Newtonian Mechanics
(Marion & Thornton Chapter Two)

Physics A300*

Fall 2002

1 A Pep Talk on Dimensional Analysis

Up to now, we've been pretty lax about the units associated with physical quantities; we've tossed around expressions like $x = 2$ and $\overline{y} = y - 3$. But from now on, we're going to think like physicists, and require every physical quantity to have the appropriate units associated with it. So just as you can't say "The distance between my house and campus is three" or "I've been waiting here for fifteen!" without specifying whether you mean three miles or three kilometers and whether you mean fifteen minutes or fifteen seconds, you'll have to say things like $x = 3\,\text{meters}$ or $m = 15\,\text{kg}$.

In this way of thinking, the units are fundamentally a part of the quantity, so Δt is not the number of seconds between events, but the time between events. Also, by using different units, you can express the same physical quantity in different ways, e.g., $5\,\text{min} = 300\,\text{sec}$.

Every physical unit in mechanics can be made up out of the basic building blocks of time, distance, and mass. (In electromagnetism, we also need to add electric charge to the list, and in thermodynamics we have to add absolute temperature.)

1.1 Dimensionally Well-Defined Expressions

The first thing we have to be able to say is what expressions even make sense. Basically, certain operations on physical quantities can only be performed if the quantities have consistent units. In particular, the following operations are only well defined if a and b have compatible units:

- Comparison, i.e., $a = b$, $a < b$, or $a > b$
- Addition and subtraction: $a + b$ or $a - b$

Compatible units means that when you count up the number of powers of time, length, and mass in each quantity, the numbers match. For instance, 3 inches and 60 centimeters both have units of length, so we can meaningfully add, subtract, or compare them. (We would have to convert from inches to centimeters to simplify the sum.) On the other hand, we cannot compare 10 kg to 12 meters, because the former has units of mass while the latter has units of length.

Some more complicated examples:

1. In the expression $15\,\mathrm{kg} \times (12\,\mathrm{m/s}) + 30\,\mathrm{N}$, we're trying to add something with units of (mass)×(length)/(time) to something with units of force = (mass)×(acceleration) = (mass)×(length)/(time)^2. These are different physical units, and so the sum doesn't make any sense.

2. If v is a velocity, t is a time, and x is a distance, we can sensibly write $x = vt$ because we are comparing something with units of length (x) to vt, which has units of velocity×time = (distance/time)×time = distance.

The requirement that two quantities to be added or compared have compatible physical units can be used to check for mistakes in problems. For instance, if I'm supposed to calculate a force, and I end up with something like $15\,\mathrm{kg} \times v^2$, where v is a velocity, I know I've gone wrone somewhere, because my answer has units of mass times velocity instead of mass times acceleration. And in this case, all I have to do is check the units on the intermediate steps. The first quantity with the wrong units is the one where the mistake was introduced. (Of course, this won't help me catch factors of two, sign errors, factors of π etc.!)

1.1.1 Transcendental Functions

Special care should be taken in the case of transcendental functions like e^x, $\ln x$, $\sin x$, and $\cos x$. In these cases, the functions are ultimately defined by their Taylor series, for example

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \tag{1}$$

but that means that when we take the transcendental function of a quantity, we're adding different powers of the quantity. In particular, the expression above only makes sense if x and x^3 have the same units. In other words, only if x^2, and hence x, is dimensionless. This is true for any transcendental function: *The argument of a transcendental function must be dimensionless.*[1] Note that this means all angles are dimensionless. We tend to say things like $\theta = \pi$ radians but that really is the same as $\theta = \pi$. Radians are what we call a "dimensionless unit", which means they don't actually have physical units, they're just there as a reminder that we're talking about an angle.

1.2 Conversion of Units

In the end, dimensional analysis is a lot like algebra: units are carried along like abstract variables. We can also use algebra to convert from one set of units to another. For example, if we know that

$$1\,\mathrm{m} = 100\,\mathrm{cm} \tag{2}$$

[1] Also, the value of the function is dimensionless.

We can use it to convert 2.54 cm into meters, by saying

$$1 = \frac{1\,\mathrm{m}}{100\,\mathrm{cm}} \tag{3}$$

and thus

$$2.54\,\mathrm{cm} = 2.54\,\mathrm{cm} \times 1 = 2.54\,\mathrm{cm} \times \frac{1\,\mathrm{m}}{100\,\mathrm{cm}} = \frac{2.54}{100}\,\mathrm{m} = 2.54 \times 10^{-2}\,\mathrm{m} \tag{4}$$

Now, this may seem like overkill in such a simple problem, but when you're doing something more complicated like converting miles per hour into meters per second, it's a nice way to keep things straight and not multiply when you were supposed to divide:

$$55\frac{\mathrm{mi}}{\mathrm{hr}} \approx 55\frac{\mathrm{mi}}{\mathrm{hr}} \times \frac{8\,\mathrm{km}}{5\,\mathrm{mi}} \times \frac{10^3\,\mathrm{m}}{\mathrm{km}} \times \frac{1\,\mathrm{hr}}{60\,\mathrm{min}} \times \frac{1\,\mathrm{min}}{60\,\mathrm{sec}} = \frac{55 \times 8 \times 10^3}{5 \times 60 \times 60}\frac{\mathrm{m}}{\mathrm{s}} = \frac{880}{36}\frac{\mathrm{m}}{\mathrm{s}} \approx 24\frac{\mathrm{m}}{\mathrm{s}} \tag{5}$$

2 Newton's Laws (M&T Section 2.2)

We are now ready to begin the study of classical mechanics. We'll use as our starting point Newton's Three Laws of Motion. Before stating them, it's worth mentioning why we still study Newtonian mechanics in spite of the fact that we now know that the world is in fact better described by other theories, such as Einstein's Special and General Theories of Relativity on the one hand, and Quantum Mechanics on the other. The reasons why classical mechanics is still relevant are:

- Newtonian mechanics, as an approximation to either Relativity or Quantum Mechanics, is perfectly adequate for describing a wide range of phenomena
- Some of the consequences of Newton's Laws will later turn out to be more easily generalized to other theories than Newton's Laws themselves are.

