Linear Oscillations
(Marion & Thornton Chapter Three)

Physics A300*

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In Chapter Two we applied Newton's laws to a number of elementary situations, and found the equation of motion and a few properties of the system in each case. In this chapter, we're basically going to study a single equation of motion in detail, and learn about many properties of its solutions. But we'll build up to the full equation, studying a couple of simpler versions of it first.

1 The Simple Harmonic Oscillator

The parts of chapter three with which we'll concern ourselves all describe one-dimensional systems. In the case of conservative force fields, the vector Newton's second law simplifies into a differential equation for the trajectory $x(t)$:

$$m\ddot{x} = F(x) \tag{1.1}$$

In one dimension, we can always describe the force in terms of a potential $U(x)$:

$$F(x) = -U'(x) \tag{1.2}$$

Note that as always, adding a constant to the potential energy would not change the force.

Now, when you've learned about harmonic oscillators, you've probably started with something called Hooke's Law:

$$F_{\text{Hooke}}(x) = -kx \tag{1.3}$$

which was probably described as a special property of springs. But actually, Hooke's Law is an approximation to just about any force field sufficiently close to a stable equilibrium. To see that, integrate the force to get the potential (arbitrarily defining the potential energy at $x = 0$ to be zero)

$$U_{\text{Hooke}}(x) = \frac{1}{2}kx^2 \tag{1.4}$$

we will see that this is generally a good approximation in some region near a stable equilibrium.

Consider a generic one-dimensional potential $U(x)$; since $F(x) = -U'(x)$, an equilibrium point, at which $F(x_{\text{eq}}) = 0$ corresponds to an extremum of $U(x)$, i.e., $U'(x_{\text{eq}}) = 0$. Now, for this to be a stable equilibrium, it should be a local minimum rather than a local maximum of $U(x)$. This is because we'd like the force to push us towards the equilibrium, so $F(x_{\text{eq}} + \delta x) = \delta x\, F'(x_{\text{eq}}) = -\delta x\, U''(x_{\text{eq}})$ is positive if $\delta x < 0$ and negative if $\delta x > 0$. This means $U''(x_{\text{eq}}) > 0$. Now, choose the origin of the coördinate so that the equilibrium point of interest lies at $x = 0$. Also exploit the arbitrary freedom to add a constant to the potential, and set $U(0) = 0$. Then if we do a Taylor expansion of $U(x)$ about $x = 0$, we'll find

$$U(x) = \overbrace{U(0)}^{0 \text{ by def'n}} + x \overbrace{U'(0)}^{0 \text{ for equilibrium}} + \frac{x^2}{2} \overbrace{U''(0)}^{>0 \text{ by stability}} + \frac{x^3}{6} U'''(0) + \dots \tag{1.5}$$

So for small enough x, i.e.,

$$x \ll \frac{3U''(0)}{U'''(0)} \tag{1.6a}$$

$$x \ll \frac{12U''(0)}{U''''(0)} \tag{1.6b}$$

$$\text{etc}$$

the potential is well approximated by

$$U(x) \approx \frac{1}{2} k x^2 \tag{1.7}$$

where

$$k = U''(0) > 0 \tag{1.8}$$

which is just the harmonic oscillator potential.

Returning to the equation of motion

$$m\ddot{x} = F(x) = -kx \tag{1.9}$$

we can write this as

$$\ddot{x} + \omega_0^2 x = 0 \tag{1.10}$$

where we have defined

$$\omega_0 = \sqrt{k/m} \tag{1.11}$$

To obtain the general solution to (1.10), we note that it's a second order linear differential equation, which means it has two important properties.

- If $x_1(t)$ and $x_2(t)$ are solutions, then the superposition $c_1 x_1(t) + c_2 x_2(t)$ is also, for any constants c_1 and c_2.
- If we have two linearly independent solutions $x_1(t)$ and $x_2(t)$, then any solution can be written as a superposition of those two.

To find the needed two independent solutions, we try a solution of the form

$$x(t) = ce^{rt} \tag{1.12}$$

differentiating gives

$$\dot{x}(t) = cre^{rt} \tag{1.13a}$$
$$\ddot{x}(t) = cr^2 e^{rt} \tag{1.13b}$$

so (1.10) becomes, for this candidate solution,

$$cr^2 e^{rt} + \omega_0^2 ce^{rt} = 0 \tag{1.14}$$

Dividing by ce^{rt}, we have

$$r^2 + \omega_0^2 = 0 \tag{1.15}$$

now, the two solutions to this are

$$r = \pm i\omega_0 \tag{1.16}$$

which would make the general solution

$$x(t) = c_+ e^{i\omega_0 t} + c_- e^{-i\omega_0 t} \tag{1.17}$$

Now, this looks funny, since it involves complex numbers, and our physical $x(t)$ will have to be real. But if we choose $c_\pm$ carefully, we can ensure that $x(t)$ is indeed real. The key is the Euler relation

