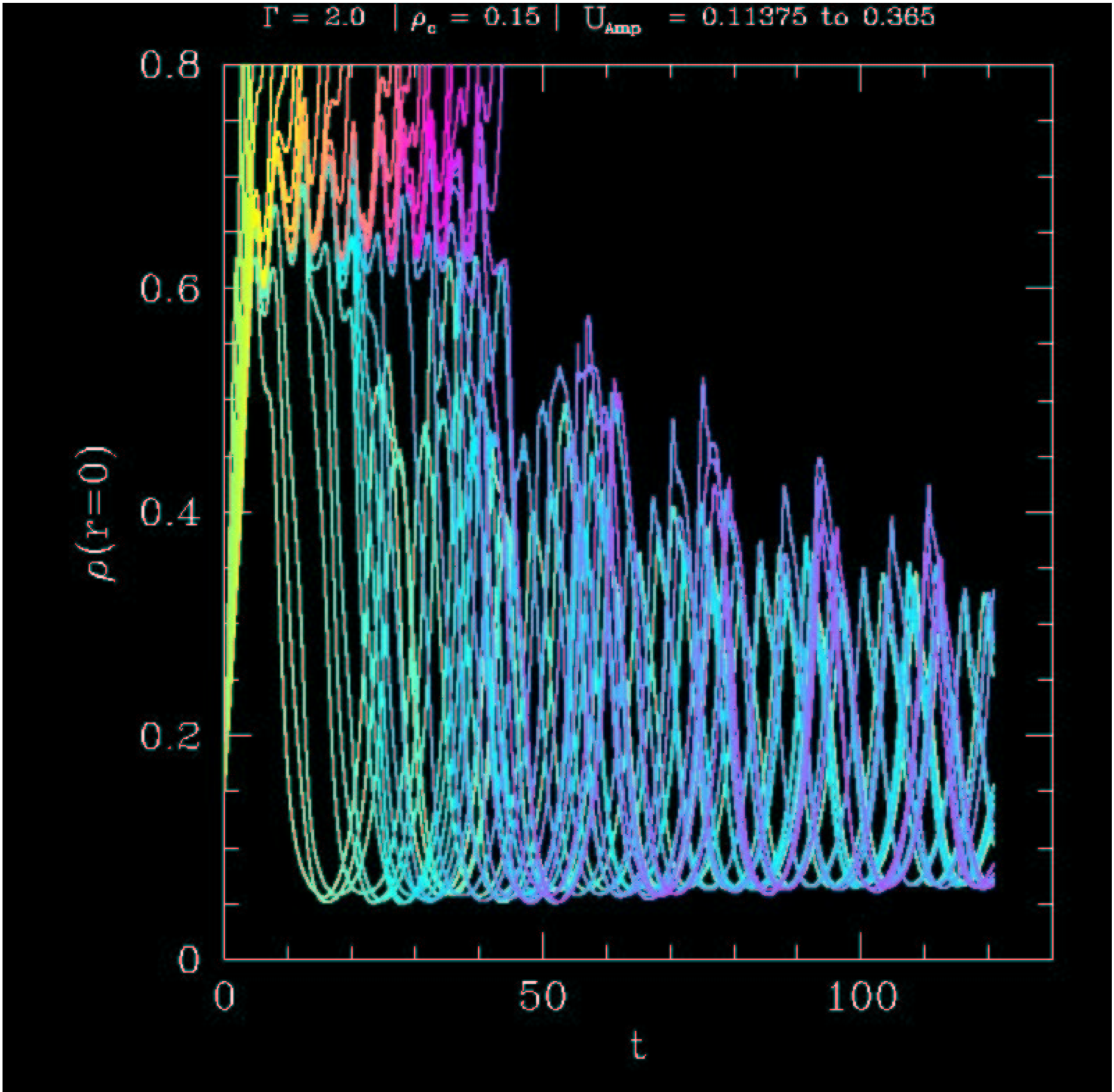
---- ζ = constant
Δ
τ ≡ |ln T|
ρ ≡ |ln R|