So here are Newton's Laws, expressed in their most familiar form:

1. Inertia: a body at rest will remain at rest, and a body in motion will continue with the same speed and direction, unless acted on by an outside force
2. The acceleration $\vec{a}$ of a body of mass m under the influence of a net force $\vec{F}$ will be given by
$$\vec{F} = m\vec{a} \tag{6}$$
3. "For every action there is an equal and opposite reaction". I.e., if object 1 exerts a force $\vec{F}_{21}$ on object 2, then object 2 must be exerting a force
$$\vec{F}_{12} = -\vec{F}_{21} \tag{7}$$
on object 1. This is equal in magnitude and opposite in direction.

Note that the first law is in some sense a special case of the second law, since it says that when a body does not have a net force acting on it, its acceleration is zero. It was a very significant development in Physics, however, since it contradicted the previous assertion of Aristotelean Physics that the natural state of any material object was to be at rest.

One of the most fundamental consequences of Newton's laws can be written in terms of the momentum

$$\vec{p} := m\vec{v} = m\frac{d\vec{x}}{dt} \tag{8}$$

of a particle. The first law implies that an isolated particle (one subject to no forces) will have a constant momentum. The second law can be rewritten as

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{9}$$

The third law says that if two isolated particles exert forces on one another, those forces will add up to zero (in a vector sense):

$$\vec{F}_{12} + \vec{F}_{21} = 0 \tag{10}$$

Substituting in the second law gives

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0 \tag{11}$$

This tells us that in two-body interactions obeying Newton's laws, the total momentum of the system is a constant.

Not all forces obey Newton's third law. For example, in electromagnetism, the magnetic force on a charge particle depends on its velocity and is therefore not always directed on a line to the charge generating the magnetic field. However, in situations like this, the concept of conservation of momentum is so useful that we keep it by associating some momentum with the electromagnetic field itself, just the right amount so the bookkeeping works out right and momentum is conserved.

3 Inertial Frames of Reference and Galilean Invariance (M&T Section 2.3)

Previously, we discussed the change of coördinate system associated with a translation of the origin. In particular, if we chose new coördinates $\{x_{\overline{k}}\} = \{\overline{x}, \overline{y}, \overline{z}\}$ whose origin $x_{\overline{k}} = 0$ corresponded in the old coördinate system $\{x_k\} = \{x, y, z\}$ to $x_k = X_k$, the new and old coördinates of any point were related by

$$x_{\overline{k}} = x_k - X_k \tag{12}$$

while the components of any proper vectors such gradients or changes in position (including velocities) didn't change.

Now consider the following physical situation: observer A defines a coördinate system $\{x, y, z\}$ and observes the motion of a system (e.g., of particles) and verifies that the observed motions obey Newton's laws. Meanwhile, observer B is coasting past at a constant speed in a constant direction so that observer A measures observer B's velocity as $\vec{\beta}$. Observer B also defines a coördinate system $\{x', y', z'\}$, with coördinate axes parallel to A's, and describes the motion of the same system in these coördinates. Assume for simplicity that the origins of the two coördinate systems are chosen so that at the instant $t = 0$, they describe the same point in space. Since the origin $x'_i = 0$ is seen in A's coördinate system to be moving with velocity β, that point will correspond to $x_i = \beta_i t$. This implies that the transformation between the two sets of coördinates is the time-dependent translation

$$x'_i = x_i - \beta_i t \tag{13}$$

or in terms of position "vectors"

$$\vec{x}' = \vec{x} - \vec{\beta} t \tag{14}$$

This relationship between the coördinates of two observers moving at constant speeds with respect to one another is called the *Galilean transformation.*

Now consider how the velocity of any given particle will be measured by each observer:

$$\vec{v}' = \frac{d\vec{x}'}{dt} = \frac{d\vec{x}}{dt} - \vec{\beta} = \vec{v} - \vec{\beta} \tag{15}$$

So, although displacements measured at a fixed time will be the same according to either observer, velocities will not. However, consider the acceleration

$$\vec{a}' = \frac{d\vec{v}'}{dt} = \frac{d\vec{v}}{dt} = \vec{a} \tag{16}$$

Since Newton's laws are formulated in terms of acceleration, they are invariant under Galilean transformations as long as the forces involved depend only on the distances between objects, and not their velocities. In other words, observer A and observer B can each perform all the mechanical measurements they like, and each of them will find Newton's laws to hold equally well. A set of coördinates for all times is called a *reference frame*, and one in which Newton's laws hold is called an *inertial reference frame.* We've just shown that any reference frame moving at a constant velocity with respect to an inertial reference frame is itself an inertial reference frame.

The invariance of the laws of physics under change from one inertial reference frame to another is known as the principle of relativity, and it's interesting to note that such a principle is satisfied not only by Einsteinian but also by Newtonian physics. So the thing that's special about Relativity is not relativity. On the other hand, Aristotelean Physics, in which there is an absolute meaning to an object being at rest rather than in motion, does not obey the principle of relativity. In fact Galilean relativity is needed to make sense of physics on a planet which is rotating with a surface speed of around 100 miles per hour and orbiting the sun at around 4 million miles per hour. Part of the resistance Galileo faced to the idea that the Earth moves around the Sun was the Aristotelean argument that if that's the case, everything on the surface should be flung off as it moves, because it won't be able to keep up.

4 Examples (M&T Section 2.4)

The next section of Marion and Thornton basically consists of a bunch of examples; you should read it carefully, since it's by working through examples that one learns the sorts of tricks needed to be able to apply the theory of classical mechanics.

