Physics A300: Classical Mechanics I

Problem Set 8

Assigned 2002 November 15
Due 2002 November 22

Show your work on all problems!

1 General Fourier Series Solution to Periodically Forced Oscillator

In class we showed that if the driving force on a harmonic oscillator

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \omega_0^2 A^{\text{in}}(t)$$

was a periodic function with the Fourier expansion

$$A^{\text{in}}(t) = \frac{a_0^{\text{in}}}{2} + \sum_{n=1}^{\infty} a_n^{\text{in}} \cos\omega_n t + \sum_{n=1}^{\infty} b_n^{\text{in}} \sin\omega_n t$$

then the steady-state solution could be written as the superposition

$$\begin{aligned} x_c(t) =& \frac{1}{2}a_0^{\text{in}} + \sum_{n=1}^{\infty} \frac{\omega_0^2 a_n^{\text{in}}}{\sqrt{(\omega_0^2-\omega_n^2)^2 + 4\beta^2\omega_n^2}} \cos(\omega_n t - \delta_n) \\ &+ \sum_{n=1}^{\infty} \frac{\omega_0^2 b_n^{\text{in}}}{\sqrt{(\omega_0^2-\omega_n^2)^2 + 4\beta^2\omega_n^2}} \sin(\omega_n t - \delta_n) \end{aligned} \tag{1}$$

where δ_n is the phase shift associated with the frequency $\omega_n = 2\pi n/T$. Using the angle difference formulas and the expressions found in class for $\cos\delta_n$ and $\sin\delta_n$, cast (1) in the standard Fourier series form

$$x_c(t) = \frac{a_0^{\text{out}}}{2} + \sum_{n=1}^{\infty} a_n^{\text{out}} \cos\omega_n t + \sum_{n=1}^{\infty} b_n^{\text{out}} \sin\omega_n t$$

and find a_n^{out} and b_n^{out} in terms of a_n^{in}, b_n^{in}, ω, ω_0, and β.

2 Gravity vs Electrostatics

a) Consider two protons separated by enough distance that one can describe their interactions in terms of a gravitational attraction due to their masses and an electrostatic repulsion due to their positive charges. Calculate to two significant figures the ratio of the magnitudes of these two forces.

b) Consider two identical ions, each with a net charge equal to the charge on a proton. How massive would the ions have to be in order for the gravitational attraction to just balance the electrostatic repulsion? Express your answer both in kilograms and in atomic mass units.

3 Flat Earth Society

Consider a disc of radius R, thickness ℓ and uniform density ρ. The gravitational acceleration of a test mass at a point a distance z directly above the center of the upper face of the disc will, by symmetry, be directed towards the center of this face.

a) Calculate (either by the potential method or directly) the magnitude of this acceleration as a function of z.

b) Take the limit of the acceleration as $\ell \to 0$, with the surface mass density $\sigma = \rho\ell$ remaining constant, and check that your result is the same as the result of Example 5.3 in Marion & Thornton for an infinitely thin disc. (M&T's ρ in that example is actually what we're calling σ.)

c) Now let ℓ be finite again, and take the limit of the acceleration as $R \to \infty$.

d) If the Earth were an infinite flat sheet of rock of density $\rho = 3.0\,\mathrm{g/cm^3}$, how thick would it need to be to produce the observed gravitational acceleration of $9.8\,\mathrm{m/s^2}$?

4 Tidal Distortion

4.1 Geometrical Preliminaries

a) Show that the surface spanned by

$$\begin{aligned} x &= x_0 + a \sin\Theta \cos\Phi \\ y &= y_0 + a \sin\Theta \sin\Phi \\ z &= z_0 + b \cos\Theta \end{aligned}$$

with $0 \leq \Theta \leq \pi$ and $0 \leq \Phi < 2\pi$ is the spheroid

$$\frac{(x - x_0)^2 + (y - y_0)^2}{a^2} + \frac{(z - z_0)^2}{b^2} = 1$$

with semiaxes a and b, centered at (x_0, y_0, z_0).

b) Show that the volume enclosed by this surface is $\frac{4\pi}{3} a^2 b$.

Hint: consider the coördinates

$$\begin{aligned} \xi &= \frac{x - x_0}{a} \\ \eta &= \frac{y - y_0}{a} \\ \zeta &= \frac{z - z_0}{b} \end{aligned}$$

4.2 Perturbative Expansion

Consider a large number of test particles which are released from rest at $t = 0$ in a sphere of radius R a distance Z away from a planet of mass M:

$$\begin{aligned} x(0) &= R \sin\Theta \cos\Phi \\ y(0) &= R \sin\Theta \sin\Phi \\ z(0) &= Z + R \cos\Theta \end{aligned}$$

where $R \ll Z$ and the planet is at $(0, 0, 0)$.

a) Write the three components $\ddot{x}(0)$, $\ddot{y}(0)$, $\ddot{z}(0)$ of the acceleration of the particle labelled by Θ and Φ, and expand each, keeping terms linear in R.

b) Consider the positions of the particles a short time δt later. Use a Taylor series to write the position $x(\delta t)$, $y(\delta t)$, $z(\delta t)$ of the particle labelled by Θ and Φ in terms of the initial position $x(0)$, $y(0)$, $z(0)$ and acceleration $\ddot{x}(0)$, $\ddot{y}(0)$, $\ddot{z}(0)$ of the particle, keeping terms of order $(\delta t)^2$ and lower.

c) Substitute the initial acceleration from part a) into the position from part b) to obtain the approximate location of each particle after time δt.

d) Show that the family of particles traces out a spheriod. What are its semi-axes and the coördinates of its center? Calculate the volume of the spheroid, again keeping terms of order $(\delta t)^2$ and lower.