Physics A300: Classical Mechanics I

Revised Problem Set 2

Assigned 2002 September 6
Due 2002 September 13

Show your work on all problems! Note that the answers to some problems are in the appendix of Marion & Thornton, but they should only be used to check your work, since the answers themselves are an insufficient solution to the problem.

1 An Identity Relating the Levi-Civita and Kronecker Delta Symbols (M & T 1-22)

a) Evaluate the sum $\sum_{k=1}^{3} \varepsilon_{ijk}\varepsilon_{\ell mk}$ (which is actually 81 different sums, each containing three terms) by considering the result for all possible combinations of i, j, ℓ, m, that is:
(a) $i = j$ **(b)** $i = \ell$ **(c)** $i = m$ **(d)** $j = \ell$ **(e)** $j = m$ **(f)** $\ell = m$
(g) $i \neq \ell$ or m **(h)** $j \neq \ell$ or m. Show that

$$\sum_{k=1}^{3} \varepsilon_{ijk}\varepsilon_{\ell mk} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$$

b) Use the result of part a) to prove

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

2 Circular Motion

Consider a particle which moves at uniform speed in a circular trajectory of radius a at angular velocity ω (so that it completes an orbit in a time $2\pi/\omega$). Let the particle move counter-clockwise in the xy-plane so that it crosses the positive y-axis at $t = 0$.

a) Write the trajectory $x(t)$, $y(t)$ in Cartesian coördinates.

b) By taking time-derivatives, calculate the velocity $\vec{v}(t) = \dot{\vec{r}}(t)$ and acceleration $\vec{a}(t) = \ddot{\vec{r}}(t)$ in Cartesian coördinates.

c) Write the trajectory $r(t)$, $\phi(t)$ in plane polar coördinates.

d) By taking time-derivatives of those expressions, re-calculate the velocity and acceleration in polar coördinates. (You will need to use the time derivatives of the $\vec{e}_r$ and $\vec{e}_\phi$ unit vectors.)

3 Expressing an Unknown Vector in Term of its Known Cross and Dot Products with a Known Vector (M & T 1-13)

Suppose that all we know about a vector $\vec{X}$ is that

$$\vec{A} \times \vec{X} = \vec{B}$$

and

$$\vec{A} \cdot \vec{X} = \varphi$$

and that we know $\vec{A}$, $\vec{B}$, and φ. Find an expression for $\vec{X}$ in terms of the known quantities $\vec{A}$, $\vec{B}$, φ, and $\left|\vec{A}\right| = \sqrt{\vec{A} \cdot \vec{A}}$.

(Hint: Consider $\vec{A} \times \vec{B}$.)

4 Invariance of the Scalar Product

Show that the same expression holds for the dot product $\vec{A} \cdot \vec{B}$ in terms of the components of $\vec{A}$ and $\vec{B}$ in any orthonormal basis as follows:

a) Consider the matrix

$$\mathbf{\Lambda} = \begin{pmatrix} \Lambda_{\overline{1}1} & \Lambda_{\overline{1}2} & \Lambda_{\overline{1}3} \\ \Lambda_{\overline{2}1} & \Lambda_{\overline{2}2} & \Lambda_{\overline{2}3} \\ \Lambda_{\overline{3}1} & \Lambda_{\overline{3}2} & \Lambda_{\overline{3}3} \end{pmatrix}$$

whose transpose is

$$\mathbf{\Lambda}^{\mathrm{T}} = \begin{pmatrix} \Lambda^{\mathrm{T}}_{1\overline{1}} & \Lambda^{\mathrm{T}}_{1\overline{2}} & \Lambda^{\mathrm{T}}_{1\overline{3}} \\ \Lambda^{\mathrm{T}}_{2\overline{1}} & \Lambda^{\mathrm{T}}_{2\overline{2}} & \Lambda^{\mathrm{T}}_{2\overline{3}} \\ \Lambda^{\mathrm{T}}_{3\overline{1}} & \Lambda^{\mathrm{T}}_{3\overline{2}} & \Lambda^{\mathrm{T}}_{3\overline{3}} \end{pmatrix}$$

Write an expression for the (i, k)th element of $\mathbf{\Lambda}^{\mathrm{T}}$ (i.e., $\Lambda^{\mathrm{T}}_{i\overline{k}}$) in terms of the elements of $\mathbf{\Lambda}$.

b) Now let $\mathbf{\Lambda}$ be orthogonal, so that

$$\mathbf{\Lambda}^{\mathrm{T}}\mathbf{\Lambda} = \mathbf{1} \ .$$

Summarize the nine components of this 3×3 matrix equation in a single equation with two free indices (i.e., write an equation relating the (i, j)th component of each side of the matrix equality.)

c) Use the result of part a) to simplify the result of part b).

d) Let the orthogonal matrix $\mathbf{\Lambda}$ define a transformation between orthonormal bases so that the components of the vector $\vec{A}$ in the new basis are given in terms of the components in the old basis by

$$A_{\overline{k}} = \sum_{i=1}^{3} \Lambda_{\overline{k}i} A_i$$

Use the result of part c) to show that

$$\sum_{k=1}^{3} A_{\overline{k}} B_{\overline{k}} = \sum_{i=1}^{3} A_i B_i$$