For the purposes of illustration, we'll go through one or two examples in class as well. The basic strategy we need to keep in mind is:

1. Consider each body in the problem individually and write down all the forces acting on it.

2. Choose a convenient coördinate system, in which most of the forces are exerted along coördinate axes (and thus have as few non-zero components as possible).

3. Use Newton's second law to write the equation of motion for each body in the system. Note that since Newton's second law is a vector equation, it will lead to a vector equation of motion for each body in the system; each component of this equation must be satisfied, so in general there will be a system of equations, even when only one body's equation of motion is of interest.

4. Solve the resulting system of equations for the unknown(s) of interest, eliminating any irrelevant unknown quantities.

4.1 Example: Ballistic Motion without Air Resistance

A classic question is the following: Let a projectile of mass m be launched from ground level with an initial speed of v_0 at an angle θ with respect to the ground. Assume no air resistance and constant downward gravitational acceleration of g.

1. How far downrange will the projectile land if the ground is flat and level?

2. For fixed initial speed, for what value of θ will the projectile travel the maximum distance downrange?

1. In this problem, there is only one body of interest: the projectile. The only force acting on it is the force of gravity, directed downward with a magnitude of mg.

2. An obvious convenient coördinate system has the y axis pointing straight up and the initial velocity of the projectile in the xy-plane. We also choose the origin of time ($t = 0$) to be the moment when the projectile is launched. This tells us that the force acting on the projectile is $\vec{F} = -mg\vec{e}_y$ and the initial velocity is $\dot{\vec{x}}(0) = v_0(\cos\theta\vec{e}_x + \sin\theta\vec{e}_y)$.

3. The equation of motion for the projectile is

   $$\vec{F} = -mg\vec{e}_y = m\ddot{\vec{x}}(t) \tag{17}$$

   or, written out component by component,

   $$\ddot{x}(t) = 0 \tag{18a}$$
   $$\ddot{y}(t) = -g \tag{18b}$$

4. To solve the problem, we first integrate the equations of motion to get the trajectory of the particle, using the initial conditions

$$\dot{x}(0) = v_0 \cos\theta \tag{19a}$$
$$\dot{y}(0) = v_0 \sin\theta \tag{19b}$$
$$x(0) = 0 \tag{19c}$$
$$y(0) = 0 \tag{19d}$$

Integrating the equations of motion once gives

$$\dot{x}(t) = \dot{x}(0) + \int_0^t \ddot{x}(t')\,dt' = \dot{x}(0) = v_0 \cos\theta \tag{20a}$$
$$\dot{y}(t) = \dot{y}(0) + \int_0^t \ddot{y}(t')\,dt' = \dot{y}(0) - gt = v_0 \sin\theta - gt \tag{20b}$$

Integrating a second time gives

$$x(t) = x(0) + \int_0^t \dot{x}(t')\,dt' = x(0) + v_0 t \cos\theta = v_0 t \cos\theta \tag{21a}$$
$$y(t) = y(0) + \int_0^t \dot{y}(t')\,dt' = y(0) + v_0 t \sin\theta - \frac{1}{2}gt^2 = v_0 t \sin\theta - \frac{1}{2}gt^2 \tag{21b}$$

If we define T to be the time at which the projectile lands, we know by definition that $y(T) = 0$. The point of the problem is to find the horizontal position, $x(T) = X$ at that time.

$$X = v_0 T \cos\theta \tag{22a}$$
$$0 = v_0 T \sin\theta - \frac{1}{2}gT^2 \tag{22b}$$

We have two equations in two unknowns (T and X). Since we don't care about T, we use the second equation to solve for T and eliminate it from the first. This gives us

$$v_0 T \sin\theta = \frac{1}{2}gT^2 \tag{23}$$

which has two solutions. One of them, $T = 0$, is not of interest, since we already know that $y(0) = 0$, and that's not the intersection with the ground that we're looking for. So instead, we use

$$T = \frac{2v_0 \sin\theta}{g} \tag{24}$$

Let's pause for a moment to check the units here. A velocity divided by an acceleration has units of

$$\frac{(\text{length})/(\text{time})}{(\text{length})/(\text{time}^2)} = (\text{time}) \tag{25}$$

so everything checks out. Now, plugging into the equation for X we have

$$X = v_0 \frac{2v_0 \sin\theta}{g} \cos\theta = \frac{2v_0^2}{g} \sin\theta \cos\theta \tag{26}$$

which is the answer to the first question. Again, we see that the dimensions are correct because

$$\frac{(\text{length})/(\text{time})^2}{(\text{length})/(\text{time}^2)} = (\text{length}) \tag{27}$$

Now for the second part of the question: We see that $X = 0$ for $\theta = 0$ (projectile fired horizontally) as well as for $\theta = \pi/2$ (projectile fired straight up). The maximum range is somewhere in between, and we can find the extremal values by looking for zeros of

$$\frac{dX}{d\theta} = \frac{2v_0^2}{g}(\cos^2\theta - \sin^2\theta) \tag{28}$$

which we see occur at $\theta = \pi/4$. This gives us the classic result that the maximum range is achieved when the projectile is fired at a $45°$ angle.

4.2 Example: Ballistic Motion with Air Resistance

Also in section 2.4 of Marion & Thornton is a discussion of retarding forces such as friction (which should be familiar from freshman Physics) and air resistance.

