Calculus of Variations
(Marion & Thornton Chapter Six)

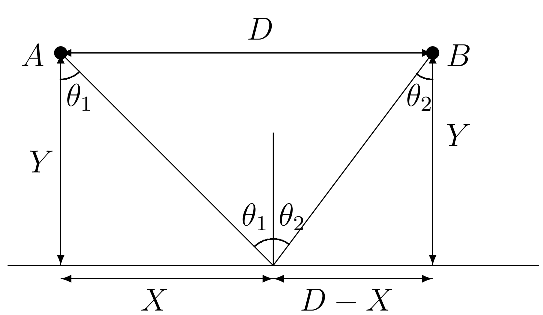
Physics A300*

Fall 2002

The general idea behind the calculus of variations, and the power of the Lagrangian formulation of Physics, is that the minimization (or more generally, extremization) of a global quantity constructed from an entire path implies a local property at each point along the path. The simplest example of this is the statement "the shortest distance between two points is a straight line". The global property is the minimization of distance, and the implied local property is that the line is straight (does not curve to the right or left). These notes motivate the general formulation of calculus of variations by exploring several simple examples

1 Example: Shortest Path Which Touches a Line

This is a actually a classic problem from introductory calculus, and one formulation is this: You live in a village (labelled by point A) a distance Y North of a river which runs straight East. A distance D to the East of you is a market (point B). You need to walk to the river, catch a fish, and bring it to the market to sell. What route do you use to minimize the distance you have to travel?



It's taken as a given that you should walk in a straight line whenever you can, so clearly the thing to do is walk straight to a point on the river, then walk straight from that point to the market. The question is, how far down the river is that point? Calling that distance X, the Pythagorean theorem that the distance you have to walk from the village to the

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river is $\sqrt{X^2+Y^2}$ while the distance you have to walk from the river to the market is $\sqrt{(D-X)^2+Y^2}$. The total distance as a function of X is

$$d(X) = \sqrt{X^2+Y^2} + \sqrt{(D-X)^2+Y^2} \tag{1.1}$$

To find the X which minimizes this, we just differentiate it with respect to X and then require that that derivative vanish:

$$d'(X) = \frac{X}{\sqrt{X^2+Y^2}} - \frac{(D-X)}{\sqrt{(D-X)^2+Y^2}} \tag{1.2}$$

The minimum thus occurs when

$$\frac{X}{\sqrt{X^2+Y^2}} = \frac{(D-X)}{\sqrt{(D-X)^2+Y^2}} \tag{1.3}$$

we could solve this directly for X (Exercise: do this), but it turns out to be more enlightening to do a little trigonometry, and note that

$$\frac{X}{\sqrt{X^2+Y^2}} = \sin\theta_1 \tag{1.4a}$$

$$\frac{D-X}{\sqrt{(D-X)^2+Y^2}} = \sin\theta_2 \tag{1.4b}$$

which means (1.3) is equivalent to

$$\sin\theta_1 = \sin\theta_2 \tag{1.5}$$

or, since θ_1 and θ_2 are clearly acute,

$$\theta_1 = \theta_2 \tag{1.6}$$

We note in passing that this is also the law of reflection from geometric optics. In fact, the calculation we've done is basically the derivation of this law from a global principle: Fermat's Principle of Least Time. On the homework, you'll show that Fermat's Principle can also be used to derive Snell's Law of Refraction, using the fact that the speed of light in a medium depends on its index of refraction.

Returning to the problem at hand, we see that

$$\tan\theta_1 = \frac{X}{Y} \tag{1.7a}$$

$$\tan\theta_2 = \frac{D-X}{Y} \tag{1.7b}$$

which means that since $\tan\theta_1 = \tan\theta_2$,

$$X = D - X \tag{1.8}$$

or

$$X = \frac{D}{2} \tag{1.9}$$

which is the standard result that the shortest path hits the river halfway in between the village and the market.

2 Example: Shortest Path Between Two Points

Now we consider how to demonstrate that the shortest path connecting two points in a plane is a straight line. For simplicity, we let the (x, y) coördinates of the first point be $(0, 0)$, and call the coördinates of the second point (x_f, y_f). We then want to consider paths $y(x)$ which satisfy

$$y(0) = 0 \tag{2.1a}$$
$$y(x_f) = x_f \tag{2.1b}$$

and find the one which is shortest.

2.1 Restricted class of paths

Before considering all of the paths, we first restrict attention to one family of paths and ask which path is the shortest out of that subset. Each path in the family consists of a straight line connecting the initial point $(0, 0)$ to an intermediate point (x_1, y_1), followed by a straight line from (x_1, y_1) to the final point (x_f, y_f). Here $x_1 = x_f/2$ and y_1 is different for each path, so minimizing the length of the path will just consist of minimizing with respect to y_1.

