Intro to/Review of Newtonian Mechanics (Symon Chapter One)

Physics A300*

Fall 2003

0 Administrata

- Physics majors party: Thursday, August 28 at 12:30 in the Physics Reading Room
- Syllabus
  - Subscribe to mailing list
  - Homework grading to be picky about writing sensible expressions, e.g., always putting vector signs on vectors and carrying units along with numerical values at intermediate steps
- Course Outline: First Seven Chapters of Symon (topics in **bold** often have their own chapters in textbooks)
  1. Review of Newtonian Mechanics (should be familiar from freshman physics)
  2. One-Dimensional Problems
     - **Harmonic oscillator**
  3. Two- and Three-Dimensional Problems
     - Vector analysis
     - **Central force motion**
  4. Systems of particles
     - Conservation laws
     - Scattering problems
  5. Rotation about an axis (will do more complicated rigid body motion next semester)
     - "one-dimensional" rotation problems
     - Will skip second half of chapter (statics)
  6. Gravity

  - Gravitational field & potential
  - This chapter & previous two will use extensively the relationship between triple integrals over a continuous mass distribution and sums over a discrete set of particles

7. Moving Coördinate Systems
  - Fictitious forces (e.g., centrifugal, Coriolis)

- Prerequisites are important!
  - Basic Physics $\longrightarrow$ We will move quickly through chapter one, which should be review.
  - Calculus $\longrightarrow$ See handout for a a possibly new perspective.
  - Will also be using linear algebra later on, so keep that handy as well.

1 Dimensional Analysis

1.1 Preamble

Begin reading chapter one of Symon. Section 1.1 is mostly narrative, and we won't go over it in class, but it should be read to understand the perspective of the text. What we're going to start with today is dimensional analysis, which is actually in section 1.6, but I'm bringing it up first, because 1) it's very important and 2) it's the background for a warning about some of the equations in section 1.3.

1.2 Prologue

From a physicist's perspective, every physical quantity has to have the appropriate units associated with it. So just as you can't say "The distance between my house and campus is three" or "I've been waiting here for fifteen!" without specifying whether you mean three miles or three kilometers and whether you mean fifteen minutes or fifteen seconds, you'll have to say things like $x = 3\,\text{meters}$ or $m = 15\,\text{kg}$.

In this way of thinking, the units are fundamentally a part of the quantity, so Δt is not the number of seconds between events, but the time between events. Also, by using different units, you can express the same physical quantity in different ways, e.g., $5\,\text{min} = 300\,\text{sec}$.

Every physical unit in mechanics can be made up out of the basic building blocks of time, distance, and mass. (In electromagnetism, we also need to add electric charge to the list, and in thermodynamics we have to add absolute temperature.)

When checking the dimensions of an expression, it's customary to talk about mass, length, time, etc., rather than kilograms, meters, seconds, etc. For example, we don't say "F has units of newtons", but "F has units of force". This is because we could just as well write F in dynes or even pounds.

1.3 Dimensionally Well-Defined Expressions

The first thing we have to be able to say is what expressions even make sense. Basically, certain operations on physical quantities can only be performed if the quantities have consistent units. In particular, the following operations are only well defined if a and b have compatible units:

- Comparison, i.e., $a = b$, $a < b$, or $a > b$
- Addition and subtraction: $a + b$ or $a - b$

Compatible units means that when you count up the number of powers of time, length, and mass in each quantity, the numbers match. For instance, 3 inches and 60 centimeters both have units of length, so we can meaningfully add, subtract, or compare them. (We would have to convert from inches to centimeters to simplify the sum.) On the other hand, we cannot compare 10 kg to 12 meters, because the former has units of mass while the latter has units of length.

Some more complicated examples:

1. In the expression $15\,\mathrm{kg} \times (12\,\mathrm{m/s}) + 30\,\mathrm{N}$, we're trying to add something with units of (mass)×(length)/(time) to something with units of force = (mass)×(acceleration) = (mass)×(length)/(time)^2. These are different physical units, and so the sum doesn't make any sense.

2. If v is a velocity, t is a time, and x is a distance, we can sensibly write $x = vt$ because we are comparing something with units of length (x) to vt, which has units of velocity×time = (distance/time)×time = distance.

The requirement that two quantities to be added or compared have compatible physical units can be used to check for mistakes in problems. For instance, if I'm supposed to calculate a force, and I end up with something like $15\,\mathrm{kg} \times v^2$, where v is a velocity, I know I've gone wrone somewhere, because my answer has units of mass times velocity squared instead of mass times acceleration. And in this case, all I have to do is check the units on the intermediate steps. The first quantity with the wrong units is the one where the mistake was introduced. (Of course, this won't help me catch factors of two, sign errors, factors of π etc.!)