To give an example, we redo the problem of calculating the range as a function of angle in the presence of a drag force proportional to the projectile's speed and directed antiparallel to the the velocity vector:

$$\vec{F}_d = -mk\vec{v} \tag{29}$$

The choice of mk rather than k for the constant is one of convenience. Note that k has units of

$$\frac{(\text{force})}{(\text{mass}) \times (\text{velocity})} = \frac{1}{(\text{time})} \tag{30}$$

The statement of the problem might as for the following to first order in the constant k:

1. The time of flight of the projectile before it strikes level ground
2. The distance downrange that the projectile lands
3. The angle θ which maximizes the downrange distance for a fixed initial speed

Now the projectile has two forces acting on it: the force of gravity and the drag force, so the equations of motion are

$$m\ddot{\vec{x}}(t) = -mg\vec{e}_y - mk\dot{\vec{x}}(t) \tag{31}$$

Dividing through by m and writing the two non-zero components of this vector equation gives

$$\ddot{x}(t) = -k\dot{x}(t) \tag{32a}$$

$$\ddot{y}(t) = -k\dot{y}(t) - g \tag{32b}$$

We need to solve these differential equations subject to the same initial condtions as before,

$$\dot{x}(0) = v_0 \cos\theta \tag{33a}$$
$$\dot{y}(0) = v_0 \sin\theta \tag{33b}$$
$$x(0) = 0 \tag{33c}$$
$$y(0) = 0 \tag{33d}$$

Since neither $x(t)$ nor $y(t)$ appears in the equations of motion, we can treat them as first-order differential equations for $\dot{x}(t)$ and $\dot{y}(t)$:

$$\frac{d\dot{x}}{dt} = -k\dot{x} \tag{34a}$$
$$\frac{d\dot{y}}{dt} = -k\dot{y} - g \tag{34b}$$

Both of these equations are integrable; we rewrite them as

$$\frac{d\dot{x}}{\dot{x}} = -k\,dt \tag{35a}$$
$$\frac{d\dot{y}}{\dot{y} - g/k} = -k\,dt \tag{35b}$$

Integrating, we find

$$-kt = \int_0^t (-k)\,dt' = \int_{\dot{x}(0)}^{\dot{x}(t)} \frac{d\dot{x}}{\dot{x}} = \ln\frac{\dot{x}(t)}{\dot{x}(0)} = \ln\frac{\dot{x}(t)}{v_0\cos\theta} \tag{36a}$$
$$-kt = \int_0^t (-k)\,dt' = \int_{\dot{y}(0)}^{\dot{y}(t)} \frac{d\dot{y}}{\dot{y} + g/k} = \ln\frac{\dot{y}(t) + g/k}{v_0\sin\theta + g/k} \tag{36b}$$

We can solve these algebraically to obtain

$$\dot{x}(t) = v_0\cos\theta e^{-kt} \tag{37a}$$
$$\dot{y}(t) = (v_0\sin\theta + g/k)e^{-kt} - g/k \tag{37b}$$

Notice that g/k has units of acceleration times time, i.e., velocity, as it must.

The expressions for the velocity can now be integrated to obtain the trajectory:

$$x(t) = x(0) + \int_0^t \dot{x}(t')\,dt' = \frac{v_0\cos\theta}{k}(1 - e^{-kt}) \tag{38a}$$
$$y(t) = y(0) + \int_0^t \dot{y}(t')\,dt' = \frac{v_0\sin\theta + g/k}{k}(1 - e^{-kt}) - gt/k \tag{38b}$$

This gives us an exact expression for the position of the projectile as a function of t. However, we cannot in this case eliminate t algebraically to get y as a function of x. Looking at the questions asked of us, we once again define the time and place where the particle lands according to

$$x(T) = X \tag{39a}$$
$$y(T) = 0 \tag{39b}$$

Explicitly, that gives

$$X = \frac{v_0 \cos\theta}{k}(1 - e^{-kT}) \tag{40a}$$
$$0 = \frac{v_0 \sin\theta + g/k}{k}(1 - e^{-kT}) - gT/k \tag{40b}$$

The second equation gives an implicit expression for T, but we cannot solve it algebraically. It is known as a *transcendental equation*. However, we don't need to throw up our hands and give up just yet. If we were given the numerical values of the constants in the problem (v_0, g, and k), we could solve the equation on a computer. Any of the methods we'd use there would boil down to guessing values for T, inserting them into the expression, and seeing if they gave an answer close to zero.

In this case, however, we're asked to obtain an answer "to first order in k". That means to use perturbation theory. To formulate this problem as simply as possible, it is useful to define the quantity

$$\kappa = \frac{kv_0}{g} \tag{41}$$

note that this is dimensionless, and it is the only product of powers of k, v_0, and g which is. This means that the value of T which solves (40b) must depend on the parameters of the problem in the following way:

$$T = \frac{v_0}{g}\tau(\kappa, \theta) \tag{42}$$

The factor out front, v_0/g, has units of time, so τ is a dimensionless quantity. Because τ is dimensionless, it can depend only on the dimensionless parameters κ and θ. It cannot depend on any other combinations of the dimensionful parameters in the problem. Written as an equation for $\tau(\kappa)$ (we leave the θ dependence implicit for the time, being, since it's important to stress that τ is a function of κ), (40b) becomes

$$0 = (1 + \kappa \sin\theta)(1 - e^{-\kappa\tau(\kappa)}) - \kappa\tau(\kappa) \tag{43}$$

This implicitly defines the function $\tau(\kappa)$. We note that $\kappa = 0$ is the case without air resistance, which we've already solved, so we know τ is perfectly well behaved as $\kappa \to 0$. Since the problem is asking us to solve things perturbatively for small k, it's reasonable to write $\tau(\kappa)$ as an expansion in powers of κ:

$$\tau(\kappa, \theta) = \tau_0(\theta) + \tau_1(\theta)\kappa + \mathcal{O}(\kappa^2) \tag{44}$$

The notation $\mathcal{O}(\kappa^2)$ refers to a term which contains only terms of order κ^2 and higher. The explicit definition is

$$f(\kappa) = \mathcal{O}(\kappa^2) \text{ iff } \lim_{\kappa\to 0} \frac{f(\kappa)}{\kappa^n} = 0 \text{ for any } n < 2 \tag{45}$$

The answer to the first part of the problem, the time of flight to first order in k, will be

$$T = \frac{v_0}{g}\left\{\tau_0(\theta) + \tau_1(\theta)\frac{kv_0}{g} + \mathcal{O}\left(\left[\frac{kv_0}{g}\right]^2\right)\right\} \tag{46}$$

(Of course, we still need to solve $\tau_0(\theta)$ and $\tau_1(\theta)$ for using (43)!)

