Gravitation
(Marion & Thornton Chapter Five)

Physics A300*

Fall 2002

1 Gravitational Force Between Two Point Masses

1. Proportional to the product of the masses
2. Inversely proportional to the square of the distance between them
3. Attractive and directed on a line from one mass to the other

Mathematically, write the force on mass 2 due to mass 1 as

$$\vec{F}_{21} = -G\frac{m_1 m_2}{r_{21}^2}\vec{e}_{21} \tag{1.1}$$

Compare electrostatics

$$\vec{F}_{21} = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r_{21}^2}\vec{e}_{21} \tag{1.2}$$

Correspondence and differences:

1. Coupling constant G vs $\frac{1}{4\pi\epsilon_0}$
2. Attractive vs repulsive for like charges
3. mass vs electric charge

Important features:

1. Satisfies Newton's third law

$$\vec{F}_{12} = -G\frac{m_2 m_1}{r_{12}^2}\vec{e}_{12} = G\frac{m_1 m_2}{r_{21}^2}\vec{e}_{21} = -\vec{F}_{21} \tag{1.3}$$

2. Because the gravitational "charge" is just the same as the inertial mass, the acceleration experience by particle 2 is

$$\vec{a}_2 = \frac{\vec{F}_{12}}{m_2} = -G\frac{m_1}{r_{21}^2}\vec{e}_{21} \tag{1.4}$$

which depends on particle 2's location but not any properties of the particle itself. This is called the *Equivalence Principle* and was instrumental in the development of Einstein's General Theory of Relativity, which describes gravity in terms of the geometry of spacetime.

In contrast, the situation in electrostatics is that the acceleration depends on the charge-to-mass ratio of the particle:

$$\vec{a}_2 = \frac{\vec{F}_{12}}{m_2} = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{r_{21}^2}\left(\frac{Q_2}{m_2}\right)\vec{e}_{21} \tag{1.5}$$

3. The coupling constant has been numerically determined as

$$G = 6.673 \times 10^{-11}\frac{\text{m}^3}{\text{kg}\,\text{s}^2} \tag{1.6}$$

We can check the units on this by considering two one-kilogram masses located one meter away from each other:

$$\left|\vec{F}\right| = 6.673 \times 10^{-11}\frac{\text{m}^3}{\text{kg}\,\text{s}^2}\frac{1\,\text{kg} \times 1\,\text{kg}}{(1\,\text{m})^2} = 6.673 \times 10^{-11}\frac{\text{kg}\,\text{m}}{\text{s}^2} = 6.673 \times 10^{-11}\,\text{N} \tag{1.7}$$

We see not only that Newton's constant G has the right units, but that the gravitational force between everyday objects is very small. This is why it is still difficult to determine Newton's constant to the same accuracy as other constants of nature. Most things which are big enough to exert an appreciable gravitational force (like moons, planets, and stars) are too big to have their masses directly compared to everyday objects whose masses we know in kilograms. So we know the combination GM pretty well for things like the Earth and the Sun, from e.g., Kepler's third law, but to get an independent measure of G (and hence of the masses of planet-sized things) required sophisticated experiments to measure the gravitational forces exerted by laboratory-sized objects.

We've defined the gravitational force in terms of the geometrical quantities r_{21} and $\vec{e}_{21}$. We'd like to take a moment to define those quantities with respect to the position vectors

$$\vec{x}_1 = x_1\vec{e}_x + y_1\vec{e}_y + z_1\vec{e}_z \tag{1.8a}$$
$$\vec{x}_2 = x_2\vec{e}_x + y_2\vec{e}_y + z_2\vec{e}_z \tag{1.8b}$$

(**NOTE:** The superscripts 1 and 2 here are labels which tell us which of the two particles we're talking about, and *not* indices referring to different components. We'll be sure to use x, y, and z rather than 1, 2, and 3 to label components, to minimize confusion.)