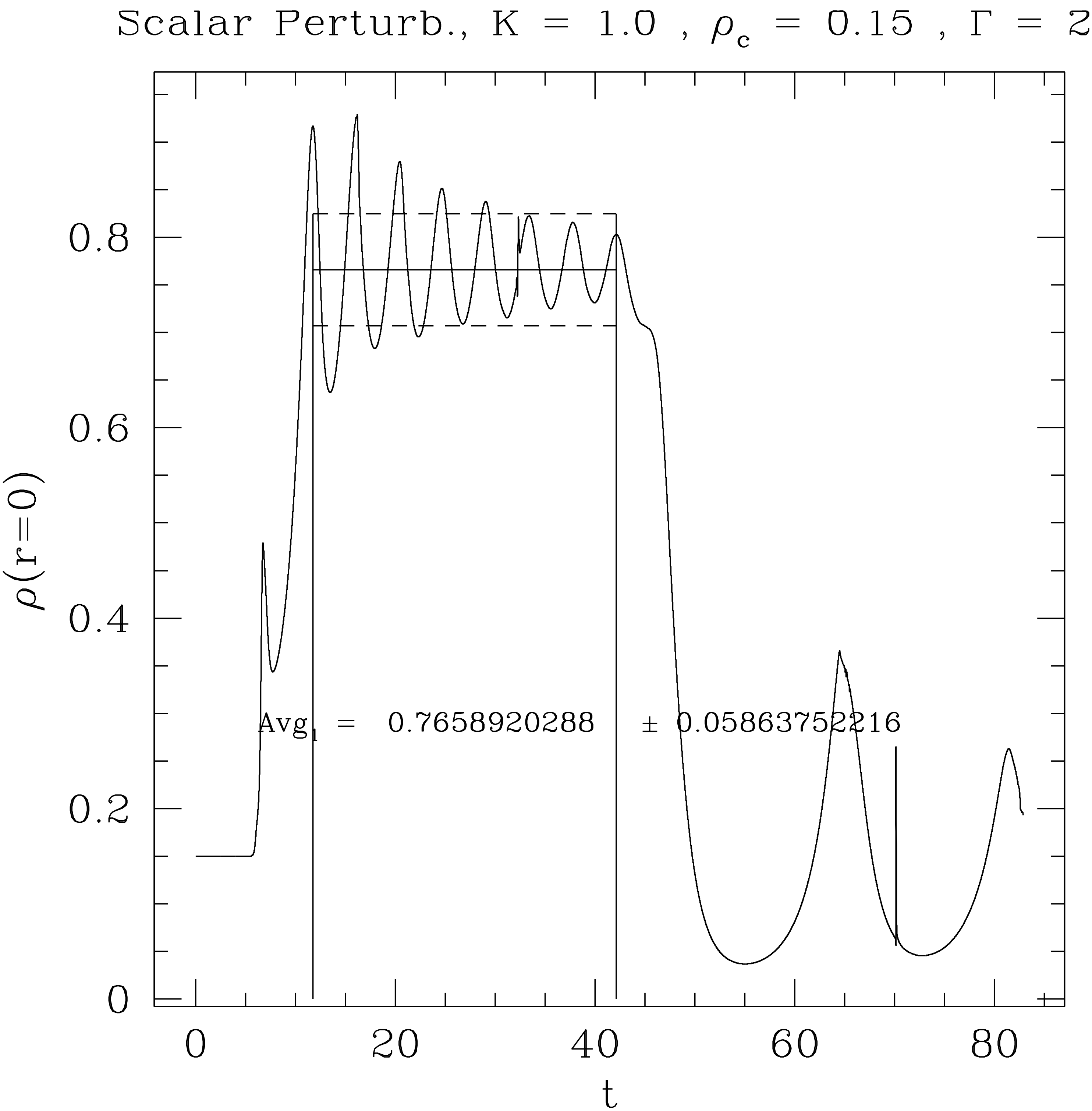


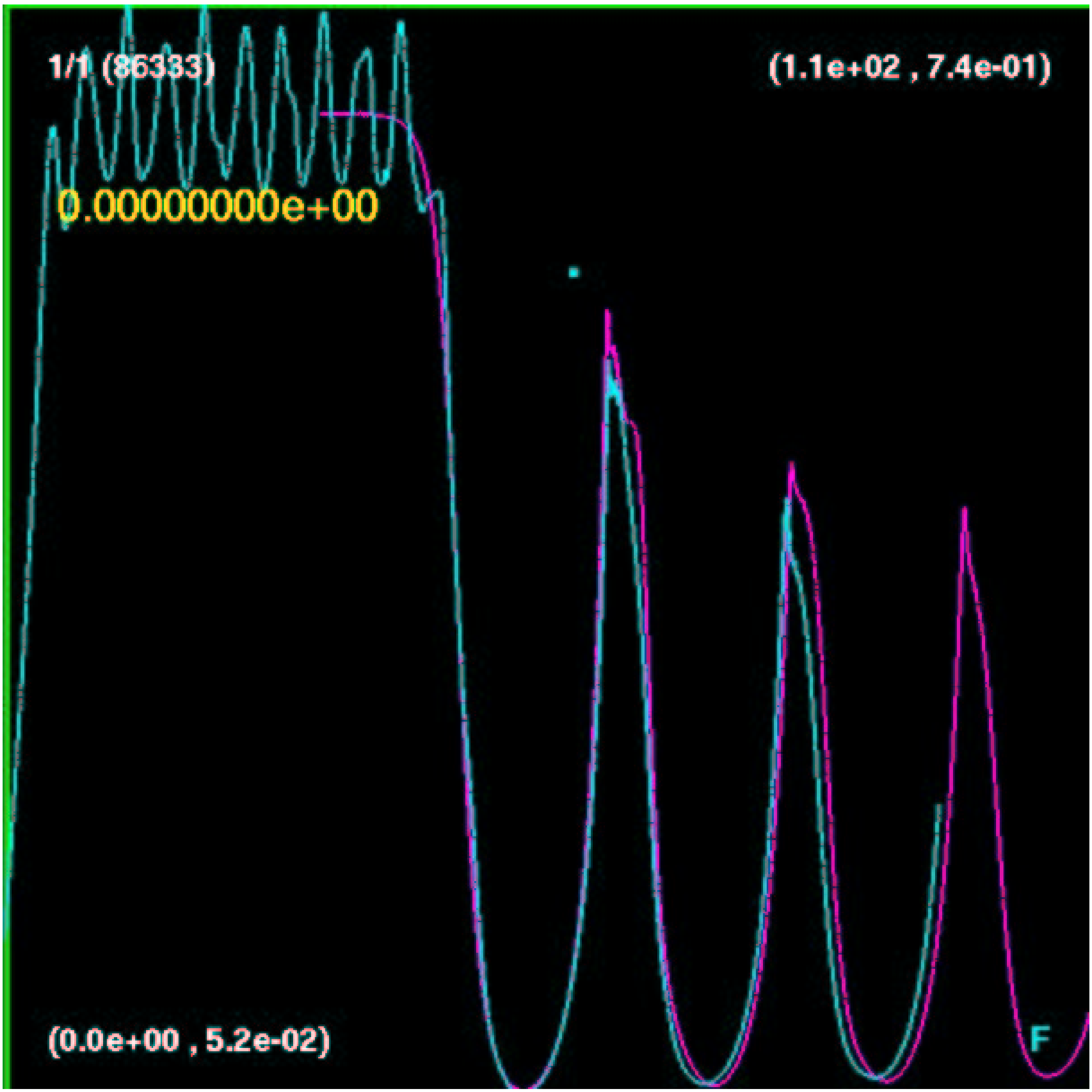
Γ = 2.0000000000000
0.15
0.1
0.05
0
M_{star}
II
I
I/II
-3
-2.9
-5
0
5
ln(ρ)



