Physics A301: Classical Mechanics II

Problem Set 8

Assigned 2003 March 24
Due 2003 March 31

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Center of Mass

Consider a pyramid $\mathcal{P}$ whose base is a square and whose faces are equilateral triangles. Let the length of each edge be a, and define a Cartesian coördinate system such that the square face lies in the xy-plane with its edges parallel to the axes and the remaining vertex lies on the positive z axis. The vertices are thus $(a/2, a/2, 0)$, $(a/2, -a/2, 0)$, $(-a/2, -a/2, 0)$, $(-a/2, a/2, 0)$, and $(0, 0, h)$, where $h > 0$ is the altitude of the pyramid (to be determined).

a) From the condition that each of the edges has length a, find h in terms of a.

b) Constant-z cross-sections of the pyramid are also squares centered at the origin with their sides parallel to the x and y axes, for $0 \leq z < h$. Find the length $A(z)$ of a side of the square cross-section for a given z.

c) Perform a triple integral to find the volume of the pyramid in terms of a; be sure to eliminate h using the result you got in part a), and note that the answer should not contain an x, y, or z.

d) Find the coördinates (X, Y, Z) of the center of mass of the pyramid, assuming its density is constant.

2 Tidal Forces in the Two-Body Problem

Consider two interacting particles moving in a non-uniform gravitational field $\vec{g}(\vec{x})$. Let m_1 and m_2 be the masses and $\vec{x}_1$ and $\vec{x}_2$ be the position vectors of the two particles and $\vec{f}_{12}$ and $\vec{f}_{21} = -\vec{f}_{12}$ be the interaction forces between them.

a) Work out the overall accelerations $\ddot{\vec{x}}_1$ and $\ddot{\vec{x}}_2$ in terms of the interaction force $\vec{f}_{12}$, the masses, and the gravitational field $\vec{g}(\vec{x})$ evaluated at appropriate values of $\vec{x}$. (I.e., your result should contain expressions like $\vec{g}(\vec{x}_1)$ and $\vec{g}(\vec{x}_2)$.)

b) Using the formalism of chapter 9 of Marion & Thornton, write an exact expression for the second time derivative $\ddot{\vec{X}}$ of the center of mass vector in terms of $\vec{f}_{12}$, the masses, and the field $\vec{g}(\vec{x})$.

c) Define the separation vector $\vec{x}_{12} = \vec{x}_1 - \vec{x}_2$ and work out the exact expression for $\ddot{\vec{x}}_{12}$ in terms of $\vec{f}_{12}$, the masses, and the field $\vec{g}(\vec{x})$. Show that in terms of the reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$,

$$\ddot{\vec{x}}_{12} = \frac{\vec{f}_{12}}{\mu} + \vec{a}_{\text{tidal}}(\vec{x}_1, \vec{x}_2)$$

and obtain an explicit expression for the tidal acceleration $\vec{a}_{\text{tidal}}(\vec{x}_1, \vec{x}_2)$ associated with the particles being at different locations in the external gravitational field. It is this tidal acceleration which causes two-body dynamics in a non-constant gravitational field to be different from those in the absence of external forces.

d) Write $\vec{x}_1$ and $\vec{x}_2$ in terms of $\vec{X}$ and $\vec{x}_{12}$.

e) For "small" values of $\left|\vec{\xi}\right|$, we can Taylor expand the vector field $\vec{g}(\vec{x})$ about a point $\vec{X}$:

$$\vec{g}(\vec{X} + \vec{\xi}) \approx \vec{g}(\vec{X}) + (\vec{\xi} \cdot \vec{\nabla})\vec{g}\Big|_{\vec{X}}$$

or explicitly in terms of components

$$g_i(\vec{X} + \vec{\xi}) \approx g_i(\vec{X}) + \sum_{j=1}^{3} \xi_j \, g_{i,j}(\vec{X})$$

where

$$g_{i,j}(\vec{x}) = \frac{\partial g_i(\vec{x})}{\partial x_j}$$

Use this Taylor expansion and the results of parts b) and d) to obtain an approximate expression for $\ddot{\vec{X}}$ whose only dependence on the field $\vec{g}(\vec{x})$ is through its value and first derivative at the center of mass location $\vec{X}$.

f) Use the Taylor expansion introduced in part e) to obtain an approximate expression for $\vec{a}_{\text{tidal}}$ which depends only on the value and first derivative of the gravitational field at the center of mass location.

g) If the gravitational field is that of a point mass at the origin:

$$\vec{g}(\vec{x}) = \frac{GM}{(x^2 + y^2 + z^2)^{3/2}}(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z)$$

and both particles lie in the xz-plane near the positive z axis so that $\vec{X} = Z\vec{e}_z$ and $\vec{x}_{12} = x_{12}\vec{e}_x + z_{12}\vec{e}_z$, explicitly evaluate the expression you got for the tidal acceleration $\vec{a}_{\text{tidal}}$ in part f) and verify that it's consistent with the results we obtained when considering tidal effects last semester.

3 Angular Momentum and Rotational Energy

Consider a right circular cylinder $\mathcal{C}$ of mass M, radius a, height h and uniform density, with its center at the origin and its axis of symmetry along the z axis, rotating counter-clockwise about the z axis with an angular speed ω.

a) Write the instantaneous velocity vector $\vec{v}$ at a point $\vec{x} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$ in Cartesian coördinates. (This should consist of an explicit expression for each of the three components of the velocity vector, involving the coördinates of the point and the angular speed ω.)

b) Work out the cross product $\vec{x} \times \vec{v}$, obtaining explicit expressions for the three components of the resulting vector, again in terms of x, y, z, and ω.

c) Use the results of part b) to write triple integrals for the three components of the angular momentum

$$\vec{L} = \iiint\limits_{\mathcal{C}} [\vec{x} \times \vec{v}(\vec{x})]\, \rho(\vec{x})\, dx\, dy\, dz$$

You don't need to write the limits of integration explicitly in this step, just leave them as "integrals over $\mathcal{C}$".

d) Change the integration variables from Cartesian to cylindrical coördinates, and perform each of the three integrals to obtain the angular momentum $\vec{L}$ in terms of M, a, h, and ω. (If you haven't done so already you'll need to use the volume of the cylinder to replace the density with the appropriate combination of M, a, and h.)

e) Repeat the process to calculate the total kinetic energy

$$T = \iiint\limits_{\mathcal{C}} \frac{1}{2}[\vec{v}(\vec{x}) \cdot \vec{v}(\vec{x})]\, \rho(\vec{x})\, dx\, dy\, dz$$

of the rotating cylinder in terms of M, a, h, and ω.