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.18}$$

which allows us to write

$$x(t) = c_+(\cos\omega_0 t + i\sin\omega_0 t) + c_-(\cos\omega_0 t - i\sin\omega_0 t) = (c_+ + c_-)\cos\omega_0 t + i(c_+ - c_-)\sin\omega_0 t \tag{1.19}$$

Now, if we define new constants

$$A_c = c_+ + c_- \tag{1.20a}$$
$$A_s = i(c_+ - c_-) \tag{1.20b}$$

and require that A_c and A_s be real, we can give the general real solution

$$x(t) = A_c \cos\omega_0 t + A_s \sin\omega_0 t \tag{1.21}$$

in terms of arbitrary real constants A_c and A_s. A slightly more useful set of constants can be obtained by defining $A \geq 0$ and ϕ so that

$$A_c = A\cos\phi \tag{1.22a}$$
$$A_s = A\sin\phi \tag{1.22b}$$

with this definition, we have

$$x(t) = A\cos\omega_0 t\cos\phi + A\sin\omega_0 t\sin\phi = A\cos(\omega_0 t - \phi) \tag{1.23}$$

Note that this is at a maximum when $\omega_0 t = \phi$, and the maximum value is A. A is called the amplitude and ϕ the phase of the oscillation.

1.1 Potential and Kinetic Energy of the Simple Harmonic Oscillator

Our starting point was

$$U(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2 x^2 \tag{1.24}$$

Using the trajectory gives

$$U(x(t)) = \frac{1}{2}m\omega_0^2 A^2 \cos^2(\omega_0 t - \phi) \tag{1.25}$$

The kinetic energy is

$$T(\dot{x}) = \frac{1}{2}m\dot{x}^2 \tag{1.26}$$

differentiating the trajectory gives

$$\dot{x}(t) = -A\omega_0 \sin(\omega_0 t - \phi) \tag{1.27}$$

and so

$$T(\dot{x}(t)) = \frac{1}{2}m\omega_0^2 A^2 \sin^2(\omega_0 t - \phi) \tag{1.28}$$

Combining the two, we find the total energy

$$E = U + T = \frac{1}{2}m\omega_0^2 A^2[\cos^2(\omega_0 t - \phi) + \sin^2(\omega_0 t - \phi)] = \frac{1}{2}m\omega_0^2 A^2 \tag{1.29}$$

which is a constant, as it should be for a conservative system.

2 The Damped Harmonic Oscillator

The next level of complexity we introduce into the system is a retarding force. You might expect us to use something like kinetic friction, with an image of a block on the end of a spring sliding along a surface, but instead we add what is called viscous damping:

$$F_{\text{damping}} = -b\dot{x} \tag{2.1}$$

This is the kind of resisting force you get when moving through a viscous medium like oil or honey, and the usual physical image is to have the mass on a spring attached to some sort of apparatus which moves an object through a sealed pot of oil (this was for some reason described as chicken fat when I was a student), generating the damping force described by (2.1).

This is not a terribly common sort of damping in what you'd think of as traditional mechanical systems, but it does come up in more complicated oscillations, and it's the natural resisting term in the analogous electric circuit. Of course, the real reason we talk about it here is that it's linear in the velocity, so it keeps the differential equation linear.

Putting together the restoring force and the damping force, the one-dimensional equation of motion becomes

$$m\ddot{x} = -b\dot{x} - kx \tag{2.2}$$

As before, we divide by m and define the natural frequency

$$\omega_0 = \sqrt{k/m} \tag{2.3}$$

We also define a damping parameter with units of inverse time

$$\beta = \frac{b}{2m} \tag{2.4}$$

Note the factor of two, which will make things more convenient later. The ODE becomes

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{2.5}$$

This is another second order linear differential equation, so we apply the same strategy to find two independent solutions, guessing the form

$$x(t) = ce^{rt} \tag{2.6}$$

This makes the differential equation

$$cr^2e^{rt} + 2\beta rce^{rt} + \omega_0^2 ce^{rt} = 0 \tag{2.7}$$

Again, we can divide by ce^{rt} to get

$$r^2 + 2\beta r + \omega_0^2 = 0 \tag{2.8}$$

Applying the quadratic equation gives the solutions

$$r_\pm = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \tag{2.9}$$

If $\beta \neq \omega_0$ this will give us our two independent solutions; we'll handle the $\beta = \omega_0$ case later. Clearly, the nature of the two solutions will depend on the sign of $\beta^2 - \omega_0^2$. We give names to the three situations as follows

$\beta^2 - \omega_0^2 < 0$	Underdamped
$\beta^2 - \omega_0^2 = 0$	Critically damped
$\beta^2 - \omega_0^2 > 0$	Overdamped