Note that homework grading will judge dimensionally nonsensical quantities more harshly than simply wrong ones.

1.3.1 Transcendental Functions

Special care should be taken in the case of transcendental functions like e^x, $\ln x$, $\sin x$, and $\cos x$. In these cases, the functions are ultimately defined by their Taylor series, for example

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \tag{1.1}$$

but that means that when we take the transcendental function of a quantity, we're adding different powers of the quantity. In particular, the expression above only makes sense if x

and x^3 have the same units. In other words, only if x^2, and hence x, is dimensionless. This is true for any transcendental function: *The argument of a transcendental function must be dimensionless.*[1] Note that this means all angles are dimensionless. We tend to say things like $\theta = \pi$ radians but that really is the same as $\theta = \pi$. Radians are what we call a "dimensionless unit", which means they don't actually have physical units, they're just there as a reminder that we're talking about an angle.

1.4 Conversion of Units

In the end, dimensional analysis is a lot like algebra: units are carried along like abstract variables. We can also use algebra to convert from one set of units to another. For example, if we know that

$$1\,\mathrm{m} = 100\,\mathrm{cm} \tag{1.2}$$

We can use it to convert 2.54 cm into meters, by saying

$$1 = \frac{1\,\mathrm{m}}{100\,\mathrm{cm}} \tag{1.3}$$

and thus

$$2.54\,\mathrm{cm} = 2.54\,\mathrm{cm} \times 1 = 2.54\,\mathrm{cm} \times \frac{1\,\mathrm{m}}{100\,\mathrm{cm}} = \frac{2.54}{100}\,\mathrm{m} = 2.54 \times 10^{-2}\,\mathrm{m} \tag{1.4}$$

Now, this may seem like overkill in such a simple problem, but when you're doing something more complicated like converting miles per hour into meters per second, it's a nice way to keep things straight and not multiply when you were supposed to divide:

$$55\frac{\mathrm{mi}}{\mathrm{hr}} \approx 55\frac{\mathrm{mi}}{\mathrm{hr}} \times \frac{8\,\mathrm{km}}{5\,\mathrm{mi}} \times \frac{10^3\,\mathrm{m}}{\mathrm{km}} \times \frac{1\,\mathrm{hr}}{60\,\mathrm{min}} \times \frac{1\,\mathrm{min}}{60\,\mathrm{sec}} = \frac{55 \times 8 \times 10^3}{5 \times 60 \times 60}\frac{\mathrm{m}}{\mathrm{s}} = \frac{880}{36}\frac{\mathrm{m}}{\mathrm{s}} \approx 24\frac{\mathrm{m}}{\mathrm{s}} \tag{1.5}$$

Note that in all these expressions, we include the units at every stage. It is a common sloppy practice to drop all the units and manipulate the numbers, then put the units back at the end. this will be marked as wrong on the homework.

2 Kinematics and Dynamics (Symon Sections 1.2 & 1.3)

Kinematics is the description of motion in terms of location, velocity, and acceleration. Dynamics is the method of determining the acceleration of a body.

2.1 Kinematics (Symon 1.2)

At this point, things are pretty straightforward. The only thing we should really mention carefully is the treatment of one-, two- and three-dimensional motion.

[1] Also, the value of the function is dimensionless.

In a one-dimensional problem, the particle's motion is described by a trajectory $x(t)$. Its time derivatives allow us to define a one-dimensional velocity and acceleration:

$$v(t) = \dot{x}(t) = \frac{dx}{dt} \tag{2.1a}$$

$$a(t) = \dot{v}(t) = \frac{dv}{dt} = \ddot{x}(t) = \frac{d^2x}{dt^2} \tag{2.1b}$$

Note that v and a, thus defined, can be positive or negative. In particular, v is a one-dimensional velocity, not a speed.

For a two- or three-dimensional problem, we need more than one coördinate to describe the particle's trajectory, and each coördinate has a corresponding velocity and acceleration. For example,

$$v_x(t) = \dot{x}(t) \tag{2.2a}$$

$$v_y(t) = \dot{y}(t) \tag{2.2b}$$

$$v_z(t) = \dot{z}(t) \tag{2.2c}$$

$$\tag{2.2d}$$

When we treat two- and three-dimensional problems in depth in chapter three, we'll introduce the notation of a vector $\vec{v}(t)$, but for the purposes of this quick review, we'll just talk about components.

