Physics A300: Classical Mechanics I

Problem Set 9

Assigned 2002 November 25
Due 2002 December 4

Show your work on all problems!

1 Tidal Acceleration in Cartesian Coördinates

a) In class, we showed that the tidal gravitational acceleration at a point $\vec{x}$ with standard Cartesian coördinates r, θ, ϕ due to a point mass M a distance $r' \gg r$ away in the z direction was

$$\delta\vec{g}(\vec{x}) = \frac{GM}{r'^3} r \left(2\cos\theta \vec{e}_z - \sin\theta \left[\cos\phi \vec{e}_x + \sin\phi \vec{e}_y\right]\right)$$

Express $\delta\vec{g}(\vec{x})$ as a function of the Cartesian coördinates

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

b) Now suppose that the external mass is a distance r' away in the x direction. By analogy with the previous case, write the resulting tidal field in Cartesian coördinates.

c) Now suppose that the external mass is a distance r' away in the y direction. By analogy with the previous cases, write the resulting tidal field in Cartesian coördinates.

2 Lunar and Solar Tides

Consider the case where a planet of radius r (centered at the origin of coördinates) is deformed by two external point masses: M_1 located a distance $r'_1 \gg r$ away and M_2 located a distance $r'_2 \gg r$ away.

a) If both masses located along the x axis, use the results of problem 1 to write the tidal field $\delta \vec{g}(x, y, z)$ at the surface of the planet in Cartesian coördinates.

b) If the first mass is on the x axis and the second mass is on the y axis, what is the tidal field in Cartesian coördinates?

c) A *spring tide* occurs at the new or full moon, when the Sun, Earth, and Moon all lie on a line; a *neap tide* occurs at first or last quarter, when the Earth-Sun direction is perpendicular to the Earth-Moon direction. Using the masses of the Sun and Moon, and their distances from the Earth, calculate the ratio of the Moon-directed component of the tidal field at spring and neap tides.

d) Calculate the ratio of the gravitational field at the center of the the Earth due to the Sun and Moon; calculate the ratio of the strengths of the tidal gravitational effects on the Earth due to the Sun and the Moon.

3 Equilibrium Tidal Height

a) Using the results of problem 1, find the tidal gravitational potential $\varphi_{\text{tidal}}(\vec{x})$ which gives

$$\delta \vec{g}(\vec{x}) = -\vec{\nabla} \varphi_{\text{tidal}}$$

where $\delta \vec{g}(\vec{x})$ is the gravitational tidal field of a point mass m (we called this M before, but in this problem we want to use M for something else) located at a position $\vec{x}' = r' \vec{e}_z$. Express this scalar field in spherical coördinates.

b) Imagine a planet which is deformed such that its radius is not R, but $R + \frac{h}{2}(3 \cos \theta - 1)$. The gravitational potential energy associated with the deformation is

$$U(h) = \rho \int_0^{2\pi} \int_0^{\pi} \int_R^{R + \frac{h}{2}(3 \cos \theta - 1)} \varphi(r, \theta, \phi) \, r^2 \sin \theta \, dr \, d\theta \, d\phi$$

where ρ is the density of the planet, assumed to be constant. Calculate the contribution $U_{\text{tidal}}(h)$ to this potential energy from the tidal potential, keeping terms up to order h^2.

c) Let the planet have a mass M and assume this is concentrated at the center. What is the gravitational potential φ_{self} corresponding to the planet's gravitational field? Calculate the contribution $U_{\text{self}}(h)$ to the gravitational potential energy arising from the potential φ_{self}, again keeping terms up to order h^2.

d) Find the tidal height h which minimizes the total potential energy $U_{\text{tidal}}(h) + U_{\text{self}}(h)$.

4 Snell's Law from Fermat's Principle

Note: There is a similar problem solved in the Student Solutions Manual. Transcriptions of that solution will receive zero credit. The problem as stated here is actually simpler than the one in the book, and thus the solution is more straightforward.

Fermat's principle states that light travels along the path which takes the shortest time. Use this to derive Snell's Law of Refraction as follows:

Consider two media with indices of refraction n_1 and n_2, respectively. Light travels in the first with a speed c/n_1 and in the second with a speed c/n_2. Now consider a planar interface between these two media. Let a light ray travel from a point A in the first medium a distance Y from the interface to a point B in the second medium a distance Y from the interface and a distance D along the interface from A. Suppose the path consists of a straight line from A to a point C a distance X along the interface, then a straight line to B.

In terms of X, Y, and D, along with c, n_1, and n_2, write

a) The distance from A to C

b) The distance from C to B

c) The time light takes to travel from A to C at a speed c/n_1

d) The time light takes to travel from C to B at a speed c/n_2

e) The total time light takes to travel along this path from A to B via C.

Consider the total time as a function of X and determine an equation which is satisfied when the total time is minimized (you don't need to solve this for X).

Write $\sin \theta_1$ and $\sin \theta_2$ (see figure) in terms of X, Y, and D, and use this to eliminate X, Y, and D from the minimum time condition. Show that this gives Snell's law

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$