For a given path, the length of the first straight segment is once again given by the Pythagorean Theorem as

$$\sqrt{x_1^2 + y_1^2} \tag{2.2}$$

while the second one is

$$\sqrt{(x_f - x_1)^2 + (y_f - y_1)^2} = \sqrt{x_1^2 + (y_f - y_1)^2} \tag{2.3}$$

so the total path length as a function of y_1 is

$$d(y_1) = \sqrt{x_1^2 + y_1^2} + \sqrt{x_1^2 + (y_f - y_1)^2} \tag{2.4}$$

and its the derivative is

$$d'(y_1) = \frac{y_1}{\sqrt{x_1^2 + y_1^2}} - \frac{y_f - y_1}{\sqrt{x_1^2 + (y_f - y_1)^2}} \tag{2.5}$$

We get a minimum when

$$\frac{y_1}{\sqrt{x_1^2 + y_1^2}} = \frac{y_f - y_1}{\sqrt{x_1^2 + (y_f - y_1)^2}} \tag{2.6}$$

i.e.,

$$y_1^2 \left[x_1^2 + (y_f - y_1)^2\right] = (y_f - y_1)^2 (x_1^2 + y_1^2) \tag{2.7}$$

which becomes

$$y_1^2 x_1^2 = (y_f - y_1)^2 x_1^2 \tag{2.8}$$

or

$$y_1 = \pm(y_f - y_1) \tag{2.9}$$

Since the negative square root would mean $y_f = 0$, which will not in general be true (and is not a property of the path at any rate), we take the positive square root, which tells us

$$y_1 = \frac{y_f}{2} \tag{2.10}$$

This is just the point we need to pass through to get a straight line from $(0,0)$ to (x_f, y_f).

2.2 Perturbations to a straight line

Any of the paths considered in the previous example can be written as the following function:

$$y(x) = \begin{cases} y_1 \frac{x}{x_1} & x \leq x_1 \\ y_1 + (y_f - y_1) \frac{x - x_1}{x_1} \end{cases} \tag{2.11}$$

In terms of the straight-line path

$$y_0(x) = y_f \frac{x}{x_f} = y_f \frac{x}{2x_1} \tag{2.12}$$

These can be written as

$$y(x) = y_0(x) + \alpha\eta(x) \tag{2.13}$$

where

$$\alpha = \frac{1}{x_1}\left(y_1 - \frac{y_f}{2}\right) \tag{2.14}$$

and

$$\eta(x) = \begin{cases} x & x \leq x_1 \\ x_f - x & x \geq x_1 \end{cases} \tag{2.15}$$

Asking which member of the family minimizes the length is thus a matter of minimizing the length of the path with respect to the size α of the departure from a straight line.

Of course, since we know the form of $\eta(x)$, we could just repeat the same calculation as before, but let's think how we would answer this question if we didn't know what shape the perturbation $\eta(x)$ had. The only thing we'll use is that it vanishes at the endpoints

$$\eta(0) = 0 = \eta(x_f) \tag{2.16}$$

which we know it has to in order that $y(x)$ pass through the same two endpoints as $y_0(x)$.

Now, given a general path $y(x)$, how do we measure its length? Well, we know how to measure the length of a straight line, so we can build up the length of a general path by looking at little segments which are so small they can be approximated as straight. (This is, after all, what we do when we calculate a derivative.) So, between a point $(x, y(x))$ and a point $(x + dx, y(x + dx))$, the distance is given by the Pythagorean Theorem as

$$ds = \sqrt{dx^2 + (y(x + dx) - y(x))^2} = \sqrt{1 + \left(\frac{y(x + dx) - y(x)}{dx}\right)^2}\, dx \tag{2.17}$$

But the term inside parentheses is just, in the limit $dx \to 0$ that we're interested in, the derivative

$$y'(x) = \lim_{dx \to 0} \frac{y(x+dx) - y(x)}{dx} \tag{2.18}$$

so the length of the path is

$$d(\alpha) = \int_0^{x_f} \sqrt{1 + y'(x)^2}\, dx \tag{2.19}$$

where the dependence on α is implicit in the definition (2.13) of $y(x)$.

To differentiate the length of the path with respect to α, we note that (taking partial derivatives with x held constant)

$$\frac{\partial y'(x)}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[y_0'(x) + \alpha \eta'(x)\right] = \eta'(x) \tag{2.20}$$

and

$$\frac{\partial}{\partial \alpha}\sqrt{1 + y'(x)^2} = \frac{y'(x)}{\sqrt{1 + y'(x)^2}} \frac{\partial y'(x)}{\partial \alpha} = \frac{\eta'(x) y'(x)}{\sqrt{1 + y'(x)^2}} \tag{2.21}$$

so

$$d'(\alpha) = \int_0^{x_f} \frac{\partial}{\partial \alpha}\sqrt{1 + y'(x)^2}\, dx = \int_0^{x_f} \frac{\eta'(x) y'(x)}{\sqrt{1 + y'(x)^2}}\, dx \tag{2.22}$$