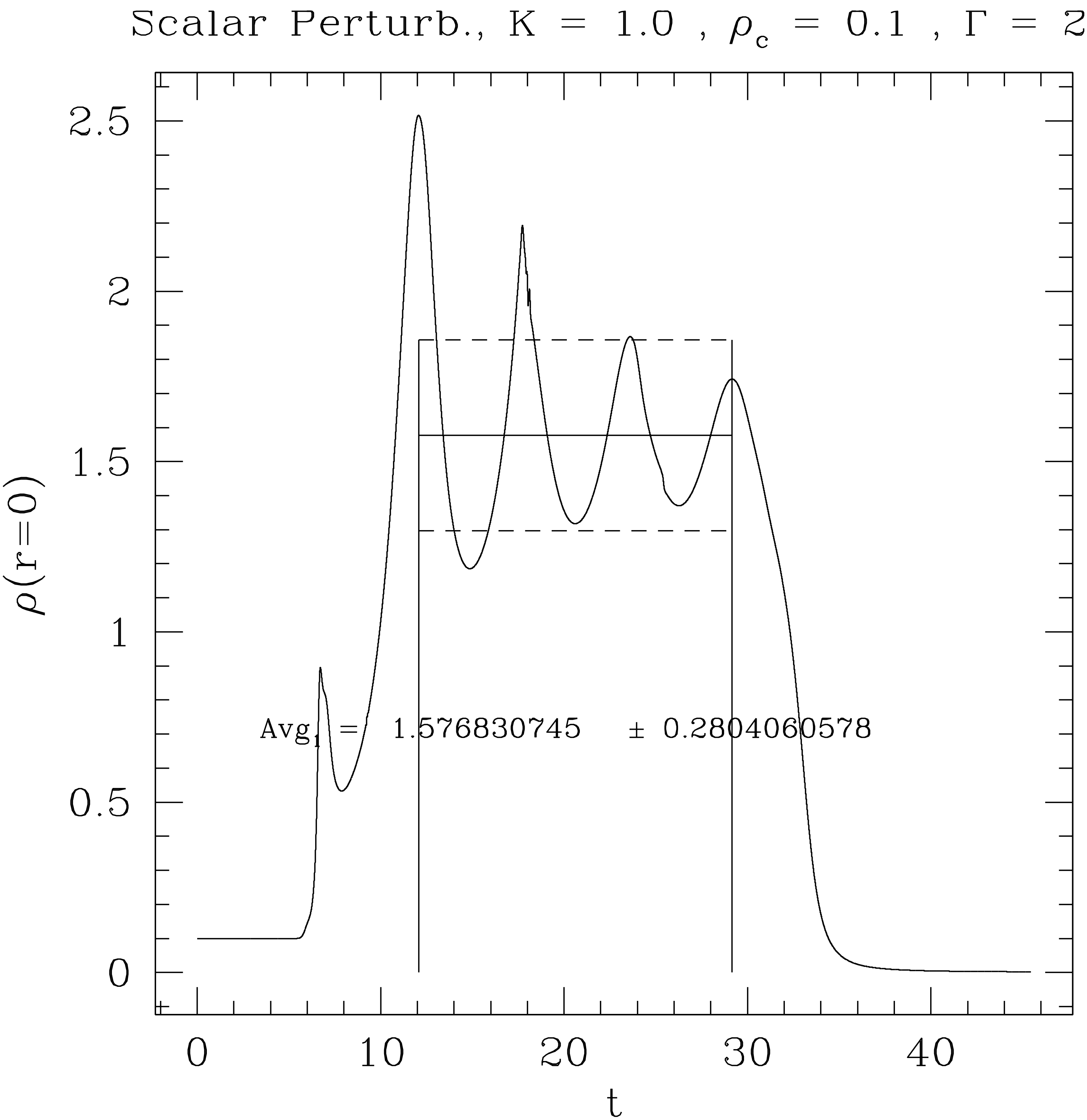
Γ = 2.0000000000000
0.15
0.1
0.05
0
Mstar
Stable
Unstable
-5
0
5
ln(ρ)


