The Grammar of Mathematical Expressions and Equations

Physics A300*

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In most cases, whether a mathematical statement is true or not cannot be determined from the statement alone. However, some expressions or equations simply do not have a sensible meaning, and can be rejected as "grammatically incorrect". An easy check on the answer to a problem, or indeed any intermediate steps, is that it obeys the grammar appropriate to the subject. This supplement is designed to spell out the rules of this grammar.

1 Summation

It is often very convenient to replace expressions like

$$a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{1.1}$$

with sums like

$$\sum_{i=1}^{3} a_i b_i \tag{1.2}$$

The variable i is called a summation variable or (because it appears as a subscript) an *index* (plural "indices"). There are a number of rules governing the sensible treatment of indices.

1. If the range of values of an index is clear from the context, it can be omitted from the summation sign. (Just about the only time this will be the case is when it labels the three components of a vector.) So we could also write (1.1) as

   $$\sum_{i} a_i b_i \tag{1.3}$$

   provided it was clear from the context that i took on the values 1, 2, and 3.

2. An index which is summed over is called a *dummy index*.

(a) A dummy index can be renamed without changing the value of an expression, as long as the new name is not used anywhere else in the expression, and every appearance of the old index is replaced with the new one. So

$$\sum_i a_i b_i \tag{1.4}$$

is the same thing as

$$\sum_j a_j b_j \tag{1.5}$$

(b) A term containing a dummy index cannot be moved outside the relevant sum, but a term not containing the dummy index can be moved inside or outside the sum at will. For example, the expression

$$\sum_j a_{ij} b_j c_k \tag{1.6}$$

is equivalent to

$$c_k \sum_j a_{ij} b_j \tag{1.7}$$

(the mathematical reason for this is actually the distributive property of multiplication over addition, i.e., that $A(B+C) = AB + AC$) but we cannot move a_{ij} or b_j outside the sum, because the resulting expression would make no sense (a dummy index would appear outside the relevant sum).

(c) You cannot use the same letter for the dummy indices associated with two different sums, or for a dummy index in a sum and an index appearing elsewhere in the same expression. So for example if

$$F = \sum_i a_i b_i \tag{1.8}$$

and

$$G = \sum_i c_i d_i \tag{1.9}$$

we have to write

$$FG = \left(\sum_i a_i b_i\right) \left(\sum_j c_j d_j\right) = \sum_i \sum_j a_i b_i c_j d_j \tag{1.10}$$

because it would be grammatically incorrect (not to mention confusing) to use i for both sums.

3. An index which is not summed over is called a *free index*.

(a) The most powerful use for a free index is to have it appear free on both sides of the equation. In that case, it's understood that the expression is true for any valid value of the index. So for example,

$$F_i = \sum_j R_{ij} G_j \tag{1.11}$$

actually represents the three equations

$$F_1 = \sum_j R_{1j} G_j = R_{11} G_1 + R_{12} G_2 + R_{13} G_3 \tag{1.12a}$$

$$F_2 = \sum_j R_{2j} G_j = R_{21} G_1 + R_{22} G_2 + R_{23} G_3 \tag{1.12b}$$

$$F_3 = \sum_j R_{3j} G_j = R_{31} G_1 + R_{32} G_2 + R_{33} G_3 \tag{1.12c}$$

(b) In general, unless something has been specified about its value, a free index must appear on both sides of an equation, and cannot be used as a dummy index on either side of the equation. (One exception to the first part is if one side of the equation is equal to zero.)

2 Integration

3 Units

4 Vectors