2.1 Underdamped Oscillations

In this case, $\sqrt{\beta^2 - \omega_0^2}$ is an imaginary number, so we define

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2} \tag{2.10}$$

so that

$$r_\pm = -\beta \pm i\omega_1 \tag{2.11}$$

Now the general solution is

$$x(t) = c_+e^{r_+t} + c_-e^{r_-t} = c_+e^{-\beta t}e^{i\omega_1 t} + c_-e^{-\beta t}e^{-i\omega_1 t} \tag{2.12}$$

Again, we use the Euler relation to write

$$\begin{aligned} x(t) =& c_+e^{-\beta t}(\cos\omega_1 t + i\sin\omega_1 t) + c_-e^{-\beta t}(\cos\omega_1 t - i\sin\omega_1 t) \\ =&(c_+ + c_-)e^{-\beta t}\cos\omega_1 t + i(c_+ - c_-)e^{-\beta t}\sin\omega_1 t \end{aligned} \tag{2.13}$$

and define new real constants

$$A_c = c_+ + c_- \tag{2.14a}$$

$$A_s = i(c_+ - c_-) \tag{2.14b}$$

which gives us a solution

$$x(t) = A_c e^{-\beta t}\cos\omega_1 t + A_s e^{-\beta t}\sin\omega_1 t \tag{2.15}$$

And as before we can define A and ϕ by

$$A_c = A\cos\phi \tag{2.16a}$$

$$A_s = A\sin\phi \tag{2.16b}$$

which gives us a general solution

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \phi) \tag{2.17}$$

Note that this differs from the solution from the undamped oscillator only in that the oscillations are multiplied by the decaying exponential $e^{-\beta t}$, and in that the oscillation frequency is

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2} \tag{2.18}$$

In the limit $\beta \to 0$, $\omega_1 \to \omega_0$ and we get back the undamped solution, as we must.

2.1.1 Example: Choosing a Particular Solution to Match Initial Conditions

To give an example of how to set the values of A and ϕ if we're given initial conditions for the problem, suppose we're told to consider an underdamped oscillator with spring constant $m\omega_0^2$ and damping parameter $2m\beta$, which is released from rest at a position x_0 away from equilibrium. We know the general solution is given by (2.17), and its derivative is

$$\dot{x}(t) = -\beta Ae^{-\beta t}\cos(\omega_1 t - \phi) - \omega_1 Ae^{-\beta t}\sin(\omega_1 t - \phi) \tag{2.19}$$

That means that the general solution has

$$x(0) = A\cos(-\phi) = A\cos\phi \tag{2.20a}$$

$$\dot{x}(0) = -\beta A\cos(-\phi) - \omega_1 A\sin(-\phi) = A(\omega_1\sin\phi - \beta\cos\phi) \tag{2.20b}$$

so we need to determine A and ϕ from

$$x_0 = A\cos\phi \tag{2.21a}$$

$$0 = A(\omega_1\sin\phi - \beta\cos\phi) \tag{2.21b}$$

Equation (2.21b) tells us

$$\omega_1 \sin\phi = \beta \cos\phi \tag{2.22}$$

or

$$\tan\phi = \frac{\beta}{\omega_1} \tag{2.23}$$

Now, to substutite this back into (2.21a), we need to define $\cos\phi$ in terms of $\tan\phi$. We can do this by using one of the only three trig identities we need to memorize:

$$\cos^2\phi + \sin^2\phi = 1 \tag{2.24}$$

We divide both sides by $\cos^2\phi$ to get

$$1 + \tan^2\phi = \frac{1}{\cos^2\phi} \tag{2.25}$$

or

$$\cos\phi = \frac{1}{\pm\sqrt{1+\tan^2\phi}} \tag{2.26}$$

we're justified in taking the positive square root in this case, because we can take A to have the same sign as x_0 in (2.21a).

Now, consider

$$\sqrt{1+\tan^2\phi} = \sqrt{1+\frac{\beta^2}{\omega_1^2}}\ . \tag{2.27}$$

For the most part, life is simplified by not writing out $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ explicitly, but here it's squared, so it actually helps a little:

$$\begin{aligned}\frac{1}{\cos\phi} &= \sqrt{1+\tan^2\phi} = \sqrt{1+\frac{\beta^2}{\omega_1^2}} = \sqrt{1+\frac{\beta^2}{\omega_0^2-\beta^2}} = \sqrt{\frac{\omega_0^2-\beta^2}{\omega_0^2-\beta^2}+\frac{\beta^2}{\omega_0^2-\beta^2}} = \sqrt{\frac{\omega_0^2}{\omega_0^2-\beta^2}} \\ &= \frac{\omega_0}{\omega_1}\ .\end{aligned} \tag{2.28}$$