Γ = 2.0 | ρ_c = 0.15 | U_{Amp} = 0.11375 to 0.365
$\rho(r=0)$
t
0.8
0.6
0.4
0.2
0
0
50
100

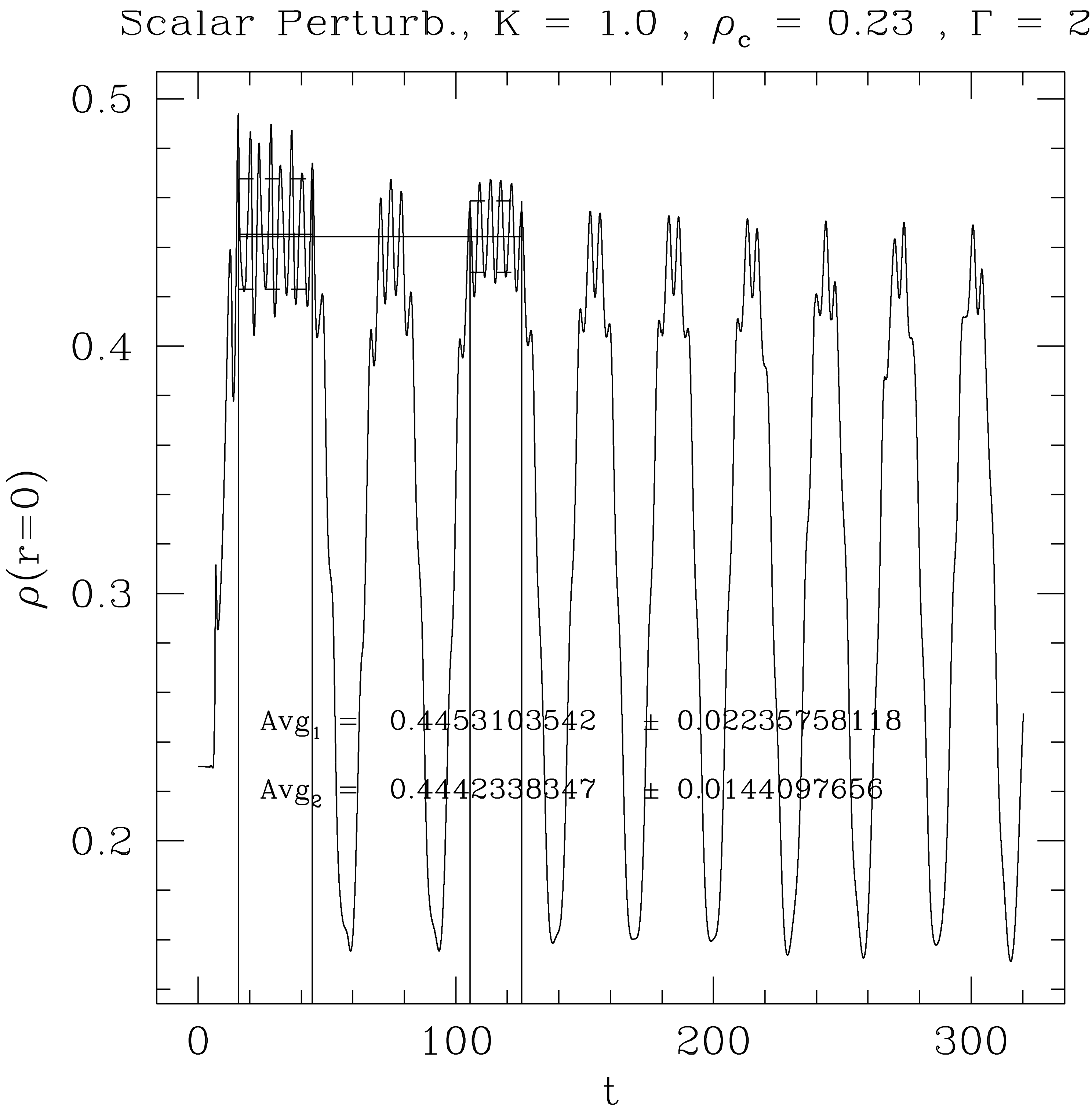

Scalar Perturb., K = 1.0 , ρ_c = 0.15 , Γ = 2
ρ(r=0)
t
0
20
40
60
80
0.2
0.4
0.6
0.8
Avg_1 = 0.7658920288 ± 0.05863752216


