

Physics A300: Classical Mechanics I

Problem Set 6

Assigned 2002 October 16

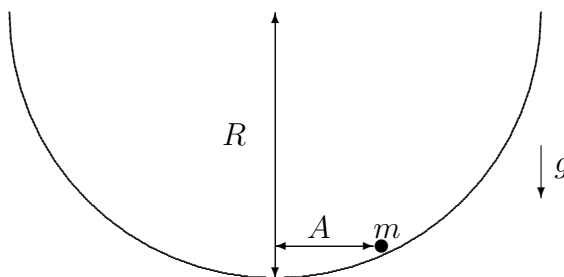
Due 2002 October 23

Show your work on all problems!

Note that solutions to this problem set will be distributed Wednesday, October 23, so no late homework can be accepted for this problem set.

1 Harmonic Oscillator Approximation

Consider a mass m moving without friction inside a hemispherical bowl of radius R under the presence of a constant downward gravitational acceleration g . Let the mass be released from rest a perpendicular distance A from the vertical axis of the bowl. (See figure.)



Show that for $A \ll R$, the system can be approximated as a simple harmonic oscillator, and calculate the natural frequency of the oscillations.

2 Euler Relation

Prove the relation

$$e^{i\theta} = \cos \theta + i \sin \theta$$

as follows:

- a) Find an expression for the n th coefficient (a_n) in the Taylor expansion

$$\cos \theta = \sum_{n=0}^{\infty} a_n \theta^n$$

(This is most easily expressed in terms of different expressions for odd and even n .)

- b) Find an expression for the n th coefficient (b_n) in the Taylor expansion

$$\sin \theta = \sum_{n=0}^{\infty} b_n \theta^n$$

(This is also most easily expressed in terms of different expressions for odd and even n .)

- c) Find an expression for the n th coefficient (c_n) in the Taylor expansion

$$e^{i\theta} = \sum_{n=0}^{\infty} c_n \theta^n$$

and show explicitly that $c_n = a_n + ib_n$

3 Parameters of a harmonic oscillator

Consider a one-kilogram mass moving on a spring obeying Hooke's law, whose restoring force has a magnitude of one Newton per four meters of displacement away from equilibrium.

- With what period does the mass oscillate?
- If the mass is released from rest at a displacement of 10 centimeters from the equilibrium point, what will its speed be when it crosses the equilibrium point (where the force due to the spring vanishes)?
- Suppose a viscous damping force is introduced into the system with a magnitude of four Newtons per 5 m/s of speed. Is the motion under-, over-, or critically damped? Assuming that at $t = 0$, mass is instantaneously stationary at a displacement of 10 centimeters from its equilibrium, write the trajectory as a function of time, specifying numerical values for all of your parameters. What is the speed of the particle when it first crosses the equilibrium point, to three significant figures?
- How large would the damping force need to be for the oscillator to be critically damped?

4 Fractional Energy Loss Per Cycle

Consider an underdamped ($\omega_0 > \beta > 0$) harmonic oscillator described by the equation of motion

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

with initial conditions

$$\begin{aligned} x(0) &= 0 \\ \dot{x}(0) &= v_0 \end{aligned}$$

for some positive v_0 .

- Use the initial conditions and the general solution for the underdamped harmonic oscillator to obtain the trajectory $x(t)$. (If your result contains any parameters other than v_0 , β , and ω_0 , be sure to specify expressions for those parameters in terms of v_0 , β , and ω_0 .)

b) Calculate the energy per unit mass

$$\mathcal{E} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega_0^2 x^2$$

of the oscillator as a function of time.

- c) Determine the time t_n of the n th zero-crossing of the oscillator. (I.e., find an expression for t_n where $x(t_n) = 0$, $t_{n+1} > t_n$, and $t_0 \geq 0$.)
- d) Evaluate the energy per unit mass $\mathcal{E}(t_n)$ at the n th zero-crossing.
- e) Write the change in \mathcal{E} between successive zero crossings

$$\Delta\mathcal{E}_{n+1,n} = \mathcal{E}(t_{n+1}) - \mathcal{E}(t_n)$$

and simplify your expression by writing it in terms of a hyperbolic sine (see Appendix D.6 of Marion & Thornton).

f) Write the average of \mathcal{E} between successive zero crossings

$$\bar{\mathcal{E}}_{n+1,n} = \frac{\mathcal{E}(t_{n+1}) + \mathcal{E}(t_n)}{2}$$

and simplify your expression by writing it in terms of a hyperbolic cosine (see Appendix D.6 of Marion & Thornton).

g) Calculate the fractional energy loss between zero crossings

$$\frac{\Delta\mathcal{E}_{n+1,n}}{\bar{\mathcal{E}}_{n+1,n}}$$

Simplify your answer so that it is manifestly independent of n .