The displacement vector from object 1 to object 2 is

$$\vec{x}_{21} = \vec{x}_2 - \vec{x}_1 = (x_2 - x_1)\vec{e}_x + (y_2 - y_1)\vec{e}_y + (z_2 - z_1)\vec{e}_z \tag{1.9}$$

The distance r_{21} between the objects is the length of this vector

$$r_{21} = |\vec{x}_2 1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{1.10}$$

The unit vector pointing from 1 to 2 can be obtained from

$$\vec{x}_{21} = r_{21}\vec{e}_{21} \tag{1.11}$$

as

$$\vec{e}_{21} = \frac{\vec{x}_{21}}{r_{21}} = \frac{(x_2 - x_1)\vec{e}_x + (y_2 - y_1)\vec{e}_y + (z_2 - z_1)\vec{e}_z}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}} \tag{1.12}$$

So we can write the gravitational force in gory detail as

$$\vec{F}_{21} = -G\frac{m_1 m_2}{r_{21}^2}\vec{e}_{21} = -Gm_1 m_2 \frac{(x_2 - x_1)\vec{e}_x + (y_2 - y_1)\vec{e}_y + (z_2 - z_1)\vec{e}_z}{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{3/2}} \tag{1.13}$$

1.1 Gravitational Field and Potential (Point Source)

The fact that the gravitational force on its object is always proportional to its mass allows us to define the acceleration that any particle will experience due to a given object's gravitational force on that particle.

In the case of a point-like object, we can gain some insight into this quantity by choosing our origin of coördinates at the location of the point source (object 1), so that $\vec{x}_1 = 0$. This makes the force on object 2

$$\vec{F}_{21} = -Gm_1 m_2 \frac{x_2\vec{e}_x + y_2\vec{e}_y + z_2\vec{e}_z}{(x_2^2 + y_2^2 + z_2^2)^{3/2}} \tag{1.14}$$

Renaming $m_1 \to M$, $m_2 \to m$ and $\vec{x}_2 \to \vec{x}$ makes things look a little cleaner:

$$\vec{F} = -GMm\frac{\vec{x}}{r^3} = -GMm\frac{\vec{e}_r}{r^2} \tag{1.15}$$

The acceleration of any mass m at a position $\vec{x}$ due to a mass M fixed at the origin is thus

$$\vec{g} = -\frac{GM}{r^2}\vec{e}_r \tag{1.16}$$

(Aside: We usually talk about the acceleration of a "test particle" of arbitrarily small mass, since a real particle of finite mass m would also attract the first object gravitationally, and cause the assumption that it's fixed to require an appeal to some outside force holding it in place.)

Now, this is a vector field, and it happens to be a conservative field. We could show this by calculating $\vec{\nabla} \times \vec{g}$ and showing it vanishes, but it's simpler just to show directly that there is a φ such that $\vec{g} = \nabla\varphi$. We do this by first calculating

$$\vec{\nabla} r = \vec{e}_x\frac{\partial r}{\partial x} + \vec{e}_y\frac{\partial r}{\partial y} + \vec{e}_z\frac{\partial r}{\partial z} \tag{1.17}$$

The x component is

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\frac{\partial}{\partial x}(x^2+y^2+z^2) = \frac{1}{2}(r^2)^{-1/2}2x = \frac{x}{r} \quad (1.18a)$$

and by identical calculations

$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad (1.18b)$$

$$\frac{\partial r}{\partial z} = \frac{z}{r} \quad (1.18c)$$

This means that

$$\vec{\nabla} r = \vec{e}_x \frac{x}{r} + \vec{e}_y \frac{y}{r} + \vec{e}_z \frac{z}{r} = \frac{\vec{x}}{r} = \vec{e}_r \quad (1.19)$$

This then tells us that

$$\vec{g} = -\frac{GM}{r^2}\vec{\nabla} r = -\vec{\nabla}\left(-\frac{GM}{r}\right) \quad (1.20)$$

Which allows us to define the *gravitational potential*

$$\varphi = -\frac{GM}{r} \quad (1.21)$$

Just as the gravitational field is the force on an object per unit mass, the gravitational potential is its energy per unit mass.

Returning to general coördinates, we see that the gravitational potential at a location $\vec{x}_2$ due to a mass m_1 located at $\vec{x}_1$ is

$$\varphi(\vec{x}_2) = -\frac{Gm_1}{r_{21}} = -\frac{Gm_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}} \quad (1.22)$$