So this means

$$\cos\phi = \frac{\omega_1}{\omega_0} \tag{2.29}$$

(As a sanity check, note that by definition $\omega_1 \leq \omega_0$.) It's then easy to see that

$$\sin\phi = \tan\phi\cos\phi = \frac{\beta}{\omega_0} \tag{2.30}$$

We can now substitute into (2.21a) to get

$$x_0 = A\cos\phi = A\frac{\omega_0}{\omega_1} \tag{2.31}$$

or

$$A = \frac{\omega_0}{\omega_1}x_0 \tag{2.32}$$

Now, we know A and ϕ, so we can substitute them into (2.17), but the answer will be a little awkward in the form

$$x(t) = x_0 \frac{\omega_0}{\omega_1} e^{-\beta t} \cos\left(\omega_1 t - \tan^{-1}\left[\frac{\beta}{\omega_1}\right]\right) \ ; \tag{2.33}$$

since we know $\cos\phi$ and $\sin\phi$, it's nicer to write

$$\begin{aligned} x(t) =& A\cos(\omega_0 t - \phi) = A(\cos\omega_1 t\cos\phi + \sin\omega_1 t\sin\phi) = x_0 \frac{\omega_0}{\omega_1} e^{-\beta t}\left(\frac{\omega_1}{\omega_0}\cos\omega_1 t + \frac{\beta}{\omega_0}\sin\omega_1 t\right) \\ =& x_0 e^{-\beta t}\left(\cos\omega_1 t + \frac{\beta}{\omega_1}\sin\omega_1 t\right) \ . \end{aligned} \tag{2.34}$$

Of course, this makes us see in retrospect that it would have been easier to start from the form (2.15), but we did learn some interesting things along the way about the phase angle in this case.

Finally, since the statement of the problem talked about ω_0 and β, but not ω_1, we should be sure to define what we mean by ω_1, when presenting the answer, so we say:

$$x(t) = x_0 e^{-\beta t}\left(\cos\omega_1 t + \frac{\beta}{\omega_1}\sin\omega_1 t\right) \qquad \text{where } \omega_1 = \sqrt{\omega_0^2 - \beta^2} \tag{2.35}$$

2.2 Overdamped Oscillations

Turning to another general class of solution, consider the case when $\beta^2 - \omega_0^2 > 0$. In this case the two roots

$$r_\pm = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \tag{2.36}$$

are real. If we define

$$\beta_1 = \sqrt{\beta^2 - \omega_0^2} < \beta \tag{2.37}$$

we have

$$r_\pm = -(\beta \mp \beta_1) < 0 \tag{2.38}$$

and the general solution

$$x(t) = c_+ e^{-(\beta-\beta_1)t} + c_- e^{-(\beta+\beta_1)t} \tag{2.39}$$

This does not oscillate, but has two terms which go to zero at different rates. (It can, however, change sign once or twice if c_+ and c_- have different signs.)

An overdamped oscillator has "too much" damping in the sense that if it starts off out of equilibrium, the damping force can resist the motion so much that it takes a long time to get back to equilibrium. An example would be a door that takes forever to close because the pneumatic cylinder provides too much resistance.

2.3 Critically Damped Oscillations

We turn at last to the special case $\beta = \omega_0$. In this case our attempt to find two independent solutions of the form

$$x(t) = ce^{rt} \tag{2.40}$$

fails, because r solves the quadratic equation

$$0 = r^2 + 2\beta r + \beta^2 = (r + \beta)^2 \tag{2.41}$$

which has only one solution, $r = \beta$

It's thus necessary to cast a wider net in looking for a pair of independent solutions, and it turns out that what works is

$$x(t) = (c_0 + c_1 t)e^{-\beta t} \tag{2.42}$$

We can verify that this is a solution for all c_0 and c_1; differentiating gives

$$\dot{x}(t) = (-\beta c_0 + c_1 - c_1 \beta t)e^{-\beta t} \tag{2.43}$$

and differentiating again gives

$$\ddot{x}(t) = (\beta^2 c_0 - \beta c_1 - c_1 \beta + c_1 \beta^2 t)e^{-\beta t} \tag{2.44}$$

So that

$$\ddot{x} + 2\beta\dot{x} + \beta^2 = \big(c_0(\beta^2 - 2\beta^2 + \beta^2) + c_1(\beta^2 t + 2\beta - 2\beta^2 t - 2\beta + \beta^2 t)\big) = 0 \tag{2.45}$$