1/1 (86333)
(1.1e+02 , 7.4e-01)
0.00000000e+00
(0.0e+00 , 5.2e-02)
F

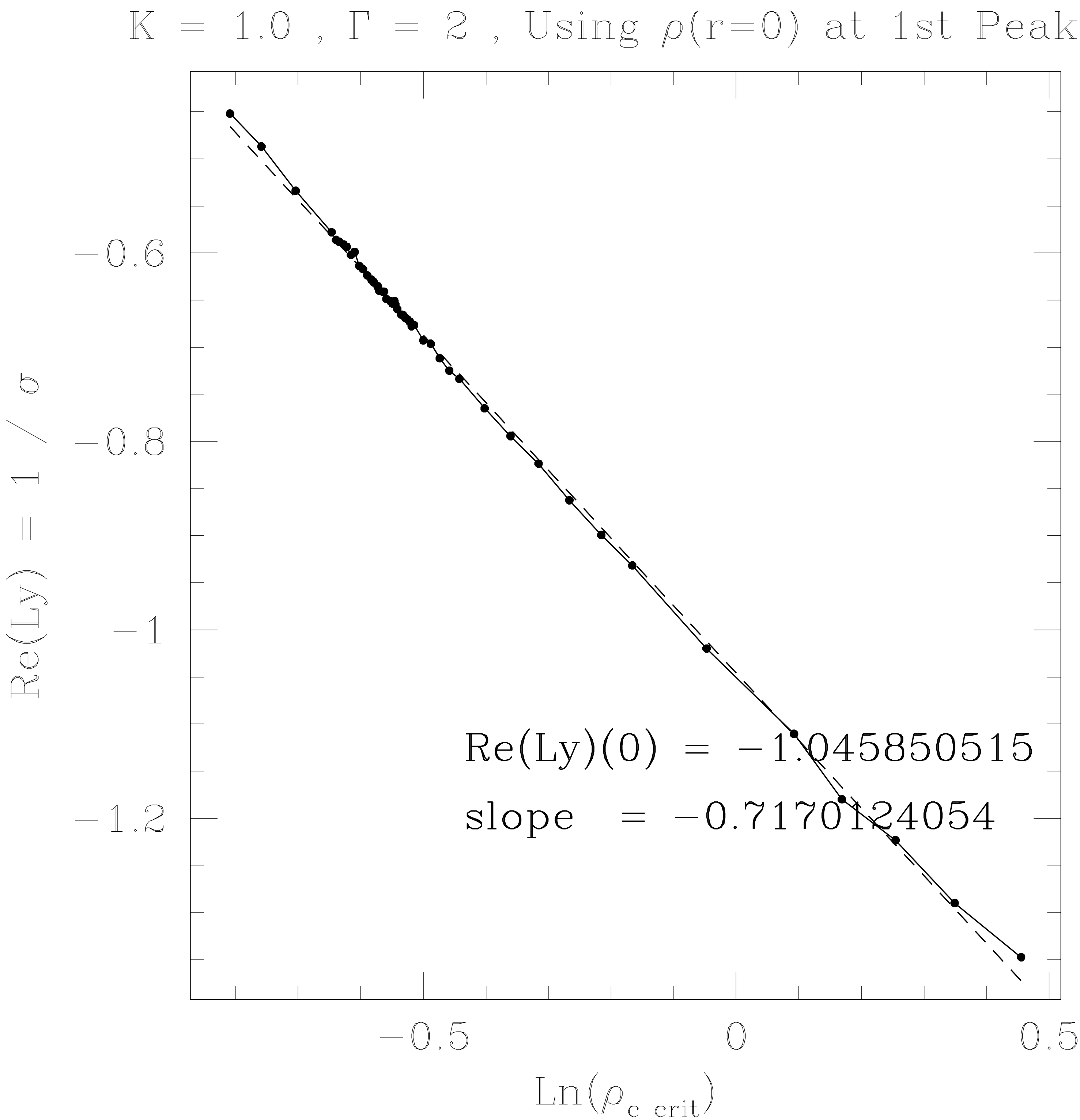
Movie #1 : `NearCrit.mpg`



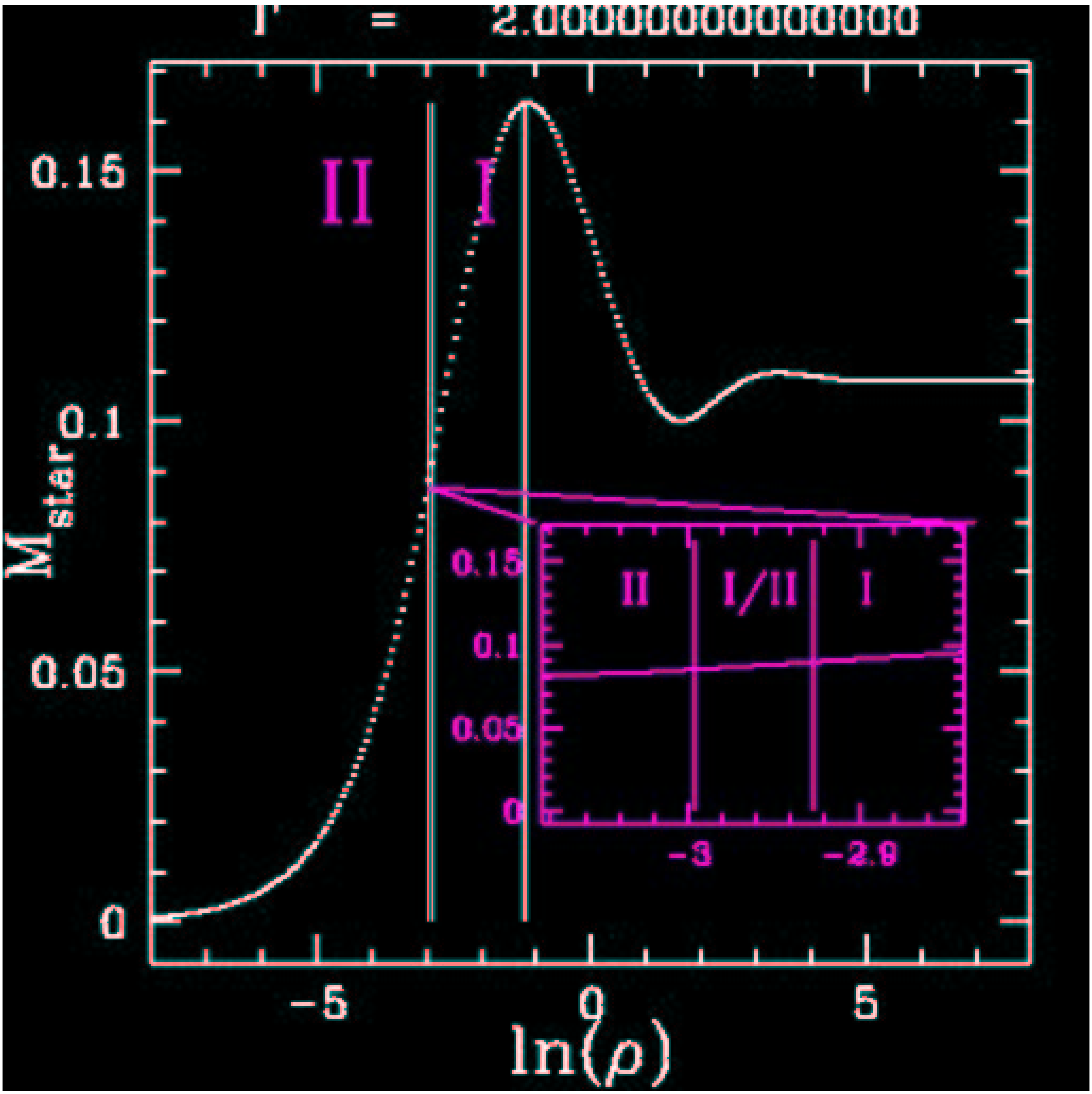
Scalar Perturb., K = 1.0 , ρ_c = 0.1 , Γ = 2
$\rho(r=0)$
t
Avg$_1$ = 1.576830745 ± 0.2804060578
0
0.5
1
1.5
2
2.5
10
20
30
40