2 Gravitational Effects of a Distribution of Mass

So far we've been talking about the force on one mass due to one other mass, and it's been convenient to use subscript labels of 1 and 2, but we're about to start considering what happens in the presence of a lot of masses, so we'll change the notation a bit. Consider the effect on a mass m located at a point $\vec{x} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$ due to a mass M located at a point $\vec{x}' = x'\vec{e}_x + y'\vec{e}_y + z'\vec{e}_z$. $\vec{x}$, which plays the role of $\vec{x}_2$ in what we've done so far, is called the *field point*, while $\vec{x}'$, which we were previously calling $\vec{x}_1$, is called the *source point*. Translating (1.13) into this notation gives us the force on the test mass m located at the field point $\vec{x}$:

$$\vec{F} = -G\frac{Mm}{|\vec{x}-\vec{x}'|^2}\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|} = -GMm\frac{(x-x')\vec{e}_x+(y-y')\vec{e}_y+(z-z')\vec{e}_z}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}} \quad (2.1)$$

the acceleration it experiences is just the gravitational field at the point $\vec{x}$:

$$\vec{a} = \frac{\vec{F}}{m} = \vec{g}(\vec{x}) = -G\frac{M}{|\vec{x}-\vec{x}'|^2}\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|} = -GM\frac{(x-x')\vec{e}_x+(y-y')\vec{e}_y+(z-z')\vec{e}_z}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}} \quad (2.2)$$

the potential field satisfying

$$\vec{g} = -\vec{\nabla}\varphi \tag{2.3}$$

is given by

$$\varphi(\vec{x}) = -G\frac{M}{|\vec{x}-\vec{x}'|} = -G\frac{M}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \tag{2.4}$$

2.1 Collection of Point Masses

Now consider the case where we have not just one source mass M located at $\vec{x}'$, but multiple sources M_1 at $\vec{x}'_1$, M_2 at $\vec{x}'_2$, M_3 at $\vec{x}'_3$, etc. In general we say that source mass M_α is at $\vec{x}'_\alpha$, where α labels which mass we're talking about. (We use α as the label rather than i to further emphasize that the subscript is not labelling different components of a vector.)

If source mass α were the only one present, we would have a gravitational field and potential of

$$\vec{g}_\alpha(\vec{x}) = -G\frac{M}{|\vec{x}-\vec{x}'|^2}\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|} = -GM\frac{(x-x')\vec{e}_x+(y-y')\vec{e}_y+(z-z')\vec{e}_z}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}} \tag{2.5}$$

and

$$\varphi_\alpha(\vec{x}) = -G\frac{M_\alpha}{|\vec{x}-\vec{x}'_\alpha|} \tag{2.6}$$

Because Newtonian gravity is linear, the total gravitational field is just the superposition of the fields coming from the individual masses:

$$\begin{aligned}\vec{g}(\vec{x}) = \sum_\alpha \vec{g}(\vec{x}) &= -G\sum_\alpha \frac{M_\alpha}{|\vec{x}-\vec{x}'_\alpha|^2}\frac{\vec{x}-\vec{x}'_\alpha}{|\vec{x}-\vec{x}'_\alpha|} \\ &= -G\sum_\alpha M_\alpha\frac{(x-x'_\alpha)\vec{e}_x+(y-y'_\alpha)\vec{e}_y+(z-z'_\alpha)\vec{e}_z}{[(x-x'_\alpha)^2+(y-y'_\alpha)^2+(z-z'_\alpha)^2]^{3/2}}\end{aligned} \tag{2.7}$$

and likewise the potential:

$$\varphi(\vec{x}) = \sum_\alpha \varphi_\alpha(\vec{x}) = -G\sum_\alpha \frac{M_\alpha}{|\vec{x}-\vec{x}'_\alpha|} = -G\sum_\alpha \frac{M_\alpha}{\sqrt{(x-x'_\alpha)^2+(y-y'_\alpha)^2+(z-z'_\alpha)^2}} \tag{2.8}$$

2.2 Continuous Mass Distribution

In many physical situations, we'll want to consider the gravitational influence of an object which is best described as a continuous distribution of matter, parametrized by a mass density $\rho(\vec{x}')$ (as before, we use $\vec{x}'$ for the source point in an effort to minimize confusion.) The mass of an infinitesimal cube of the source, located at (x', y', z') and measuring dx', dy' and dz' on its three sides is

$$d^3M = \rho(x', y', z')\, dx'\, dy'\, dz' \tag{2.9}$$

The gravitational field of the whole body is given by the superposition of the fields from all of these little cubes; in the limit that the size of the cubes goes to zero (and their number goes to infinity), the sum in (2.7) becomes an integral:

$$\vec{g}(x,y,z) = -G \iiint \frac{(x-x')\vec{e}_x + (y-y')\vec{e}_y + (z-z')\vec{e}_z}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}} \rho(x',y',z')\, dx'\, dy'\, dz' \tag{2.10}$$

where the volume integral is over the entire solid generating the gravitational field. In vector notation, this is

$$\vec{g}(\vec{x}) = -G \iiint \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|^2} \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|} d^3V' \tag{2.11}$$