3 The Forced, Damped Harmonic Oscillator

To add the last major feature to the equation we've been studying this chapter, we add an exernal driving force of the form

$$F_{\text{driving}} = F_0 \cos\omega t \tag{3.1}$$

with some given amplitude F_0 and frequency ω. (The choice of cosine rather than sine is somewhat arbitrary, but as we saw on the exam, the phase of the trig function can be chosen by making a convenient choice of the origin of the time coördinate.) The equation of motion for the oscillator is now

$$m\ddot{x} = F_{\text{Hooke}} + F_{\text{damping}} + F_{\text{driving}} = -kx - b\dot{x} + F_0 \cos\omega t \tag{3.2}$$

defining $\omega_0 = \sqrt{k/m}$ and $\beta = b/2m$ as before, and also

$$A_{\text{in}} = \frac{F_0}{m\omega_0^2} \tag{3.3}$$

we get the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \omega_0^2 A_{\text{in}} \cos\omega t \tag{3.4}$$

A_{in} is a nice measure of the amplitude of the driving force, since it's the amplitude with which the oscillator would be displaced if the force were applied with $\omega = 0$. It's also got units of length, so it can be directly compared to the output with which the oscillator actually oscillates.

3.1 Nature of the Differential Equation

The differential equation (3.4) is unlike those we've considered so far, in that it is *not* a homogeneous linear equation; it has terms which are not linear in $x(t)$ or its derivatives, so for example $x(t) = 0$ is not a solution. The good news is, the left-hand side is still linear, so we can make up the general solution to (3.4) out of a general solution $x_c(t)$ to the corresponding homogeneous equation and *any* function $x_p(t)$ which solves (3.4):

$$x(t) = x_p(t) + x_c(t) \tag{3.5}$$

where

$$\ddot{x}_c + 2\beta\dot{x}_c + \omega_0^2 x_c = 0 \tag{3.6a}$$

$$\ddot{x}_p + 2\beta\dot{x}_p + \omega_0^2 x_p = \omega_0^2 A_{\text{in}} \cos\omega t \tag{3.6b}$$

We've already found a two-parameter family of solutions to (3.6a); it's just the general solution for the damped harmonic oscillator, so if we find any solution to (3.6a), their sum will be a solution to (3.4) with two specifiable constants which can be used to match initial conditions.

3.2 Solving the Differential Equation

In the usual tradition of Physicists solving differential equations, we guess an answer and see if it works. A reasonable thing to try is an oscillating solution; we know that if the driving force keeps going forever, the oscillator will never settle down to zero displacement, so we should try a solution which just oscillates and doesn't decay; and since the driving force is what's keeping it going, let's assume it oscillates at that frequency, but not necessarily in phase. The solution we try is thus

$$x_p(t) = A_{\text{out}} \cos(\omega t - \delta) \tag{3.7}$$

The output A_{out} and phase offset δ are not arbitrary; we need to figure out which values are needed for (3.7) to be a solution to (3.6b).

Now, we could calculate $\dot{x}_p$ and $\ddot{x}_p$ and substitute those into (3.6b) (which is done in the book) to find out δ and A_{out}, but it turns out the math is easier if we play a little trick and use the Euler relation to write

$$x_p(t) = A_{\text{out}} \frac{e^{i(\omega t-\delta)} - e^{-i(\omega t-\delta)}}{2} = \underbrace{\frac{A_{\text{out}}e^{-i\delta}}{2} e^{i\omega t}}_{x_+(t)} + \underbrace{\frac{A_{\text{out}}e^{i\delta}}{2} e^{-i\omega t}}_{x_-(t)} \tag{3.8}$$

and

$$\ddot{x}_p + 2\beta\dot{x}_p + \omega_0^2 x_p = \omega_0^2 A_{\text{in}} \frac{e^{i\omega t} - e^{-i\omega t}}{2} \tag{3.9}$$

which becomes

$$\ddot{x}_+ + 2\beta\dot{x}_+ + \omega_0^2 x_+ + \ddot{x}_- + 2\beta\dot{x}_- + \omega_0^2 x_- = \frac{\omega_0^2 A_{\text{in}}}{2} e^{i\omega t} + \frac{\omega_0^2 A_{\text{in}}}{2} e^{-i\omega t} \tag{3.10}$$

where

$$x_\pm = \frac{A_{\text{out}} e^{\mp i\delta}}{2} e^{\pm i\omega t} \tag{3.11}$$

We will have a solution if A_{out} and δ are chosen so that both of the equations represented by

$$\ddot{x}_\pm + 2\beta\dot{x}_\pm + \omega_0^2 x_\pm = \frac{\omega_0^2 A_{\text{in}}}{2} e^{\pm i\omega t} \tag{3.12}$$

are independently satisfied. It may seem that we've traded in one real equation for two complex ones, but note that by construction x_+ and x_- are complex conjugates, and that the equations we're requiring them to satisfy are complex conjugates of each other. So we can just solve one of them and the other will automatically be satisfied.