It's not obvious in this problem, but we can cast this into a more useful form if we integrate by parts:

$$d'(\alpha) = \left.\frac{\eta(x) y'(x)}{\sqrt{1 + y'(x)^2}}\right|_0^f - \int_0^{x_f} \eta'(x) \frac{d}{dx}\left(\frac{y'(x)}{\sqrt{1 + y'(x)^2}}\right) dx \tag{2.23}$$

The first term vanishes because of (2.16), and performing the derivative in the second gives

$$\begin{aligned} d'(\alpha) &= -\int_0^{x_f} \eta'(x) y''(x) \left(\frac{(1 + y'(x)^2) - y'(x)^2}{(1 + y'(x)^2)^{3/2}}\right) dx \\ &= -\int_0^{x_f} \eta'(x) y''(x) \left(\frac{1}{(1 + y'(x)^2)^{3/2}}\right) dx \end{aligned} \tag{2.24}$$

The factor in parentheses is always positive, so evidently the path with $d'(\alpha) = 0$ is the one with $y''(x) = 0$. But that's just the condition for a straight line (which is the path with $\alpha = 0$), so we see once again that the straight-line path is the shortest one in this family.

But now we note that we didn't say anything special about the perturbation $\eta(x)$, except that it vanishes at the endpoints. So that means that whatever distortion we choose to define our family of paths, the shortest path will always be the one with $\alpha = 0$, the straight line.

3 The General Problem; Euler's Equation

The length of a path is a special case of something called a *functional*, a process for calculating a number given a path. The functionals we'll be interested will all be of the form

$$J[y] = \int_{x_i}^{x_f} \mathcal{J}\left(y(x), y'(x), x\right) dx \tag{3.1}$$

where $\mathcal{J}$ is some specified function. We'll be interested in finding the function y which minimizes or maximizes the functional J. We'll assume that the endpoints $y(x_i)$ and $y(x_f)$ are fixed, but in between we can consider any $y(x)$ we like.

The way we "differentiate with respect to the function y" is to consider an arbitrary family

$$y_\alpha(x) = y_0(x) + \alpha\eta(x) \tag{3.2}$$

where $\eta(x)$ is any function with $\eta(x_i) = 0 = \eta(x_f)$. We say that $y_0(x)$ is a path which extremizes $J[y]$ if and only if, for *any* choice of $\eta(x)$,

$$J(\alpha) = J[y_\alpha] = \int_{x_i}^{x_f} \mathcal{J}\left(y_\alpha(x), y'_\alpha(x), x\right) dx \tag{3.3}$$

is extremized by $\alpha = 0$, i.e. $J'(0) = 0$.

We can actually find the condition that makes this true in general, just by taking the appropriate derivative:

$$J'(\alpha) = \int_{x_i}^{x_f} \frac{\partial \mathcal{J}}{\partial \alpha} dx = \int_{x_i}^{x_f} \left(\frac{\partial \mathcal{J}}{\partial y}\frac{\partial y_\alpha}{\partial \alpha} + \frac{\partial \mathcal{J}}{\partial y'}\frac{\partial y'_\alpha}{\partial \alpha}\right) dx = \int_{x_i}^{x_f} \left(\eta(x)\frac{\partial \mathcal{J}}{\partial y} + \eta'(x)\frac{\partial \mathcal{J}}{\partial y'}\right) dx \tag{3.4}$$

Here it's a little easier to see that integrating by parts will be useful, since it will mean that only $\eta(x)$ and not $\eta'(x)$ will appear in the integral:

$$J'(\alpha) = \int_{x_i}^{x_f} \eta(x) \left(\frac{\partial \mathcal{J}}{\partial y} - \frac{d}{dx}\frac{\partial \mathcal{J}}{\partial y'}\right) dx \tag{3.5}$$

So we see that to get $J'(0) = 0$ independent of the choice of $\eta(x)$, we need $y_0(x)$ to satisfy

$$\frac{\partial \mathcal{J}}{\partial y} - \frac{d}{dx}\frac{\partial \mathcal{J}}{\partial y'} = 0 \tag{3.6}$$

This is known as the Euler equation, and forms the basis of the Euler-Lagrange equations of Lagrangian mechanics.

3.1 Straight-Line Example from Euler Equations

Returning to the special case we considered, the minimization of path length, we see from (2.19) that the integrand of the functional in this case is

$$\mathcal{J}(y(x), y'(x), x) = \sqrt{1 + y'(x)^2} \tag{3.7}$$

so that

$$\frac{\partial \mathcal{J}}{\partial y} = 0 \tag{3.8a}$$

$$\frac{\partial \mathcal{J}}{\partial y'} = \frac{y'(x)}{\sqrt{1 + y'(x)^2}} \tag{3.8b}$$

This makes the Euler-Lagrange equation

$$0 = -\frac{d}{dx}\frac{y'(x)}{\sqrt{1 + y'(x)^2}} = -y''(x)\frac{1}{(1 + y'(x)^2)^{3/2}} \tag{3.9}$$

which was exactly the condition we got before.