Physics A301: Classical Mechanics II

Problem Set 5

Assigned 2003 February 24
Due 2003 March 7

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Lorentz Force Law from an Action Principle

Consider the Lagrangian

$$L = \sum_{j=1}^{3} \frac{1}{2} m \dot{x}_j{}^2 - Q\varphi(t,\vec{x}) + Q \sum_{j=1}^{3} \dot{x}_j A_j(t,\vec{x})$$

where $\varphi(t,\vec{x})$ is some scalar field and $\vec{A}(t,\vec{x})$ is some vector field.

a) Calculate the partial derivatives $\frac{\partial L}{\partial x_i}$ and $\frac{\partial L}{\partial \dot{x}_i}$.

b) Work out the total derivatives $\dot{\varphi} = \frac{d\varphi}{dt}$ and $\dot{A}_i = \frac{dA_i}{dt}$ in terms of the partial derivatives $\frac{\partial \varphi}{\partial t}$, $\frac{\partial \varphi}{\partial x_j}$, $\frac{\partial A_i}{\partial t}$ and $\frac{\partial A_i}{\partial x_j}$.

c) Write the Euler-Lagrange equation

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0$$

using the results of part b) to expand all total time derivatives.

d) Solve the Euler-Lagrange equations for $m\ddot{x}_i$, simplifying as much as possible.

e) Show that the resulting equation of motion is just the Lorentz force law

$$m\ddot{x}_i = Q\left(E_i + \sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\dot{x}_j B_k\right)$$

where ϵ_{ijk} is the Levi-Civita symbol (see Chapter One) and the electric and magnetic fields are defined from our scalar and vector potential fields by

$$E_i = -\frac{\partial \varphi}{\partial x_i} - \frac{\partial A_i}{\partial t}$$

$$B_k = \sum_{\ell=1}^{3}\sum_{m=1}^{3} \epsilon_{k\ell m}\frac{\partial A_m}{\partial x_\ell}$$

In order to show that $\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}\dot{x}_j B_k$ equals the corresponding term in the equations of motion, you'll need to use several properties of the Levi-Civita symbol from last semester (and Chapter One), notably that

$$\sum_{k=1}^{3}\epsilon_{kij}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$$

2 Two-Body System in an External Gravitational Field

Assume that two point masses m_1 and m_2, whose position vectors are $\vec{x}_1$ and $\vec{x}_2$, respectively, move under the influence not only of a central force interaction described by a potential $U_{\text{int}}(|\vec{x}_1 - \vec{x}_2|)$, but also a constant gravitational field $\vec{g} = -g\vec{e}_z$ in the negative z-direction.

a) Write the gravitational potential energies $U_1(\vec{x}_1)$ and $U_2(\vec{x}_2)$ of the two masses due to the external gravitational field, and construct the Lagrangian.

b) Show that the change of coördinates to

$$\vec{X} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2}$$
$$\vec{x} = \vec{x}_1 - \vec{x}_2$$

once again allows the separation of the Lagrangian into two non-interacting pieces:

$$L = L_X(\vec{X}, \dot{\vec{X}}) + L_x(\vec{x}, \dot{\vec{x}})$$

c) Find the equations of motion for the center of mass $\vec{X}$ and describe its motion in words.

d) How would this procedure break down if the gravitational field were not constant?

3 Mass Ratios in Two-Body Motion

a) Consider the motion of two point-like objects mass m_1 and m_2, respectively, under the influence only of a central force between the two objects. If the first object is moving in a circular orbit of radius r_1, what is the radius of the second object's orbit?

b) The nature of a two-body problem is often described by the reduced mass ratio $\nu = \mu/M$. Write ν as a function of the ratio m_1/m_2 and plot ν versus m_1/m_2. What is the range of possible values of ν? When is it a minimum or maximum, and what are those values?

c) Calculate ν for the following systems

  i) The Sun and the Earth
  ii) The Sun and Jupiter
  iii) The Earth and the Moon
  iv) Saturn and Titan
  v) Pluto and Charon
  vi) A hypothetical system of identical twin planets orbiting each other

A good place to look up planetary bodies is at `http://www.nineplanets.org/`