Physics A300: Classical Mechanics I

Problem Set 7

Assigned 2002 November 6
Due 2002 November 13

Show your work on all problems!

1 Alternate Forms of General Solution

For the damped harmonic oscillator satisfying

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{1}$$

we showed in class that the general solution could be written as

$$x(t) = (A_c \cos\omega_1 t + A_s \sin\omega_1 t)e^{-\beta t} \qquad \text{if } \beta < \omega_0 \text{ (underdamped)} \tag{2a}$$

$$x(t) = (A_+ e^{\beta_1 t} + A_- e^{-\beta_1 t})e^{-\beta t} \qquad \text{if } \beta > \omega_0 \text{ (overdamped)} \tag{2b}$$

$$x(t) = (A_0 + A_1 t)e^{-\beta t} \qquad \text{if } \beta = \omega_0 \text{ (critically damped)} \tag{2c}$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$, $\beta_1 = \sqrt{\beta^2 - \omega_0^2}$, (M&T calls this ω_2) and A_c, A_s, A_0, A_1, A_+, and A_- are all real constants which can be set to match the desired initial conditions.

a) Show that the overdamped solution (2b) can be written in the form

$$x(t) = (B_c \cosh\beta_1 t + B_s \sinh\beta_1 t)e^{-\beta t} \tag{3}$$

and determine the new constants B_c and B_s in terms of the old constants A_+ and A_-

b) By extending the definition of ω_1 to the case where $\beta > \omega_0$ as $\omega_1 = i\beta_1$, and allowing A_c and A_s to be complex, show that the underdamped solution (2a) is equivalent to the overdamped solution (3) and determine the resulting B_c and B_s in terms of A_c and A_s. What conditions must A_c and A_s satisfy in order for $x(t)$ to be real in this case?

c) Show that in the limit $\beta \to \omega_0$, i.e., $\omega_1 \to 0$, the underdamped solution (2a) becomes the critically damped solution (2c), and determine expressions for the resulting A_0 and A_1 in terms of A_c and A_s (and ω_1). What has to happen to A_c and A_s in order for the solution to be finite in the limit $\omega_1 \to 0$?

2 General Solution in Terms of Initial Conditions

a) Consider a damped harmonic oscillator with $x(0) = x_0$ and $\dot{x}(0) = v_0$. Find the A_c and A_s needed to make the general solution (2a) satisfy these initial conditions, and use this to write $x(t)$ as a function of only x_0, v_0, t, β, and ω_1. Simplify your answer as much as possible.

b) Without using the results of problem 1, show that your solution to part a) of this problem gives a real $x(t)$ in the overdamped case by making the substitution $\omega_1 = i\beta_1$ and rewriting the solution in terms of hyperbolic trig functions.

c) Without using the results of problem 1, show that your solution to part a) of this problem gives a finite $x(t)$ in the critically-damped case by taking the limit $\omega_1 \to 0$ and deriving an expression for $x(t)$ in the critically damped case which does not contain ω_1.

3 Energy Transfer

Consider the steady-state solution

$$x(t) = A_{\text{out}} \cos(\omega t - \delta)$$

to a forced, damped harmonic oscillator.

a) Calculate the work done on the oscillator *per unit time per unit mass* by the three forces in the problem:

   i) $F_{\text{Hooke}} = -m\omega_0^2 x$
   ii) $F_{\text{damping}} = -2m\beta\dot{x}$
   iii) $F_{\text{driving}} = m\omega_0^2 A_{\text{in}} \cos \omega t$

   This is the net power per unit mass deposited into the oscillator by each force.

b) Calculate the net work per unit mass done by each individual force over a complete cycle of the oscillator. Show *explicitly* that no net work is done by the restoring force F_{Hooke} and that the amount of energy dissipated by the retarding force F_{damping} is equal to the amount of work done by the driving force F_{driving}. Calculate the frequency ω at which this energy transfer per cycle is greatest, for a fixed driving amplitude A_{in}.

4 Fourier Analysis

Consider a square wave of period T and amplitude x_0, which is given inside the interval $-T/2 < t < T/2$ by

$$A^{\text{in}}(t) = \begin{cases} -x_0 & \text{when } -T/2 < t < -T/4 \\ x_0 & \text{when } -T/4 < t < T/4 \\ -x_0 & \text{when } T/4 < t < T/2 \end{cases}$$

and which is defined outside that interval by its periodicity: $A^{\text{in}}(t+T) = A^{\text{in}}(t)$.

a) Find the coëfficients a_n^{in} and b_n^{in} in the Fourier series

$$A^{\text{in}}(t) = \frac{1}{2} a_0^{\text{in}} + + \sum_{n=1}^{\infty} a_n^{\text{in}} \cos\left(\frac{2\pi n t}{T}\right) + \sum_{n=1}^{\infty} b_n^{\text{in}} \sin\left(\frac{2\pi n t}{T}\right)$$

Express your answers in a form involving no sines or cosines.

b) If the square wave is applied to a harmonic oscillator of natural frequency ω_0 and damping parameter β, i.e.:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \omega_0^2 A^{\text{in}}(t),$$

find the values of ω_n, A_n^{out}, and δ_n in the expansion

$$x(t) = \sum_{n=0}^{\infty} A_n^{\text{out}} \cos(\omega_n t - \delta_n)$$

of the steady-state solution.