2.2 Dynamics (Symon 1.3)

In this section Symon motivates Newton's second and third laws by describing a series of experiments to measure the masses of particles relative to some reference mass. Unfortunately, he chooses to give this reference object a mass of one, thereby rather needlessly throwing dimensional analysis out the window. For the purposes of this course, you should consider expressions like Symon's (1.5) to be incorrect, because they make no dimensional sense. To carry out the same demonstration while respecting dimensional analysis, you'd need to invent a unit of mass (call it a mass unit, or "mu"), and define it as the mass of the reference body: $m_1 = 1\,\text{mu}$. Then equation (1.5) would become

$$m_i = k_{1i} \times (1\,\text{mu}) = -\frac{\ddot{x}_1}{\ddot{x}_i} \times (1\,\text{mu}) \tag{2.3}$$

but in the end the mu's would cancel out of Symon's (1.7).

It's interesting to note that in the real world, there actually *is* a reference mass, kept at the International Bureau of Weights and Measures in Paris. Its mass is defined to be one kilogram (not just "one").

3 Newton's Laws (Symon Section 1.4)

With the preliminaries out of the way, we now recall Newton's Three Laws of Motion. Before stating them, it's worth mentioning why we still study Newtonian mechanics in spite of the fact that we now know that the world is in fact better described by other theories, such as Einstein's Special and General Theories of Relativity on the one hand, and Quantum Mechanics on the other. The reasons why classical mechanics is still relevant are:

- Newtonian mechanics, as an approximation to either Relativity or Quantum Mechanics, is perfectly adequate for describing a wide range of phenomena
- Some of the consequences of Newton's Laws will later turn out to be more easily generalized to other theories than Newton's Laws themselves are.

So here are Newton's Laws, expressed in their most familiar form:

1. Inertia: a body at rest will remain at rest, and a body in motion will continue with the same speed and direction, unless acted on by an outside force
2. The accelerations $\{\ddot{x}, \ddot{y}, \ddot{z}\}$ of a body of mass m under the influence of a net force with components $\{F_x, F_y, F_z\}$ will be given by

$$F_x = m\ddot{x} \tag{3.1a}$$
$$F_y = m\ddot{y} \tag{3.1b}$$
$$F_z = m\ddot{z} \tag{3.1c}$$

3. "For every action there is an equal and opposite reaction". I.e., if object 1 exerts a force with components $\{F_{21x}, F_{21y}, F_{21z}\}$ on object 2, then object 2 must be exerting a force with components

$$F_{12x} = -F_{21x} \tag{3.2a}$$
$$F_{12y} = -F_{21y} \tag{3.2b}$$
$$F_{12z} = -F_{21z} \tag{3.2c}$$

on object 1. This force is equal in magnitude and opposite in direction.

Note that the first law is in some sense a special case of the second law, since it says that when a body does not have a net force acting on it, its acceleration is zero. It was a very significant development in Physics, however, since it contradicted the previous assertion of Aristotelean Physics that the natural state of any material object was to be at rest.

One of the most fundamental consequences of Newton's laws can be written in terms of the momentum

$$p_x := mv_x = m\frac{dx}{dt} \tag{3.3a}$$
$$p_y := mv_y = m\frac{dy}{dt} \tag{3.3b}$$
$$p_z := mv_z = m\frac{dz}{dt} \tag{3.3c}$$
$$\tag{3.3d}$$

of a particle. The first law implies that an isolated particle (one subject to no forces) will have a constant momentum. The second law can be rewritten as

$$F_x = \frac{dp_x}{dt} \tag{3.4a}$$
$$F_y = \frac{dp_y}{dt} \tag{3.4b}$$
$$F_z = \frac{dp_z}{dt} \tag{3.4c}$$

The third law says that if two isolated particles exert forces on one another, the components of those forces will add up to zero:

$$F_{12x} + F_{21x} = 0 \tag{3.5a}$$
$$F_{12y} + F_{21y} = 0 \tag{3.5b}$$
$$F_{12z} + F_{21z} = 0 \tag{3.5c}$$

Substituting in the second law gives

$$\frac{dp_{1x}}{dt} + \frac{dp_{2x}}{dt} = \frac{d}{dt}(p_{1x} + p_{2x}) = 0 \tag{3.6a}$$
$$\frac{dp_{1y}}{dt} + \frac{dp_{2y}}{dt} = \frac{d}{dt}(p_{1y} + p_{2y}) = 0 \tag{3.6b}$$
$$\frac{dp_{1z}}{dt} + \frac{dp_{2z}}{dt} = \frac{d}{dt}(p_{1z} + p_{2z}) = 0 \tag{3.6c}$$

This tells us that in two-body interactions obeying Newton's laws, the total momentum of the system is a constant.