We could have written T as an expansion in powers of k instead, but then checking the dimensions would have been trickier, since each coëfficient in the expansion would have had different units. Also, strictly speaking we shouldn't say a dimensionful quantity is small; as we've noted, $k \ll 1$ is not a meaningful expression. What really matters is that k is small compared to combination of other parameters in the problem with units of inverse time: $k \ll v_0/g$ or in other words $\kappa \ll 1$.

Now, we can substitute (44) into (43) to derive the equations for τ_0, τ_1, etc., but first we'll want to expand the exponential in powers of κ. To do that, we'll need to use the Taylor series

$$e^{\xi} = \sum_{n=0}^{\infty}\frac{\xi^n}{n!} = 1 + \xi + \frac{1}{2}\xi^2 + \frac{1}{6}\xi^3 + \mathcal{O}(\xi^4) \tag{47}$$

With this expansion, (43) becomes

$$\begin{aligned}
0 =&(1+\kappa\sin\theta)\left(1 - \left[1 - \kappa\tau + \frac{1}{2}\kappa^2\tau^2 - \frac{1}{6}\kappa^3\tau^3 + \mathcal{O}(\kappa^4)\right]\right) - \kappa\tau \\
=&(1+\kappa\sin\theta)\left(\kappa\tau - \frac{1}{2}\kappa^2\tau^2 - \frac{1}{6}\kappa^3\tau^3 + \mathcal{O}(\kappa^4)\right) - \kappa\tau \\
=&-\frac{1}{2}\kappa^2\tau^2 - \frac{1}{6}\kappa^3\tau^3 + \mathcal{O}(\kappa^4) \\
&+\sin\theta\left(\kappa^2\tau - \frac{1}{2}\kappa^3\tau^2 + \mathcal{O}(\kappa^4)\right) \\
=&\kappa^2\left(-\frac{1}{2}\tau(\kappa)^2 + \tau(\kappa)\sin\theta\right) + \kappa^3\left(\frac{1}{6}\tau(\kappa)^3 - \frac{1}{2}\tau(\kappa)^2\sin\theta\right) + \mathcal{O}(\kappa^4)
\end{aligned} \tag{48}$$

Note that in order to get enough information to find τ to second order in κ, we've had to expand the equation for the trajectory out to fourth order (since the first two terms in the expansion vanish). We need to substitute the expansion (44) into the latest form of the equation to get:

$$\begin{aligned}
0 =&\kappa^2\left(-\frac{1}{2}\left[\tau_0 + \tau_1\kappa + \mathcal{O}(\kappa^2)\right]^2 + \left[\tau_0 + \tau_1\kappa + \mathcal{O}(\kappa^2)\right]\sin\theta\right) \\
&+\kappa^3\left(\frac{1}{6}\left[\tau_0 + \mathcal{O}(\kappa)\right]^3 - \frac{1}{2}\left[\tau_0 + \mathcal{O}(\kappa)\right]^2\sin\theta\right) + \mathcal{O}(\kappa^4) \\
=&\kappa^2\left(-\frac{1}{2}\left[\tau_0^2 + 2\tau_0\tau_1\kappa\right] + \left[\tau_0 + \tau_1\kappa\right]\sin\theta\right) + \kappa^3\left(\frac{1}{6}\tau_0^3 - \frac{1}{2}\tau_0^2\sin\theta\right) + \mathcal{O}(\kappa^4) \\
=&\kappa^2\left(-\frac{1}{2}\tau_0^2 + \tau_0\sin\theta\right) + \kappa^3\left(-\tau_0\tau_1 + \tau_1\sin\theta + \frac{1}{6}\tau_0^3 - \frac{1}{2}\tau_0^2\sin\theta\right) + \mathcal{O}(\kappa^4)
\end{aligned} \tag{49}$$

Now here's the key to doing perturbation theory: (49) holds for any value of κ, so the coëfficients of each of the powers of κ must vanish independently. (It was important that we rearrange the expression so that all of these coëfficients were independent of κ.) This means that

$$0 = -\frac{1}{2}\tau_0^2 + \tau_0 \sin\theta \tag{50a}$$

and

$$0 = -\tau_0\tau_1 + \tau_1 \sin\theta + \frac{1}{6}\tau_0^3 - \frac{1}{2}\tau_0^2 \sin\theta \tag{50b}$$

We've solved (50a) before;

$$\tau_0 = 2\sin\theta \tag{51}$$

means that

$$T = 2\frac{v_0}{g}\sin\theta + \mathcal{O}(kv_0/g) \tag{52}$$

this is the time of flight to zeroth order in k, i.e., without air resistance.

The next order gives us

$$\tau_1 = \frac{\frac{1}{6}\tau_0^3 - \frac{1}{2}\tau_0^2\sin\theta}{\tau_0 - \sin\theta} = \frac{\frac{8}{6}\sin^3\theta - \frac{4}{2}\sin^3\theta}{2\sin\theta - \sin\theta} = -\frac{2}{3}\sin^2\theta \tag{53}$$

which tells us

$$T = \frac{v_0}{g}\left(2\sin\theta - \frac{2}{3}\frac{kv_0}{g}\sin^2\theta + \mathcal{O}([kv_0/g]^2)\right) \tag{54}$$

