# Physics A301: Classical Mechanics II

Problem Set 4

Assigned 2003 February 7
Due 2003 February 14

**Show your work on all problems!** Be sure to give credit to any collaborators, or outside sources used in solving the problems.

## 1 The Atwood Machine

Consider an Atwood machine, which consists of two blocks of mass $m_1$ and $m_2$ connected by a massless rope which hangs over a pulley suspended from a fixed point in a constant gravitational field (see figure 2-11(a) of M&T). Let $x_1$ and $x_2$ be the distances of the two blocks below the pulley, and let the rope have total length $\ell$.

a) Construct the modified Lagrangian for the system, with $x_1$ and $x_2$ as the generalized coördinates, and with a Lagrange multiplier $\lambda$ to enforce the constraint $x_1 + x_2 = \ell$ which says that the total length of the rope doesn't change.

b) Find all three modified Euler-Lagrange equations (two equations of motion and one constraint).

c) Eliminate $\lambda$ from the equations of motion to obtain a single equation containing $x_1$, $x_2$, and their time derivatives.

d) Use the constraint (and its time derivatives) to eliminate $x_2$ and its time derivatives from the equation you found in part b) and produce a single differential equation in $x_1$.

e) Compare this approach to the balance of forces used to attack this problem in example 2.9; how is the Lagrange multiplier $\lambda$ of this problem related to the tension $T$ in the rope?

## 2 Time-Dependent Lagrangian

Consider the Lagrangian from the accelerated-pendulum problem

$$L(\theta, \dot{\theta}, t) = \frac{1}{2} m\ell^2\dot{\theta}^2 + m(v_0 + at)\ell\dot{\theta}\cos\theta + \frac{1}{2}m(v_0 + at)^2 + mg\ell\cos\theta$$

a) Calculate the partial derivatives $\frac{\partial L}{\partial \theta}$, $\frac{\partial L}{\partial \dot{\theta}}$, and $\frac{\partial L}{\partial t}$.

b) Calculate the total derivative

$$\frac{d}{dt}L(\theta(t),\dot\theta(t),t) = \frac{\partial L}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial \dot\theta}\frac{d\dot\theta}{dt} + \frac{\partial L}{\partial t}$$

c) Use the equation of motion

$$\ddot\theta = -\frac{g}{\ell}\sin\theta - \frac{a}{\ell}\cos\theta$$

to replace $\ddot\theta$ in your expression for the total derivative $\frac{dL}{dt}$ and demonstrate that $\frac{dL}{dt} \neq \frac{\partial L}{\partial t}$

d) Construct the Hamiltonian

$$H = \dot\theta\frac{\partial L}{\partial \dot\theta} - L$$

as a function of $\theta$, $\dot\theta$, and $t$.

e) Using the kinetic and potential energies

$$T = \frac{1}{2}m\ell^2\dot\theta^2 + m(v_0 + at)\ell\dot\theta\cos\theta + \frac{1}{2}m(v_0+at)^2$$

$$U = mg\ell\cos\theta$$

construct the total energy $E = T + U$, and calculate $E - H$.

# 3 Hamilton's Equations of Motion

Consider a particle of mass $m$ moving in the gravitational field of a point source of mass $M$ which is fixed at the origin of coördinates. Use spherical coördinates $(r, \theta, \phi)$ throughout the problem.

a) Write the potential energy $U(r, \theta, \phi)$ of the orbiting particle, with the zero of potential energy defined to lie at $r \to \infty$.

b) Using the line element in Appendix F.3, write the kinetic energy $T(r, \theta, \phi, \dot r, \dot\theta, \dot\phi)$ in spherical coördinates.

c) Write the Lagrangian $L(r, \theta, \phi, \dot r, \dot\theta, \dot\phi)$ for the problem.

d) Find the canonically conjugate momenta $p_r = \frac{\partial L}{\partial \dot r}$, $p_\theta = \frac{\partial L}{\partial \dot\theta}$, and $p_\phi = \frac{\partial L}{\partial \dot\phi}$ as functions of $(r, \theta, \phi, \dot r, \dot\theta, \dot\phi)$.

e) Invert those functions to write the generalized velocities $\dot r$, $\dot\theta$, and $\dot\phi$ as functions of $(r, \theta, \phi, p_r, p_\theta, p_\phi)$.

f) Construct the Hamiltonian

$$H(r, \theta, \phi, p_r, p_\theta, p_\phi) = p_r\dot r + p_\theta\dot\theta + p_\phi\dot\phi - L$$

where all the velocities appearing on the right-hand side have been replaced by the functions you found in part e).

g) Calculate $\frac{\partial H}{\partial p_r}$, $\frac{\partial H}{\partial p_\theta}$, and $\frac{\partial H}{\partial p_\phi}$ and verify that the results are just the expressions you got in part e) for $\dot r$, $\dot\theta$, and $\dot\phi$, respectively.

h) Use the other three Hamilton's equations to find expressions for $\dot p_r$, $\dot p_\theta$, and $\dot p_\phi$ in terms of $(r, \theta, \phi, p_r, p_\theta, p_\phi)$.