Similarly, the expression (2.8) for the potential becomes, in the continuous limit:

$$\varphi(x,y,z) = -G \iiint \frac{\rho(x',y',z')\, dx'\, dy'\, dz'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \tag{2.12}$$

or in vector notation

$$\varphi(\vec{x}) = -G \iiint \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3V' \tag{2.13}$$

2.2.1 Example: Gravitational Field of a Spherical Shell

Let $\mathcal{S}$ be a spherical shell of constant density and total mass M with inner radius a and outer radius b, centered on the origin. Find the gravitational field $\vec{g}(\vec{x})$ resulting from this shell, at an arbitrary position $\vec{x}$.

For economy of notation, we'll call the density ρ. We can relate ρ to M by performing a volume integral:

$$M = \iiint\limits_{\mathcal{S}} \rho\, d^3V = 4\pi \frac{b^3 - a^3}{3} \rho \tag{2.14}$$

In the calculations, we'll just work with a constant ρ and then related it to M at the end.

We'll eventually want to know the field at any position, but for starters, let's look at a position along the positive z axis. (We'll eventually be able to deduce the field everywhere by rotating our coördinate system.) Along the positive z axis, the field is, from (2.10),

$$\vec{g}(0,0,z) = -G\rho \iiint\limits_{\mathcal{S}} \frac{-x'\vec{e}_x - y'\vec{e}_y + (z-z')\vec{e}_z}{[x'^2 + y'^2 + (z-z')^2]^{3/2}} d^3V' \tag{2.15}$$

where we have pulled the constant $\rho(x',y',z') = \rho$ out of the integral.

By symmetry, the gravitational field along the z axis can't have any x or y component, since there's nothing about the geometry to pick out the positive or negative direction along either of those axes. But it's instructive to verify explicitly that for this source $g_x(0,0,z) = 0$. The spherical shell is a little tricky to describe in Cartesian coördinates, but it's easier if we define $\mathcal{B}_R$ to be a solid sphere (a ball) of radius R centered at the origin, and note that the shell $\mathcal{S}$ is just that part of a ball of radius b which is not also in a ball of radius a:

$$\mathcal{B}_b = \mathcal{B}_a \cup \mathcal{S} \tag{2.16}$$

This means that the integral of anything function $f(\vec{x}')$ over $\mathcal{B}_b$ is

$$\iiint\limits_{\mathcal{B}_b} f(\vec{x}')\, d^3V' = \iiint\limits_{\mathcal{B}_a} f(\vec{x}')\, d^3V' + \iiint\limits_{\mathcal{S}} f(\vec{x}')\, d^3V' \tag{2.17}$$

so

$$\iiint\limits_{\mathcal{S}} f(\vec{x}')\, d^3V' = \iiint\limits_{\mathcal{B}_b} f(\vec{x}')\, d^3V' - \iiint\limits_{\mathcal{B}_a} f(\vec{x}')\, d^3V' \tag{2.18}$$

This means that the first step in calculating

$$g_x(0,0,z) = -G\rho \iiint\limits_{\mathcal{S}} \frac{-x'\, d^3V'}{[x'^2 + y'^2 + (z-z')^2]^{3/2}} \tag{2.19}$$

is to calculate the integral over a ball of arbitrary radius R:

$$-G\rho \iiint\limits_{\mathcal{B}_R} \frac{-x'\, d^3V'}{[x'^2 + y'^2 + (z-z')^2]^{3/2}} \tag{2.20}$$

The limits of integration have to be chosen to just cover the ball

$$x'^2 + y'^2 + z'^2 \le R^2 \tag{2.21}$$

We tackle them from the outside integral in:

1. For each z' that contains any of the sphere, we integrate over x' and y'. The range of z' for which there is some piece of the ball present at some x' and y' is

$$z'^2 \le R^2 \tag{2.22}$$

i.e.

$$-R \le z' \le R \tag{2.23}$$

2. Inside the integral over z' is an integral over y'; this is performed at a fixed value of z'; the range of y' values such that there is some x' for which (x', y', z') is in the ball is

$$y'^2 + z'^2 \le R^2 \tag{2.24}$$

i.e.