Having found the time of flight to first order in k, it's relatively easy to do the rest of the problem. Returning to (40a), we have

$$\begin{aligned}
X =& \frac{v_0\cos\theta}{k}(1 - e^{-kT}) = \frac{v_0^2}{g}\cos\theta\kappa^{-1}(1 - e^{\kappa\tau(\kappa)}) = \frac{v_0^2}{g}\cos\theta\left(\tau(\kappa) - \frac{1}{2}\kappa\tau(\kappa)^2 + \mathcal{O}(\kappa^2)\right) \\
=& \frac{v_0^2}{g}\cos\theta\left(\tau_0 + \kappa\left[\tau_1 - \frac{1}{2}\tau_0^2\right] + \mathcal{O}(\kappa^2)\right) = \frac{v_0^2}{g}\cos\theta\left(2\sin\theta + \kappa\left[\frac{2}{3}\sin^2\theta - 2\sin^2\theta\right] + \mathcal{O}(\kappa^2)\right) \\
=& \frac{v_0^2}{g}\cos\theta\left(2\sin\theta - \frac{4}{3}\frac{kv_0}{g}\sin^2\theta + \mathcal{O}([kv_0/g]^2)\right)
\end{aligned} \tag{55}$$

To find the θ which gives maximum range, we consider X as a function of θ, and note that it is a maximum when $X'(\theta) = 0$. To make it easier to take this derivative, we use the trigonometric identities

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{56}$$

and

$$\sin^2\theta = 1 - \cos^2\theta \tag{57}$$

to write

$$X(\theta) = \frac{v_0^2}{g}\left(\sin 2\theta + \frac{4}{3}\kappa(\cos^3\theta - \cos\theta) + \mathcal{O}(\kappa^2)\right) \tag{58}$$

Differentiating gives

$$X'(\theta) = \frac{v_0^2}{g}\left(2\cos 2\theta + \frac{4}{3}\kappa(-3\sin\theta\cos^2\theta + \sin\theta) + \mathcal{O}(\kappa^2)\right) \tag{59}$$

And we define Θ to be the angle θ which maximizes X, i.e., which satisfies

$$0 = \frac{g}{2v_0^2}X'(\Theta) = \cos 2\Theta + \kappa 2\sin\Theta\left(\frac{1}{3} - \cos^2\Theta\right) + \mathcal{O}(\kappa^2) \tag{60}$$

Now, Θ, considered to be defined by this equation, also depends on κ, and we are also supposed to expand it to first order:

$$\Theta = \Theta_0 + \kappa\Theta_1 + \mathcal{O}(\kappa^2) \tag{61}$$

The question is, what are Θ_0 and Θ_1. We substitute (44) into (60) and find

$$0 = \cos(2\Theta_0 + 2\kappa\Theta_1) + \kappa 2\sin\Theta_0\left(\frac{1}{3} - \cos^2\Theta_0\right) + \mathcal{O}(\kappa^2) \tag{62}$$

Now,

$$\cos(2\Theta_0 + 2\kappa\Theta_1) = \cos 2\Theta_0 + 2\kappa\Theta_1\sin 2\Theta_0 + \mathcal{O}(\kappa^2) \tag{63}$$

(either by Taylor-expanding or by using the angle sum formula and small angle formula) so

$$0 = \cos 2\Theta_0 + 2\kappa\left(\Theta_1\sin 2\Theta_0 + \left[\frac{1}{3} - \cos^2\Theta_0\right]\sin\Theta_0\right) + \mathcal{O}(\kappa^2) \tag{64}$$

Once again, this equation must be satisfied for any value of κ and therefore the coëfficient of each power of κ must vanish. The zeroth-order term gives

$$\cos 2\Theta_0 = 0 \tag{65}$$

or

$$\Theta_0 = \pi/4 \tag{66}$$

which was the answer we got in section 4.1 with no air resistance. The first-order term tells us

$$0 = \Theta_1\sin 2\Theta_0 + \left[\frac{1}{3} - \cos^2\Theta_0\right]\sin\Theta_0 = \Theta_1 + \frac{1}{\sqrt{2}}\left(\frac{1}{3} - \frac{1}{2}\right) \tag{67}$$

Which means

$$\Theta_1 = \frac{1}{\sqrt{2}}(1/2 - 1/3) = \frac{1}{6\sqrt{2}} \tag{68}$$

or finally

$$\Theta = \frac{\pi}{4} + \frac{1}{6\sqrt{2}}\frac{kv_0}{g} + \mathcal{O}([kv_0/g]^2) \tag{69}$$

5 Work and Energy (M&T Section 2.5–2.6)

Working with forces an accelerations works well if the forces in the problem are constant or vary in some simple way as the trajectories of the system progress, but sometimes it's easier to consider the accumulated effect of the action of a force. This can be accomplished with a concept called work.

If a particle moves from a point P to a point Q while being acted on by a force $\vec{F}$, the *work* done by the force on the particle over the course of that movement is defined by the line integral

$$W_{P\to Q} = \int\limits_{P\to Q} \vec{F}\cdot d\vec{\ell} \tag{70}$$

Note that the force $\vec{F}$ can be different at different points of the path (so it really belongs under the integral), and that the work done can in general depend on the circumstances under which the particle moved from P to Q, such as the path taken and the time at which it occurred, so it's not necessarily a function only of the endpoints P and Q.