Scalar Perturb., K = 1.0 , ρ_c = 0.23 , Γ = 2
ρ(r=0)
0.5
0.4
0.3
0.2
t
0
100
200
300
Avg_1 = 0.4453103542 ± 0.02235758118
Avg_2 = 0.4442338347 ± 0.0144097656



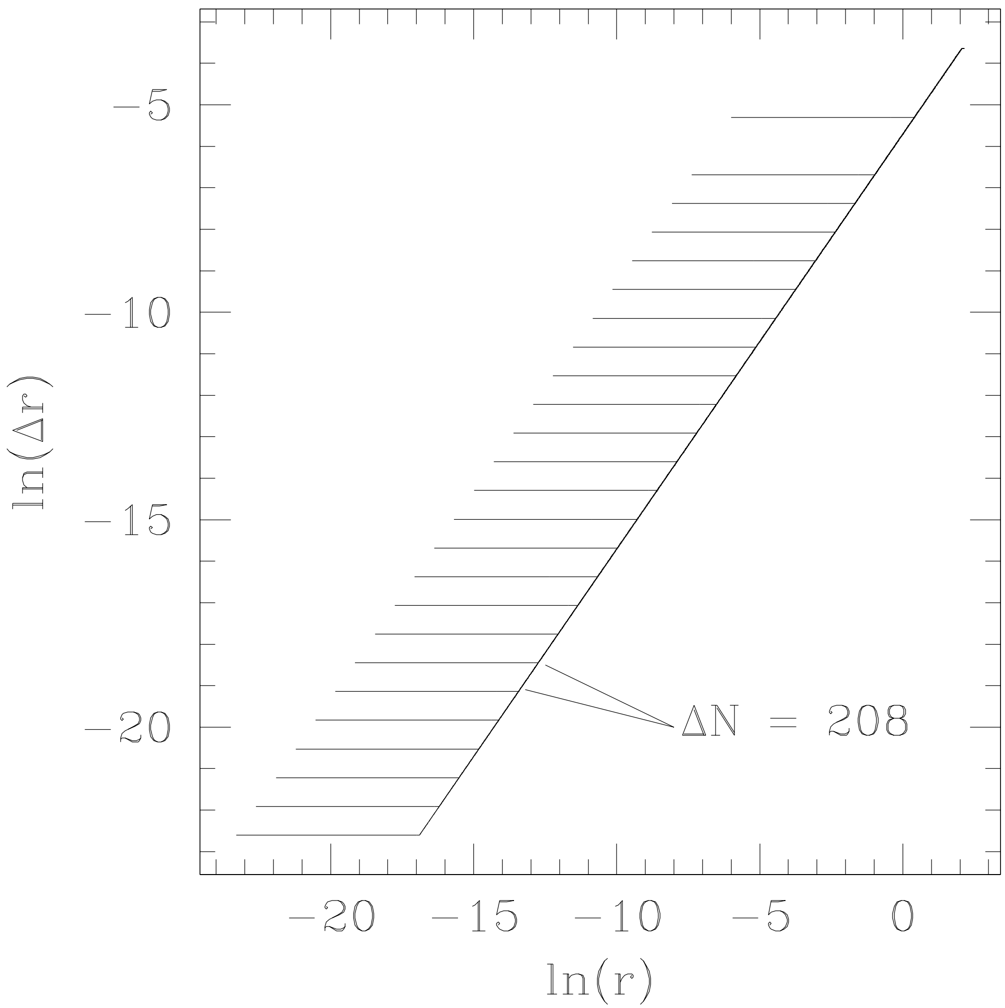
K = 1.0 , Γ = 2 , Using ρ(r=0) at 1st Peak
Re(Ly) = 1 / σ
Ln(ρ_c crit)
−0.6
−0.8
−1
−1.2
−0.5
0
0.5
Re(Ly)(0) = −1.045850515
slope = −0.7170124054