We concentrate on x_+, noting that

$$x_+ = \frac{A_{\text{out}} e^{-i\delta}}{2} e^{i\omega t} \tag{3.13a}$$

$$\dot{x}_+ = i\omega \frac{A_{\text{out}} e^{-i\delta}}{2} e^{i\omega t} \tag{3.13b}$$

$$\ddot{x}_+ = -\omega^2 \frac{A_{\text{out}} e^{-i\delta}}{2} e^{i\omega t} \tag{3.13c}$$

Substituting this into the "plus" version of (3.12), we find

$$(-\omega^2 + 2i\beta\omega + \omega_0^2) \frac{A_{\text{out}} e^{-i\delta}}{2} e^{i\omega t} = \frac{\omega_0^2 A_{\text{in}}}{2} e^{i\omega t} \tag{3.14}$$

or, cancelling out the $e^{i\omega t}/2$ and rearranging,

$$\left[(\omega_0^2 - \omega^2) + 2i\beta\omega\right] A_{\text{out}} e^{-i\delta} = \omega_0^2 A_{\text{in}} \tag{3.15}$$

Now, (3.15) is a complex equation, which states that one complex number is equal to another. Recall that if $z = x + iy$ and $w = u + iv$ are two complex numbers, the complex equation

$$z = w \tag{3.16}$$

is equivalent to the two real equations

$$x = u \tag{3.17a}$$

$$y = v \tag{3.17b}$$

i.e., if two complex expressions are equal, then the real parts are equal and the imaginary parts are equal. Another real equation, which is not independent of the other two, is that the magnitude-squared of the two equations is equal:

$$z^* z = x^2 + y^2 = u^2 + v^2 = w^* w \tag{3.18}$$

In this case, the most useful pair of equations is the equality of the imaginary parts (which A_{out} drops out of) and of the squared magnitudes (which doesn't involve δ).

3.2.1 Determination of the Amplitude

The most direct way to find the amplitude A_{out} is to take the square of the magnitude of each side of (3.15) by multiplying it by its complex conjugate:

$$\left\{\left[(\omega_0^2-\omega^2)+2i\beta\omega\right]A_{\text{out}}e^{-i\delta}\right\}\left\{\left[(\omega_0^2-\omega^2)-2i\beta\omega\right]A_{\text{out}}e^{i\delta}\right\}=\left\{\omega_0^2A_{\text{in}}\right\}\left\{\omega_0^2A_{\text{in}}\right\} \tag{3.19}$$

or

$$\left[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2\right]{A_{\text{out}}}^2=\omega_0^4{A_{\text{in}}}^2 \tag{3.20}$$

which tells us

$$A_{\text{out}}=A_{\text{in}}\frac{\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}} \tag{3.21}$$

We have chosen to take the positive square root, but this is a reasonable thing to do, since it just means we choose A_{out} to have the same sign as A_{in}. We'll see that there is always a choice of δ which makes this work.

3.2.2 Determination of the Phase

To find δ, we concentrate on the imaginary part of (3.15), which can be extracted by writing it as

$$\begin{aligned}\omega_0^2A_{\text{in}}&=\left[(\omega_0^2-\omega^2)+2i\beta\omega\right]A_{\text{out}}(\cos\delta-i\sin\delta)\\&=(\omega_0^2-\omega^2)\cos\delta+2\beta\omega\sin\delta+i\left[2\beta\omega\cos\delta-(\omega_0^2-\omega^2)\sin\delta\right]A_{\text{out}}\end{aligned} \tag{3.22}$$

the imaginary part of the left-hand side vanishes, so setting the imaginary part of the right-hand side to zero gives

$$\tan\delta=\frac{2\beta\omega}{\omega_0^2-\omega^2} \tag{3.23}$$

Now, knowing the tangent of an angle only tells us that angle modulo π (because $\tan(\theta+\pi)=\cos(\theta+\pi)/sin(\theta+\pi)=(-\cos\theta)/(-\sin\theta)=\tan\theta$) so we should pause and make sure we know which branch of the arctangent we want to take when we calculate δ. Fortunately, once we figure out the value of δ for $\omega=0$, we can vary ω smoothly, and since we can see from (3.21) that the amplitude A_{out} doesn't go through zero for any ω, we don't have to worry about δ suddenly jumping by π.

At $\omega=0$, (3.23) tells us that $\tan\delta=0$. This means $\sin\delta=0$ and $\delta=0$ or $\delta=\pi$ (we'll choose δ to lie between $-\pi$ and π, since adding 2π to the phase angle doesn't affect anything). But if we look at the real part of (3.15), we see that for $\omega=0$

$$\omega_0^2A_{\text{in}}=2\beta\omega\cos\delta A_{\text{out}} \tag{3.24}$$

Since we have chosen A_{out} to have the same sign as A_{in}, $\cos\delta$ has to be 1 when $\omega=0$, rather than -1. Thus the phase lag is zero when $\omega=0$.