5.1 Kinetic Energy

One thing we can say about the total work done on a particle by all of the forces acting on it between a time t_1 and another time t_2 comes from Newton's second law

$$\vec{F} = m\ddot{\vec{x}} \tag{71}$$

We write

$$W_{t_1\to t_2} = \int\limits_{\vec{x}(t_1)\to\vec{x}(t_2)} \vec{F}\cdot d\vec{\ell} = \int\limits_{\vec{x}(t_1)\to\vec{x}(t_2)} m\ddot{\vec{x}}\cdot d\vec{\ell} \tag{72}$$

Now recall the practical implementation of the line integral in terms of a parametrized curve:

$$\int\limits_{\mathcal{C}} \vec{A}\cdot d\vec{\ell} = \int_{s_P}^{s_Q} \left(\vec{A}\cdot\frac{d\vec{x}}{ds}\right) ds \tag{73}$$

in this case, the obvious parameter to choose is the time t. The parametrized curve is just the trajectory $\vec{x}(t)$ of the particle. Using that and writing $\dot{\vec{x}}$ for $\frac{d\vec{x}}{dt}$, we have

$$W_{t_1\to t_2} = \int_{t_1}^{t_2} m\ddot{\vec{x}}\cdot\dot{\vec{x}}dt \tag{74}$$

Now, the nice thing about this expression is that we can recognize

$$\ddot{\vec{x}}\cdot\dot{\vec{x}} = \frac{1}{2}\frac{d}{dt}(\dot{\vec{x}}\cdot\dot{\vec{x}}) \tag{75}$$

so that

$$W_{t_1\to t_2} = \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{1}{2}m\dot{\vec{x}}\cdot\dot{\vec{x}}\right) dt = \frac{1}{2}m\left|\dot{\vec{x}}(t_2)\right|^2 - \frac{1}{2}m\left|\dot{\vec{x}}(t_1)\right|^2 \tag{76}$$

in other words, the total work done by all the forces acting on the particle is the change in the quantity

$$T = \frac{1}{2}mv^2 \tag{77}$$

we call this the *Kinetic Energy*.

5.2 Potential Energy

Another reason why the concept of work is so useful comes up in the case of forces which can be described as vector fields. Very often, we'll be able to say that the force a particular particle experiences just depends on where the particle is, and possibly when it's there: $\vec{F}(\vec{x}, t)$. This is a vector field at each instant of time. In most cases of interest, the same vector force field will apply at any instant of time.

There is a wide range of forces for which the vector field $\vec{F}$ can actually be written as the gradient of a time-independent scalar field

$$\vec{F}(\vec{x}) = -\vec{\nabla}U(\vec{x}) \tag{78}$$

(U is something we're going to call the *potential energy* for reasons to be made clear in a moment.) Forces described by these special force fields are called *conservative forces*. You showed on a previous problem set that a necessary condition for this is that $\vec{\nabla} \times \vec{F} = 0$. It turns out to be a sufficient condition as well. To reiterate why a vector field being a gradient of some scalar field is equivalent to its curl vanishing, consider

$$(\vec{\nabla} \times \vec{F})_z = \sum_j \sum_k \epsilon_{zjk} \frac{\partial F_k}{\partial x_j} = \epsilon_{zxy} \frac{\partial F_y}{\partial x} + \epsilon_{zyx} \frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \tag{79}$$

The vanishing of the z component of $\vec{\nabla} \times \vec{F}$ is equivalent to

$$\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y} \tag{80}$$

But if there is some scalar field $U(x, y, z)$ for which we can write $\vec{F} = -\vec{\nabla}U$, then these two partial derivatives have to be equal, because that's just

$$-\frac{\partial}{\partial x}\frac{\partial U}{\partial y} = -\frac{\partial}{\partial y}\frac{\partial U}{\partial x} \tag{81}$$

which is a basic property of partial derivatives.

So to show that a force $\vec{F}(x, y, z)$ is conservative, you can do one of two things:

1. Find an explicit $U(x, y, z)$ such that $F_i(x, y, z) = \partial_i U(x, y, z)$, or

2. Calculate $\vec{\nabla} \times \vec{F}$ and show that it vanishes.

To show that a force is non-conservative, you can calculate $\vec{\nabla} \times \vec{F}$ and show that it's non-zero, which is the same thing as showing that at least one pair of would-be mixed partial derivatives don't match.

What does a conservative force do for us? Consider the work done by a conservative force in moving a particle from point P to point Q:

$$W_{P\to Q} = \int_{P\to Q} \vec{F}\cdot d\vec{\ell} = -\int_{P\to Q} \vec{\nabla}U\cdot d\vec{\ell} = -[U(Q)-U(P)] \tag{82}$$

Now, this *does* depend only on the endpoints of the path, and if the only force acting on a particle is a conservative one, the work done on that particle by the force can be written in two different ways, either as the gain in kinetic energy or the loss in potential energy:

$$W_{t_1\to t_2} = T(t_2)-T(t_1) = \Delta T = -[U(\vec{x}(t_2)-U(\vec{x}(t_1)] = -\Delta U \tag{83}$$

This can be rewritten as

$$\Delta T + \Delta U = \Delta(T+U) = 0 \tag{84}$$

This means that **for a conservative force**, the total energy

$$E = T + U \tag{85}$$

is conserved, i.e., does not change with time.

5.3 Interpretation of Potential Energy

It may seem a little strange that the work done by a force is minus the change in potential energy, but it makes sense if we think about it in the following ways:

- As the particle moves under the influence of the conservative force, if the particle is accelerated, the force does positive work on the particle, which means potential energy is lost and kinetic energy is gained. The total energy remains the same, but it is coverted from potential to kinetic.

- It's natural to think about the work needed to move a particle from one point in a potential field to another, and expect that positive work must be done to move the particle to a position of higher potential energy. That is the way things work, since in that case there are two forces present: the force due to the potential field and an external force doing the pushing. If the particle is at rest when you start, and then you push it to a position of higher potential where it is also at rest, the kinetic energy of the particle will not have changed, so by (76), the total work done by both forces combined must be zero. The force associated with the potential will have done negative work (since the change in potential energy will be positive), which means the force doing the pushing will have done positive work (of the same magnitude), as expected.

The most familiar example of potential energy is also the standard way of gaining an intuitive grasp of the concept. Consider a particle of mass m moving in a constant gravitational field. The force on the particle will be

$$\vec{F}_g(x,y,z) = -mg\vec{e}_z \tag{86}$$

where the unit vector $\vec{e}_z$ points straight up. This makes the potential energy

$$U(x, y, z) = mgz \tag{87}$$

which is proportional to the height. So when a particle moves to a position of higher potential energy, it's something like moving it to a higher altitude in a constant gravitational field; if it's released and falls to a lower position, it will have been accelerated and have some kinetic energy by the time it gets there.