$$-\sqrt{R^2 - z'^2} \le y' \le \sqrt{R^2 - z'^2} \tag{2.25}$$

3. Inside the y' and z' integrals is an integral over x'; this is performed at a fixed y' and z'. The range of x' values which cover the resulting one-dimensional slice of the ball is given by

$$x'^2 + y'^2 + z'^2 \le R^2 \tag{2.26}$$

i.e.

$$-\sqrt{R^2 - y'^2 - z'^2} \le y' \le \sqrt{R^2 - y'^2 - z'^2} \tag{2.27}$$

this allows us to set the specific limits of integration

$$
\begin{aligned}
&- G\rho \iiint\limits_{\mathcal{B}_R} \frac{-x'\,d^3V'}{[x'^2+y'^2+(z-z')^2]^{3/2}} \\
&= -G\rho \int_{-R}^{R} \left(\int_{-\sqrt{R^2-z'^2}}^{\sqrt{R^2-z'^2}} \left(\int_{-\sqrt{R^2-y'^2-z'^2}}^{\sqrt{R^2-y'^2-z'^2}} \frac{-x'}{[x'^2+y'^2+(z-z')^2]^{3/2}}\,dx' \right) dy' \right) dz'
\end{aligned}
\tag{2.28}
$$

The innermost function is the integral of an odd function of x' over a symmetric interval, so it vanishes. Explicitly, if we change variables to $\xi = -x'$, it becomes

$$
\begin{aligned}
\int_{-\sqrt{R^2-y'^2-z'^2}}^{\sqrt{R^2-y'^2-z'^2}} \frac{-x'}{[x'^2+y'^2+(z-z')^2]^{3/2}}\,dx' &= \int_{\sqrt{R^2-y'^2-z'^2}}^{-\sqrt{R^2-y'^2-z'^2}} \frac{\xi}{[\xi^2+y'^2+(z-z')^2]^{3/2}}\,(-d\xi) \\
&= -\int_{-\sqrt{R^2-y'^2-z'^2}}^{\sqrt{R^2-y'^2-z'^2}} \frac{-\xi}{[\xi^2+y'^2+(z-z')^2]^{3/2}}\,d\xi
\end{aligned}
\tag{2.29}
$$

which is just minus what we started with (up to a renaming of the integration variable), so it has to vanish.

But if the integral over a ball of any radius vanishes, than the integral over the shell vanishes, and we've confirmed

$$g_x(0,0,z) = 0 \tag{2.30}$$

By a similar calculation, we can show

$$g_y(0,0,z) = 0 \tag{2.31}$$

and hence

$$\vec{g}(0,0,z) = g_z(0,0,z)\vec{e}_z \tag{2.32}$$

$$g_z(0,0,z) = -G\rho \iiint\limits_{\mathcal{S}} \frac{(z-z')\,dx'dy'dz'}{[x'^2+y'^2+(z-z')^2]^{3/2}} \tag{2.33}$$

This integral is much easier if we change to spherical coördinates

$$x' = r'\sin\theta'\cos\phi' \tag{2.34a}$$

$$y' = r'\sin\theta'\sin\phi' \tag{2.34b}$$

$$z' = r'\cos\theta' \tag{2.34c}$$

in terms of which the volume element becomes[1]

$$dx'dy'dz' = r'^2\sin\theta'\,dr'\,d\theta'\,d\phi' \tag{2.35}$$

[1]Note that we can get the volume element from the usual geometric construction of an almost-cube with dimensions dr. $r\,d\theta$, and $r\sin\theta\,d\phi$, or by noting that

$$dx'\,dy'\,dz' = |\det \mathbf{J}|\,dr'\,d\theta'\,d\phi'$$

This makes the integral

$$
\begin{aligned}
g_z(0,0,z) &= -G\rho \int_0^{2\pi} \int_0^{\pi} \int_a^{b} \frac{(z - r'\cos\theta')}{[r'^2 \sin^2\theta' + (z - r'\cos\theta')^2]^{3/2}} r'^2, dr' \, \sin\theta' \, d\theta' \, d\phi' \\
&= -2\pi G\rho \int_a^{b} \int_0^{\pi} \frac{(z - r'\cos\theta')}{[r'^2 \sin^2\theta' + (z - r'\cos\theta')^2]^{3/2}} \sin\theta' \, d\theta' \, r'^2 \, dr'
\end{aligned} \tag{2.36}
$$