Not all forces obey Newton's third law. For example, in electromagnetism, the magnetic force on a charge particle depends on its velocity and is therefore not always directed on a line to the charge generating the magnetic field. However, in situations like this, the concept of conservation of momentum is so useful that we keep it by associating some momentum with the electromagnetic field itself, just the right amount so the bookkeeping works out right and momentum is conserved.

4 Examples

4.1 Some Familiar Forces

4.1.1 Gravity

There's a whole chapter devoted to the gravitational fields of general objects, but there are two useful approximations:

1. A mass m a distance r away from a point mass (or a spherical body) of mass M experiences a force of magnitude

$$|F| = \frac{GMm}{r^2} \tag{4.1}$$

where $G = 6.673 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$. The force is directed towards the attracting body. We write $|F|$ to emphasize that the *direction* of the force (and thus the sign of its projection along a particular coördinate axis) needs to be determined from the geometry of the problem.

2. Close to the surface of a planet, this simplifies. If you're a height h above the surface of a planet of radius R, the magnitude of the force becomes

$$|F| = \frac{GMm}{(R+h)^2} = m\frac{GM}{R^2}\left(1+\frac{h}{R}\right)^{-2} \tag{4.2}$$

If $h \ll R$, the term in parentheses is approximately equal to one, and the magnitude of the force becomes

$$|F| \approx mg \tag{4.3}$$

where g is the constant acceleration

$$g = \frac{GM}{R^2} \tag{4.4}$$

A common mistake is to assume that the force due to gravity is always mg; whether this is true or not depends on the problem, and in particular if the motion is confined to a small enough range of altitudes for the gravitational attraction to be roughly constant.

4.1.2 Normal Force

Both normal force and friction are simplified empirical descriptions of complicated microscopic forces. The normal force represents all of the complicated surface and internal interactions which keep an object from passing through another when they're in contact. It can basically be summed up by two properties:

1. It is directed perpendicular to the surface of contact

2. Its magnitude is just as large as needed to prevent one object from falling through the other

Usually condition 2 means that the normal force cancels out all the other forces acting in that direction. However, for a curved surface this will not be the case, since following the surface will involve some perpendicular acceleration.

4.1.3 Friction

The empirical definition of sliding friction is naturally divided into *kinetic* and *static* friction.

Kinetic Friction is the force which resists the sliding motion of two surfaces along each other, when the objects are actually sliding. Empirically, its magnitude is proportional to the normal force

$$|F_{fk}| = \mu_k N \tag{4.5}$$

where μ_k is a constant (a property of the surface interface) known as the coëfficient of kinetic friction. The direction is always against the direction of motion, which means that if v is the velocity in the direction parallel to the interface,

$$F_{fk} = -\mu_k N \frac{|v|}{v} \tag{4.6}$$

Static Friction is the force which prevents sliding, and it's a little trickier to describe, because, like the normal force, it's only as large as it needs to be to cancel out a net force along the surface. But just as any surface will eventually buckle under a weight put on it (which sets a limit on the normal force, which is typically too large to matter in an elementary problem), there is a limit to how strong the frictional force restraining motion can be. This is once again proportional to the normal force, so we write

$$|F_{fs}| \le \mu_s N \tag{4.7}$$

where μ_s is another constant called the coëfficient of static friction. For ordinary materials $\mu_s > \mu_k$.

4.2 Examples

The rest of Chapter One basically consists of a bunch of examples; you should read it carefully, since it's by working through examples that one learns the sorts of tricks needed to be able to apply the theory of classical mechanics.

For the purposes of illustration, we'll go through one example in class as well. The basic strategy we need to keep in mind is:

1. Consider each body in the problem individually and write down all the forces acting on it.

2. Choose a convenient coördinate system, in which most of the forces are exerted along coördinate axes (and thus have as few non-zero components as possible).

3. Use Newton's second law to write the equations of motion for each body in the system. Note that since Newton's second law has components in all the perpendicular directions relevant to the problem, each component equation must be satisfied, so in general there will be a system of equations, even when only one body's equation of motion is of interest.

4. Solve the resulting system of equations for the unknown(s) of interest, eliminating any irrelevant unknown quantities.

4.2.1 Example: Ballistic Motion without Air Resistance

A classic question is the following: Let a projectile of mass m be launched from ground level with an initial speed of v_0 at an angle θ with respect to the ground. Assume no air resistance and constant downward gravitational acceleration of g.

1. How far downrange will the projectile land if the ground is flat and level?
2. For fixed initial speed, for what value of θ will the projectile travel the maximum distance downrange?