6 Conservation Laws (M&T Section 2.5)

As I mentioned at the beginning of the discussion of energy, the approach we've used in examples so far, describing the forces acting on a particle at each instant of time and using Newton's second law to derive an equation of motion, works well when the forces are easy to model. Often, however, detailed consideration of all the forces in a problem is difficult or impossible, and a more fruitful approach to solving problems is to use consequences of Newton's laws, which tell us that under certain circumstances, certain quantities do not change as the system evolves. We say that those quantities are *conserved*, and that they obey *conservation laws*. In that case, if a problem involves a simple initial condition and a simple final condition, with a complicated interaction in between, we can use the fact that even the complicated interaction conserves certain quantities, and solve the problem by setting the values of those quantities before and after the interaction equal to one another

6.1 Conservation of Momentum

We've already considered this back in Section 2. There we showed that Newton's first law means that a body subject to no forces has a constant momentum $\vec{p} = m\dot{\vec{x}}$. We also showed that Newton's third law means that if two bodies exert forces on each other and don't interact with anything else, the total momentum of the two is conserved:

$$\vec{p}_1 + \vec{p}_2 = \text{ constant} \tag{88}$$

6.2 Conservation of Angular Momentum

Before we can discuss conservation of angular momentum, we have to define angular momentum itself. If a particle is at position $\vec{x}$ and moving with momentum $\vec{p}$, the angular momentum is a vector quantity defined by

$$\vec{L} = \vec{x} \times \vec{p} \tag{89}$$

Note that the definition of $\vec{L}$ includes the dreaded "position vector" $\vec{x}$ without any time derivatives. This means that $\vec{L}$ is only defined with respect to a particular choice of origin $\vec{x} = 0$. We are not free to work out the angular momentum of a particle in one coördinate system, then translate our coördinates and expect that expression to be valid in the new coördinate system.

Note also that $\vec{L}$ is a vector of magnitude

$$\left|\vec{L}\right| = |\vec{x}|\,|\vec{p}|\sin\alpha = (r)(mv)\sin\theta \tag{90}$$

where α is the angle between $\vec{x}$ and $\vec{p}$. This reduces to the familiar definition of mass times velocity times perpendicular distance from an axis, with $r\sin\alpha$ playing the role of the perpendicular distance.

Now, to work out the conservation law for angular momentum, we define the torque $\vec{\tau}$ resulting from a force $\vec{F}$ as

$$\vec{\tau} = \vec{x} \times \vec{F} \tag{91}$$

Now consider the time derivative of the angular momentum of a particle:

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{x} \times \vec{p}) = \dot{\vec{x}} \times \vec{p} + \vec{x} \times \dot{\vec{p}} = m\underbrace{\dot{\vec{x}} \times \dot{\vec{x}}}_{0} + \vec{x} \times \vec{F} = \vec{\tau} \tag{92}$$

So if the total torque on a particle vanishes (which can happen if $\vec{F} = 0$ or $\vec{x} = 0$ or $\vec{F} \parallel \vec{x}$), its angular momentum is conserved.

6.3 Conservation of Energy

As we described in our discussion of energy in Section 5, when a particle moves under the influence of a conservative force, its total energy (kinetic plus potential) is constant. To re-derive this with a slightly different approach, assume that the force is again associated with a vector field $\vec{F}$ which can be derived from some scalar field U, but now let them both be explicitly time-dependent:

$$\vec{F}(\vec{x}, t) = -\vec{\nabla}U(\vec{x}, t) \tag{93}$$

Define the kinetic energy to be

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{\vec{x}}(t) \cdot \dot{\vec{x}}(t) \tag{94}$$

and the total energy to be

$$E = T + U \tag{95}$$

Now consider the time derivative of this:

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt} \tag{96}$$

The time dervative of the kinetic energy is

$$\frac{dT}{dt} = \frac{1}{2}m\frac{d}{dt}(\dot{\vec{x}} \cdot \dot{\vec{x}}) = m\dot{\vec{x}} \cdot \ddot{\vec{x}} \tag{97}$$

The potential energy can depend on time either explicitly, or implicitly through its position dependence and the trajectory of the particle:

$$\frac{dU(x,y,z,t)}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt} + \frac{\partial U}{\partial t} = \sum i = 1^3 \frac{\partial U}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial U}{\partial t} = \vec{\nabla}U \cdot \dot{\vec{x}} + \frac{\partial U}{\partial t} = -\vec{F} \cdot \dot{\vec{x}} + \frac{\partial U}{\partial t} \tag{98}$$

Combining them, we find

$$\frac{dE}{dt} = m\dot{\vec{x}} \cdot \ddot{\vec{x}} - \vec{F} \cdot \dot{\vec{x}} + \frac{\partial U}{\partial t} = \underbrace{(m\ddot{\vec{x}} - \vec{F})}_{\text{0 by Newton's 2nd law}} \cdot \dot{\vec{x}} + \frac{\partial U}{\partial t} = \frac{\partial U}{\partial t} \tag{99}$$

So, if the potential does not depend explicitly on time, energy is conserved.

6.4 Conservation Laws in General

E, $\vec{p}$ & $\vec{L}$ always conserved *but* sometimes they are transferred into a part of the universe we are not modelling in a particular problem. Examples:

- Projectile motion, $\vec{p}$ of projectile changes, but so does momentum of the Earth
- Air resistance: momentum and energy transferred to air molecules
- Friction: momentum transferred into object we're sliding along; energy transferred into heat (vibrational motion of solid)
- Inelastic collision: "lost" energy goes into heat and deformation of bodies

A Appendix: Correspondence to Class Lectures

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