The quantity inside the square brackets is

$$
r'^2 \sin^2\theta' + (z - r'\cos\theta')^2 = r'^2 \sin^2\theta' + r'^2 \cos^2\theta' + z^2 - 2r'z\cos\theta' = r'^2 + z^2 - 2r'z\cos\theta' \tag{2.37}
$$

Note that this is always a positive number, which is equal to $(r' - z)^2$ at $\theta' = 0$, increasing with θ' all the way to $(r' + z)^2$ at $\theta' = \pi$. So we can change variables, replacing θ' with

$$
u = \sqrt{r'^2 + z^2 - 2r'z\cos\theta'} \tag{2.38}
$$

to get the differential, we note that

$$
2u\,du = d(u^2) = d(r'^2 + z^2 - 2r'z\cos\theta') = 2r'z\sin\theta d\theta \tag{2.39}
$$

so

$$
\sin\theta d\theta = \frac{u\,du}{r'z} \tag{2.40}
$$

We also note that

$$
r'\cos\theta' = \frac{r'^2 + z^2 - u^2}{2z} \tag{2.41}
$$

which tells us

$$
\begin{aligned}
g_z(0,0,z) &= -2\pi G\rho \int_a^{b} \int_{|z-r'|}^{z+r'} \left(z - \frac{r'^2 + z^2 - u^2}{2z} \right) u^{-3} \frac{u\,du}{r'z} r'^2 \, dr' \\
&= -\frac{2\pi G\rho}{z^2} \int_a^{b} \underbrace{\left(\int_{|z-r'|}^{z+r'} \frac{z^2 - r'^2 + u^2}{2u^2} du \right)}_{I(r')} r' \, dr'
\end{aligned} \tag{2.42}
$$

Looking at the integral

$$
I(r') = \frac{1}{2} \int_{|z-r'|}^{z+r'} \left(\frac{z^2 - r'^2}{u^2} + 1 \right) du = \frac{1}{2} \left[\frac{r'^2 - z^2}{u^2} + u \right]_{|z-r'|}^{z+r'} \tag{2.43}
$$

---

where $\mathbf{J}$ is the *Jacobian matrix*

$$
\mathbf{J} = \begin{pmatrix} \frac{\partial x'}{\partial r'} & \frac{\partial x'}{\partial \theta'} & \frac{\partial x'}{\partial \phi'} \\ \frac{\partial y'}{\partial r'} & \frac{\partial y'}{\partial \theta'} & \frac{\partial y'}{\partial \phi'} \\ \frac{\partial z'}{\partial r'} & \frac{\partial z'}{\partial \theta'} & \frac{\partial z'}{\partial \phi'} \end{pmatrix}
$$

A straightforward calculation shows that $\det \mathbf{J} = r'^2 \sin\theta$. (If you've never done this calculation, you should do it as an exercise.)

we see the value depends on whether r' is greater or less than z. Considering each case separately, we find

$$I(r' > z) = \frac{1}{2}\left(\frac{(r'+z)(r'-z)}{r'+z} + (r'+z) - \frac{(r'+z)(r'-z)}{r'-z} - (r'-z)\right) = 0 \tag{2.44}$$

and

$$\begin{aligned} I(r' < z) &= \frac{1}{2}\left(-\frac{(z+r')(z-r')}{z+r'} + (z+r') + \frac{(z+r')(z-r')}{z-r'} - (z-r')\right) \\ &= \frac{-z+r'+z+r'+z+r'-z+r'}{2} = 2r' \end{aligned} \tag{2.45}$$

Armed with the result that

$$I(r') = \begin{cases} 2r' & r' < z \\ 0 & r' > z \end{cases} \tag{2.46}$$

we need to think about how the possible values of r' compare to z in the integral

$$g_z(0,0,z) = -\frac{2\pi G\rho}{z^2}\int_a^b I(r')r'\,dr' \tag{2.47}$$

There are three cases, depending on the value of z

$\mathbf{0 < z < a}$ In this case, all of the values $a \le r' \le b$ in the integral are larger than z, and therefore

$$g_z(0,0,z) = -\frac{2\pi G\rho}{z^2}\int_a^b (0)r'\,dr' = 0 \qquad \text{when } 0 < z < a \tag{2.48}$$

$\mathbf{z > b}$ Here, all the possible values of r' are smaller than z and thus

$$g_z(0,0,z) = -\frac{2\pi G\rho}{z^2}\int_a^b (2r')r'\,dr' = -\frac{4\pi G\rho}{z^2}\frac{b^3-a^3}{3} = -\frac{GM}{z^2} \qquad \text{when } z > b \tag{2.49}$$