Armed with this knowledge, we can follow the behavior of δ as a function of ω:

$\omega=0$	$\sin\delta=0$	$\cos\delta=1$	$\delta=0$
$0<\omega<\omega_0$	$0<\sin\delta<1$	$0<\cos\delta<1$	$0<\delta<\pi/2$
$\omega=\omega_0$	$\sin\delta=1$	$\cos\delta=0$	$\delta=\pi/2$
$\omega>\omega_0$	$0<\sin\delta<1$	$-1<\cos\delta<0$	$\pi/2<\delta<\pi$
$\omega\to\infty$	$\sin\delta\to0$	$\cos\delta\to-1$	$\delta\to\pi$

This means that $\sin\delta$ always has the same sign as ω and $\cos\delta$ always has the same sign as $\omega_0^2 - \omega^2$, which allows us to use (3.23) and the fact that $\sin^2\delta + \cos^2\delta = 1$ to write

$$\sin\delta = \frac{2\beta\omega}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}} \tag{3.25a}$$

$$\cos\delta = \frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}} \tag{3.25b}$$

3.3 Analysis of the Solution: Amplitude Resonance

Consider the behavior of the output amplitude (3.21) as a function of the driving frequency ω:

- At $\omega = 0$, $A_{\text{out}} = A_{\text{in}}$,
- When $\omega \to \infty$, $A_{\text{out}} \to 0$

To see what happens in between, we take the derivative $\frac{\partial A_{\text{out}}}{\partial\omega}$ at constant A_{in} to see if there is a maximum output amplitude at some frequency:

$$\frac{\partial A_{\text{out}}}{\partial\omega} = -\frac{1}{2}A_{\text{in}}\frac{\omega_0^2}{\left[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2\right]^{3/2}}\left[-4\omega(\omega_0^2-\omega^2)+8\beta^2\omega\right] \tag{3.26}$$

This vanishes when

$$\omega_0^2 - \omega^2 = 2\beta^2 \tag{3.27}$$

i.e.,

$$\omega^2 = \omega_0^2 - 2\beta^2 \tag{3.28}$$

If $2\beta^2 \geq \omega_0^2$, there is no maximum for positive ω, and the output amplitude simply decreases with increasing frequency.

If $2\beta^2 < \omega_0^2$, the amplitude increases with frequency until it reaches a maximum, then decreases from there on. The maximum occurs at a frequency known as the *resonant frequency*.

$$\omega_R := \sqrt{\omega_0^2 - 2\beta^2} \tag{3.29}$$

If we expand out the contents of the square root in the denominator of (3.21), we see that

$$(\omega_0^2-\omega^2)^2+4\beta^2\omega^2 = \omega_0^4 - 2\omega^2(\omega_0^2-2\beta^2)+\omega^4 = \omega_0^4 - 2\omega^2\omega_R^2+\omega^4 \tag{3.30}$$

and so

$$A_{\text{out}} = A_{\text{in}}\frac{\omega_0^2}{\sqrt{\omega_0^4-2\omega^2\omega_R^2+\omega^4}} \tag{3.31}$$

The maximum value of $A_{\text{out}}/A_{\text{in}}$, which occurs when $\omega = \omega_R$, is

$$\left(\frac{A_{\text{out}}}{A_{\text{in}}}\right)_{\text{max}} = \frac{\omega_0^2}{\sqrt{\omega_0^4-\omega_R^4}} = \frac{\omega_0^2}{\sqrt{\omega_0^4-(\omega_0^4-4\beta^2\omega_0^2+4\beta^4)}} = \frac{\omega_0^2}{2\beta\sqrt{\omega_0^2-\beta^2}} = \frac{\omega_0^2}{2\beta\omega_1} \tag{3.32}$$

Now, in the limit $\beta \ll \omega_0$, when $\omega_1 \approx \omega_0$ (to lowest order in β/ω_0), this is approximately

$$\frac{\omega_0}{2\beta} \tag{3.33}$$

which is, in the same limit, equal to something called the *quality factor* Q. While everyone agrees on the definition of Q in this limit, there seems to be some difference in conventions as to what happens when β is comparible to ω_0 and Q is thus small. The book chooses the definition

$$Q = \frac{\omega_R}{2\beta} \tag{3.34}$$

which has the advantage that $Q \to 0$ when the damping becomes so small that there is no resonance.

4 Linear Superposition and Fourier Methods

We've solved the problem of a forced damped harmonic oscillator where the driving force is a sinusoid of a fixed frequency. You might worry that this is not terribly general, and we'd have to solve the equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \omega_0^2 A^{\text{in}}(t) \tag{4.1}$$

over and over again for different external driving forces. Fortunately, there are two facts which mean that we can reuse our answer from the sinusoidal case in the presence of a wide variety of driving forces.