1. In this problem, there is only one body of interest: the projectile. The only force acting on it is the force of gravity, directed downward with a magnitude of mg.
2. An obvious convenient coördinate system has the y axis pointing straight up and the initial velocity of the projectile in the xy-plane. We also choose the origin of time ($t = 0$) to be the moment when the projectile is launched. This tells us that the force acting on the projectile is

$$F_x = 0 \tag{4.8a}$$
$$F_y = -mg \tag{4.8b}$$

   and the initial velocity is

$$\dot{x}(0) = v_0 \cos\theta \tag{4.9a}$$
$$\dot{y}(0) = v_0 \sin\theta \tag{4.9b}$$

3. The equations of motion for the projectile are

$$F_x = 0 = m\ddot{x} \tag{4.10a}$$
$$F_y = -mg = m\ddot{y} \tag{4.10b}$$

   or, solving algebraically for the acceleration,

$$\ddot{x}(t) = 0 \tag{4.11a}$$
$$\ddot{y}(t) = -g \tag{4.11b}$$

4. To solve the problem, we first integrate the equations of motion to get the trajectory of the particle, using the initial conditions

$$\dot{x}(0) = v_0 \cos\theta \tag{4.12a}$$
$$\dot{y}(0) = v_0 \sin\theta \tag{4.12b}$$
$$x(0) = 0 \tag{4.12c}$$
$$y(0) = 0 \tag{4.12d}$$

Integrating the equations of motion once gives

$$\dot{x}(t) = \dot{x}(0) + \int_0^t \ddot{x}(t')\,dt' = \dot{x}(0) = v_0 \cos\theta \tag{4.13a}$$

$$\dot{y}(t) = \dot{y}(0) + \int_0^t \ddot{y}(t')\,dt' = \dot{y}(0) - gt = v_0 \sin\theta - gt \tag{4.13b}$$

Note that we have chosen to call the integration variable t' rather than t because we would like to have the arbitrary time t as one of our limits of integration.

Integrating a second time gives

$$x(t) = x(0) + \int_0^t \dot{x}(t')\,dt' = x(0) + v_0 t \cos\theta = v_0 t \cos\theta \tag{4.14a}$$

$$y(t) = y(0) + \int_0^t \dot{y}(t')\,dt' = y(0) + v_0 t \sin\theta - \frac{1}{2}gt^2 = v_0 t \sin\theta - \frac{1}{2}gt^2 \tag{4.14b}$$

If we define T to be the time at which the projectile lands, we know by definition that $y(T) = 0$. The point of the problem is to find the horizontal position, $x(T) = X$ at that time.

$$X = v_0 T \cos\theta \tag{4.15a}$$

$$0 = v_0 T \sin\theta - \frac{1}{2}gT^2 \tag{4.15b}$$

We have two equations in two unknowns (T and X). Since we don't care about T, we use the second equation to solve for T and eliminate it from the first. This gives us

$$v_0 T \sin\theta = \frac{1}{2}gT^2 \tag{4.16}$$

which has two solutions. One of them, $T = 0$, is not of interest, since we already know that $y(0) = 0$, and that's not the intersection with the ground that we're looking for. So instead, we use

$$T = \frac{2v_0 \sin\theta}{g} \tag{4.17}$$

Let's pause for a moment to check the units here. A velocity divided by an acceleration has units of

$$\frac{(\text{length})/(\text{time})}{(\text{length})/(\text{time}^2)} = (\text{time}) \tag{4.18}$$

so everything checks out. Now, plugging into the equation for X we have

$$X = v_0 \frac{2v_0 \sin\theta}{g} \cos\theta = \frac{2v_0^2}{g} \sin\theta \cos\theta \tag{4.19}$$

which is the answer to the first question. Again, we see that the dimensions are correct because

$$\frac{(\text{length})/(\text{time})^2}{(\text{length})/(\text{time}^2)} = (\text{length}) \tag{4.20}$$

Now for the second part of the question: We see that $X = 0$ for $\theta = 0$ (projectile fired horizontally) as well as for $\theta = \pi/2$ (projectile fired straight up). The maximum range is somewhere in between, and we can find the extremal values by looking for zeros of

$$\frac{dX}{d\theta} = \frac{2v_0^2}{g}(\cos^2\theta - \sin^2\theta) \tag{4.21}$$

which we see occur at $\theta = \pi/4$. This gives us the classic result that the maximum range is achieved when the projectile is fired at a 45° angle.

A Appendix: Correspondence to Class Lectures

Date	Sections	Pages
2003 August 26	0–1	1–4
2003 August 28	2–4.2	4–9
2003 September 2	4.2.1	10–12