$\mathbf{a < z < b}$ In this case, r' can be larger or smaller than z, but the only non-zero contributions are from $r' < z$ so the integral becomes

$$g_z(0,0,z) = -\frac{2\pi G\rho}{z^2}\int_a^z (2r')r'\,dr' = -\frac{4\pi G\rho}{z^2}\frac{z^3-a^3}{3} = -\frac{GM}{z^2}\frac{z^3-a^3}{b^3-a^3} \qquad \text{when } z > b \tag{2.50}$$

Putting it all together, we get

$$\vec{g}(0,0,z) = \begin{cases} 0 & 0 \le z \le a \\ -\frac{GM}{z^2}\frac{z^3-a^3}{b^3-a^3}\vec{e}_z & a \le z \le b \\ -\frac{GM}{z^2}\vec{e}_z & z \ge b \end{cases} \tag{2.51}$$

Now, as promised, we'll use this to get the field at a general point. Consider the two non-rotationally-invariant things appearing in the field (2.51): z and $\vec{e}_z$. In each case, there is a corresponding coördinate-invariant concept. For a point on the positive z axis, z is the distance from the origin; for a general point this is r. For a point on the positive z axis, $\vec{e}_z$ is a unit vector which points directly away from the origin; for a general point this is $\vec{e}_r$. Given that we could have chosen our z axis to point in any direction (due to the spherical symmetry of the source), if we want to talk about the field at a point, we could temporarily set up coördinates where the point is on the positive z axis, work out the gravitational field in terms of the rotationally invariant quantities r and $\vec{e}_r$, and then rotate them back where the came from. This sort of construction gives us the entire gravitational field of the shell:

$$\vec{g}(\vec{x}) = \begin{cases} 0 & 0 \le r \le a \\ -\frac{GM}{r^2}\frac{r^3-a^3}{b^3-a^3}\vec{e}_r & a \le r \le b \\ -\frac{GM}{r^2}\vec{e}_r & r \ge b \end{cases} \tag{2.52}$$

where $r\,|\vec{x}|$ and $\vec{e}_r = \vec{x}/r$ as usual.

Note that this means that outside a spherically symmetric distribution of matter, the gravitational field is the same as if the whole mass were concentrated at the center, while inside a spherical shell, there is no gravitational field due to the shell.

3 Tidal Effects

3.1 The Equivalence Principle

Everything falls at the same rate in a gravitational field; this means that if you're falling freely in a constant gravitational field, the relative distance between you and other objects is unaffected; local experiments in a freely-falling lab with no windows cannot distinguish it from a lab floating in space.

But most gravitational fields are not constant. If you are falling in the gravitational field of the Earth, and the lab is big enough, you can tell you're not floating in space by looking at the behavior of objects near the edges of the lab. Objects near the floor will be closer to the Earth than those at the center, and will be accelerated more; objects near the ceiling will be accelerated less. Objects near the sides will be accelerated towards the center of the Earth, which is not straight down in lab coördinates, but rather a little towards the center. Subtract off the average motion, and you'll see the objects near the floor experience a net downward acceleration, those near the ceiling a net upward acceleration, and those near the sides a net inward acceleration.

Note: In general relativity, this variation in gravitational attraction is the only absolute measure of gravitational attraction.

Effects due to spatial variations of the gravitational field are called *tidal effects*. As the name suggests, they are responsible for ocean tides. Recall the qualitative explanation that the moon attracts the water nearest to it more and farthest from it less strongly, which explains why high tides are approximately 12 hours apart; at any time, two places on the Earth are experiencing high tide; one with the moon directly overhead and one with the moon directly "underfoot". We will now consider this problem quantitatively.

3.2 Tidal Field of a Point Mass

Consider the tidal gravitational effects of a point mass (or a spherically-symmetric mass distribution, as we've seen from the last example) of mass M a distance r' away from a spherical body of radius $r \ll r'$. We choose the coördinates such that the source mass is at a position $\vec{x}' = r'\vec{e}_z$ and the tidally influenced body is (initially) at the origin of coördinates. The gravitational field at some point on the surface

$$\vec{x} = r\vec{e}_r = r(\sin\theta\cos\phi\vec{e}_x + \sin\theta\sin\phi\vec{e}_y + \cos\theta\vec{e}_z) \tag{3.1}$$

is given, using (2.2), by

$$\vec{g}(\vec{x}) = -G\frac{M}{|\vec{x}-\vec{x}'|^2}\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|} = -G\frac{M\vec{\xi}}{\xi^3} \tag{3.2}$$

where we have defined

$$\vec{\xi} = \vec{x} - \vec{x}' \tag{3.3}$$

Given the coördinates we've defined, z is special, but there's not a lot of difference between x and y, so it's easier to lump them together and talk about the unit vector $\vec{e}_z$ towards the source and a unit vector

$$\vec{e}_\rho = \cos\phi\vec{e}_x + \sin\phi\vec{e}_y \tag{3.4}$$

perpendicular to it.