Γ = 2.0000000000000
M_{star}
0.15
0.1
0.05
0
−5
0
5
ln(ρ)
II
I
0.15
0.1
0.05
0
II
I/II
I
−3
−2.9

Movie #2 : `v_crit_physcoords.mpg`

Mesh Refinement

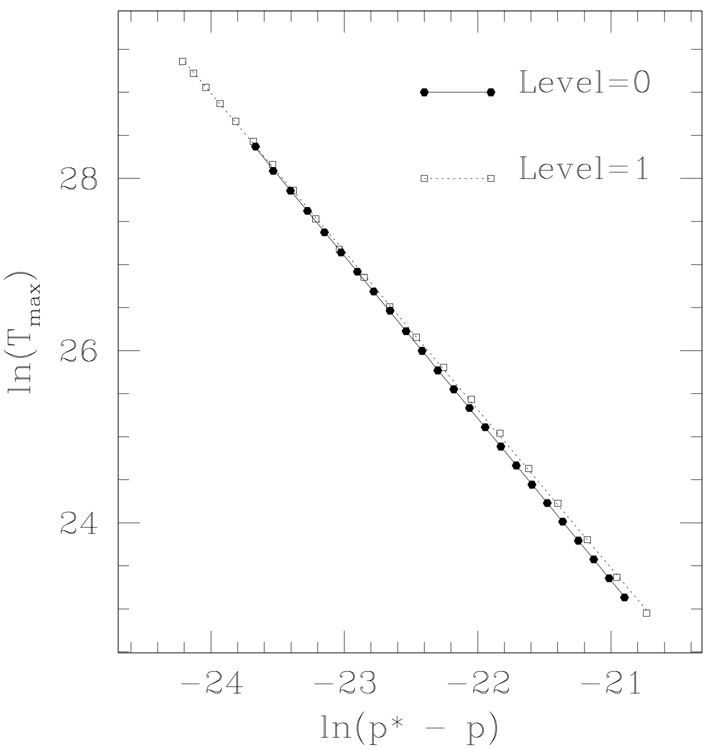


- Adds points near origin

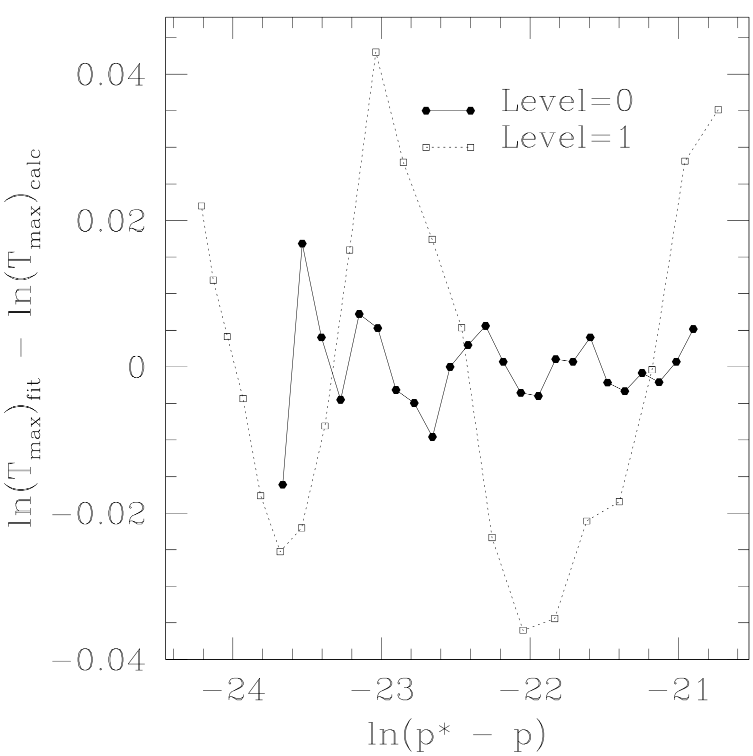
- Tracks features of the solution to determine refinement;