1. Linear superposition. The differential equation (4.1) contains a linear differential operator

   $$\mathcal{L} = \frac{d^2}{dt^2} + 2\beta\frac{d}{dx} + \omega_0^2 \tag{4.2}$$

   The linearity means

   $$\mathcal{L}(x_1(t) + x_2(t)) = \mathcal{L}x_1(t) + \mathcal{L}x_2(t) \tag{4.3}$$

   This is just the principle that we used to allow us to write the general solution of the inhomogeneous equation (3.4) as a superposition of the general solution to the inhomogeneous equation (2.5) and any solution to the inhomogeneous equation. That means that if we break up the driving force $A^{\text{in}}(t)$ into two terms:

   $$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \omega_0^2(A_1^{\text{in}}(t) + A_2^{\text{in}}(t)) \tag{4.4}$$

   then the general solution will be a sum of the complementary function $x_c(t)$ and steady-state solutions corresponding to the two components of the driving force:

   $$x(t) = x_c(t) + x_1(t) + x_2(t) \tag{4.5}$$

   where

   $$\ddot{x}_{1,2} + 2\beta\dot{x}_{1,2} + \omega_0^2 x_{1,2} = \omega_0^2 A_{1,2}^{\text{in}}(t) \tag{4.6}$$

So if the driving force is a sum of sines and cosines, we can write down the solution. For instance, if

$$A^{\text{in}}(t) = x_0 \cos\omega t + 4x_0 \sin 3\omega t \tag{4.7}$$

then the steady-state solution is

$$x_p(t) = \frac{\omega_0^2 x_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \cos(\omega t - \delta_1) + \frac{4\omega_0^2 x_0}{\sqrt{(\omega_0^2 - 9\omega^2)^2 + 36\beta^2\omega^2}} \sin(3\omega t - \delta_2) \tag{4.8}$$

where

$$\delta_1 = \tan^{-1} \frac{2\beta\omega}{\omega_0^2 - \omega^2} \tag{4.9a}$$

$$\delta_2 = \tan^{-1} \frac{6\beta\omega}{\omega_0^2 - 36\omega^2} \tag{4.9b}$$

$$\tag{4.9c}$$

2. By using Fourier series and Fourier transforms, *any* driving force can be written as an infinite sum of sines and cosines.

4.1 Periodic Driving Forces (Fourier Series)

Consider first the case where the driving force has some periodicity $A^{\text{in}}(t) = A^{\text{in}}(t+T)$. Not every sine or cosine will have this periodicity, just those with angular frequencies

$$\omega_n = \frac{2\pi n}{T} \tag{4.10}$$

The methods of Fourier series (see supplemental exercises) show us that any periodic function can be written as an infinite series of sines and cosines:

$$A^{\text{in}}(t) = \frac{a_0^{\text{in}}}{2} + \sum_{n=0}^{\infty} a_n^{\text{in}} \cos\omega_n t + \sum_{n=0}^{\infty} b_n^{\text{in}} \sin\omega_n t \tag{4.11}$$

with the coëfficients given by

$$a_n^{\text{in}} = \frac{2}{T} \int_{-T/2}^{T/2} A^{\text{in}}(t) \cos\omega_n t \, dt \tag{4.12a}$$

$$b_n^{\text{in}} = \frac{2}{T} \int_{-T/2}^{T/2} A^{\text{in}}(t) \sin\omega_n t \, dt \tag{4.12b}$$

By superposition, the steady-state solution to (4.1) is thus

$$\begin{aligned} x_c(t) =& \frac{1}{2} a_0^{\text{in}} \sum_{n=1}^{\infty} \frac{\omega_0^2 a_n^{\text{in}}}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\beta^2\omega_n^2}} \cos(\omega_n t - \delta_n) \\ &+ \sum_{n=1}^{\infty} b_n^{\text{out}} \frac{\omega_0^2 b_n^{\text{in}}}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\beta^2\omega_n^2}} \sin(\omega_n t - \delta_n) \end{aligned} \tag{4.13}$$

where

$$\delta_n = \tan^{-1} \frac{2\beta\omega_n}{\omega_0^2 - \omega_n^2} \tag{4.14}$$

Note that

- We have used the zero-frequency behavior to write $\delta_0 = 0$ and $a_0^{\text{out}} = a_0^{\text{in}}$; in fact this means the constant term just effectively shifts the equilibrium position.
- To write (4.13) as a traditional Fourier series, we need to use the angle difference formulas to write e.g.,

$$\cos(\omega_n t - \delta_n) = \cos\delta_n \cos\omega_n t + \sin\delta_n \sin\omega_n t \tag{4.15}$$

and then reorganize the sums to give

$$x_c(t) = \frac{a_0^{\text{out}}}{2} + \sum_{n=0}^{\infty} a_n^{\text{out}} \cos\omega_n t + \sum_{n=0}^{\infty} b_n^{\text{out}} \sin\omega_n t \tag{4.16}$$

Excercise: work out the general formula for a^{out} and b^{out}.

A Appendix: Correspondence to Class Lectures

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