With this notation, we can calculate

$$\xi = \sqrt{\vec{\xi}\cdot\vec{\xi}} = \sqrt{|\vec{x}|^2 + \left|\vec{x}'\right|^2 - 2\vec{x}\cdot\vec{x}'} = \sqrt{r^2 + r'^2 - 2rr'\cos\theta} \tag{3.5}$$

so that

$$\vec{g}(\vec{x}) = -GM(r\cos\theta\vec{e}_z + r\sin\theta\vec{e}_\rho - r'\vec{e}_z)(r^2 + r'^2 - 2rr'\cos\theta)^{-3/2} \tag{3.6}$$

On the other hand, the gravitational field at the center of the planet (i.e., the origin of coördinates) is

$$\vec{g}(0) = -GMr'^{-2}\vec{e}_z \tag{3.7}$$

In order to consider the *difference* between the gravitational field at the surface and at the center, we introduce the *tidal field*

$$\begin{aligned}\delta\vec{g}(\vec{x}) &= \vec{g}(\vec{x}) - \vec{g}(0)\\ &= -GMr'^{-2}\left(\left[\left\{-1+\frac{r}{r'}\cos\theta\right\}\vec{e}_z + \frac{r}{r'}\sin\theta\vec{e}_\rho\right]\left[1+\frac{r'^2}{r^2}-2\frac{r}{r'}\cos\theta\right]^{-3/2} + \vec{e}_z\right)\end{aligned} \tag{3.8}$$

Since we are assuming $r \ll r'$, we are only interested in the tidal field to first order in r/r'; to that order, we have

$$\left(1+\frac{r'^2}{r^2}-2\frac{r}{r'}\cos\theta\right)^{-3/2} \approx \left(1-2\frac{r}{r'}\cos\theta\right)^{-3/2} \approx \left(1-3\frac{r}{r'}\cos\theta\right) \tag{3.9}$$

and

$$
\begin{aligned}
\delta\vec{g}(\vec{x}) = \vec{g}(\vec{x}) - \vec{g}(0) &= -GMr'^{-2}\left(\left[-\vec{e}_z + \frac{r}{r'}\left\{\cos\theta\vec{e}_z + \sin\theta\vec{e}_\rho\right\}\right]\left[1 - 3\frac{r}{r'}\cos\theta\right] + \vec{e}_z\right) \\
&= -GMr'^{-2}\left(+\frac{r}{r'}\left[\cos\theta\vec{e}_z + \sin\theta\vec{e}_\rho\right] - 3\frac{r}{r'}\cos\theta\vec{e}_z\right) \\
&= \frac{GM}{r'^3} r\left[2\cos\theta\vec{e}_z - \sin\theta\vec{e}_\rho\right]
\end{aligned}
\tag{3.10}
$$

Notice:

- When $\theta = 0$ (which corresponds to the side of the planet near the external mass), the tidal gravitational field is pointed in the positive z direction (towards the external mass, away from the center of the planet)
- When $\theta = \pi/2$ (which corresponds to a direction perpendicular to the line between the planet and the external mass), the tidal gravitational field is in the negative ρ direction (inward towards the center of the planet)
- When $\theta = \pi$ (which corresponds to the far side of the planet away from the external mass), the tidal gravitational field is pointed in the negative z direction (away from the external mass, , away from the center of the planet)

this corresponds to what we said qualitatively in Section 3.1, that there are two tidal bulges (differential force directed outwards) at any given time.

Note also that the strength of the tidal field falls off like r'^{-3}, as opposed to the strength of the average gravitational field, which falls off like r'^{-2}. This is why (as you will show on the homework) the Moon's tidal effects on the Earth are slightly stronger than the Sun's, while the Sun's gravitational attraction of the Earth as a whole is stronger than that of the Moon.