Movie #3 : `v_crit.mpg`

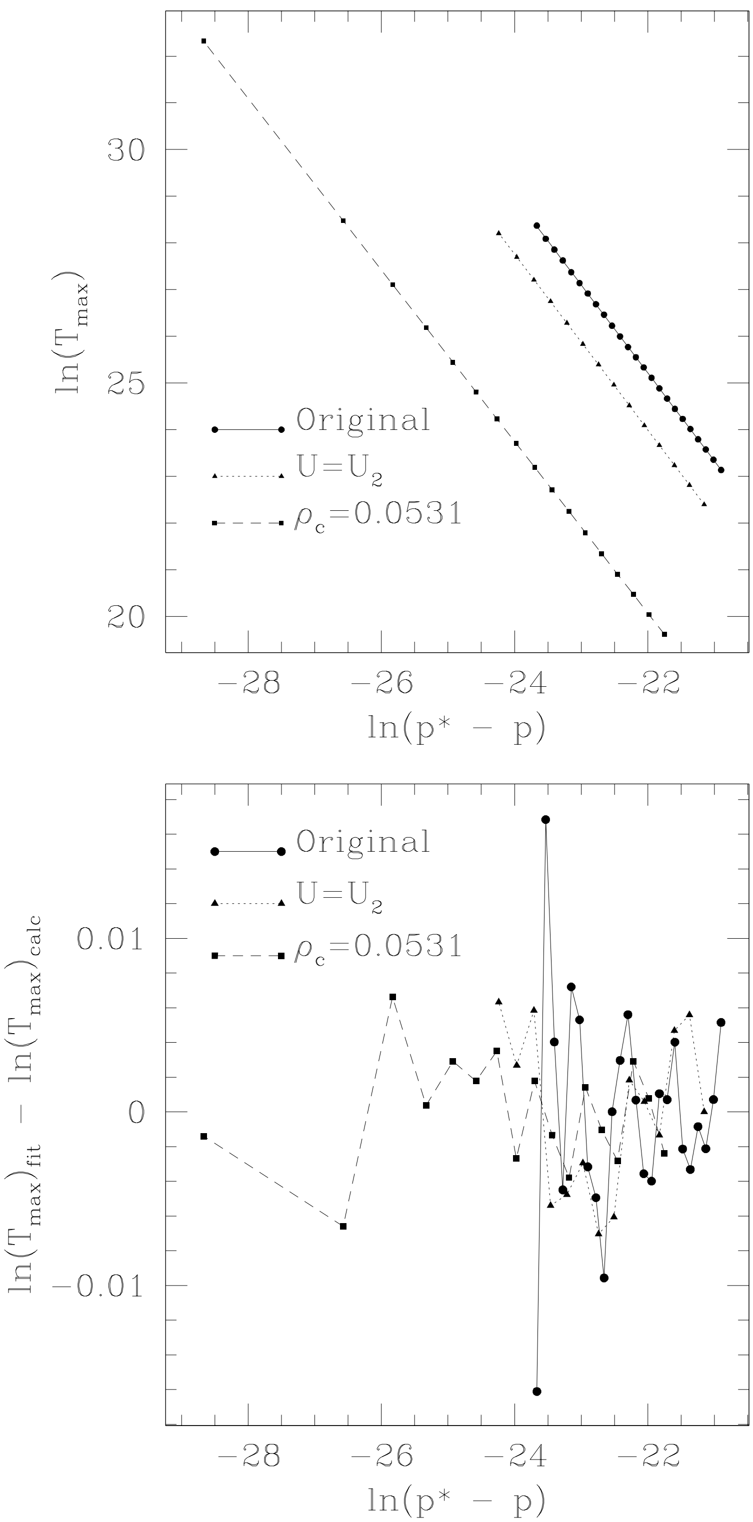
Movie #4 : `v_ultra_crit.mpg`



Level=0
Level=1
28
26
24
ln(T_max)
−24
−23
−22
−21
ln(p* − p)



Level=0
Level=1
0.04
0.02
0
−0.02
−0.04
ln(T_max)_fit − ln(T_max)_calc
−24
−23
−22
−21
ln(p* − p)



ln(T_max)
30
25
20
−28
−26
−24
−22
ln(p* − p)
Original
U=U_2
ρ_c=0.0531
ln(T_max)_fit − ln(T_max)_calc
0.01
0
−0.01
−28
−26
−24
−22
ln(p* − p)
Original
U=U_2
ρ_c=0.0531

Universality of Type-II Scaling Behavior

ρ_c	l	U	γ	p^*
0.05	0	U_1	0.94272	0.4687536738322
0.0531	0	U_1	0.918693	0.448204742983571
0.05	1	U_1	.9191	0.468290309415
0.05	0	U_2	0.94234392	0.42990315097513

Conclusions

- Type-I and Type-II Phen. observed in TOV solutions;
- Some evidence for overlap in param. space of Type-I/Type-II;
- Type-I Crit. Sol'n = Unstable Sol'n w/ same mass;
- Type-II Crit. Sol'n = Crit. Sol'n of Self-similar EOS fluid;