One-Dimensional Motion (Symon Chapter Two)

Physics A300[*]

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Contents

Part I
Consequences of Newton's Second Law

Newton's second law, in its one-dimensional form, says

$$F = m\frac{d^2x}{dt^2} = \frac{dp}{dt} \tag{0.1}$$

In general, the force F experienced by the particle can depend on the location x of the particle, its velocity $v = \frac{dx}{dt}$, and the time t itself. So we write the equation of motion as

$$m\ddot{x} = F(x, v, t) \tag{0.2}$$

What we do with the equation of motion depends on the nature of the problem. Sometimes we are to assume the trajectory $x(t)$ as given, and make deductions using the form of $F(x(t), \dot{x}(t), t)$. Sometimes we need to derive the form of $x(t)$ itself by solving the differential equation (0.2). The method depends on the form of F. Following sections 2.1 and 2.3–2.5, we consider several different possibilities.

1 Momentum and Energy

First, we work from the perspective that the trajectory $x(t)$ is given, and ask about the implications of Newton's second law, which can be written

$$\frac{dp}{dt} = F(x(t), \dot{x}(t), t) \tag{1.1}$$

This is a first-order ordinary differential equation in one variable, and can be solved by integration

$$\int_{t_1}^{t_2} F(x(t), \dot{x}(t), t)\, dt = \int_{t_1}^{t_2} \frac{dp}{dt}\, dt = p(t_2) - p(t_1) = p_2 - p_1 \tag{1.2}$$

The integral $\int F\, dt$, which gives the change in momentum as a result of an applied force, is called the *impulse*. This is an especially useful quantity in cases where the detailed time dependence of the force is not known, but the change in momentum is, such as a ball bouncing elastically off a wall.

Another useful integral of the force is the *work* done between two times

$$W_{21} = \int_{t_1}^{t_2} F(x(t), \dot{x}(t), t)\, \dot{x}(t)\, dt \tag{1.3}$$

again applying Newton's Second Law we see

$$W_{21} = \int_{t_1}^{t_2} F\, \dot{x}\, dt = \int_{t_1}^{t_2} m\ddot{x}\dot{x}\, dt = \int_{t_1}^{t_2} \frac{d}{dt}\underbrace{\left(\frac{1}{2}m\dot{x}(t)^2\right)}_{T(t)} dt = T(t_2) - T(t_1) = T_2 - T_1 \tag{1.4}$$

The quantity

$$T(t) = \frac{1}{2} m \dot{x}^2 \tag{1.5}$$

is called the *kinetic energy*; we've just derived the *Work-Energy Theorem* (1.4) which shows that the work done on a particle over part of its trajectory is equal to the change in its kinetic energy.

If $x(t)$ is a monatonic function between two times, we can change variables in the integral defining the work from t to x:

$$W_{21} = \int_{t_1}^{t_2} F\,\dot{x}\,dt = \int_{x_1}^{x_2} F\,dx \tag{1.6}$$

If $x(t)$ is not a monatonic function, we can still break up the integral over time into intervals over which it is, and then write the contribution to the work for each interval as an integral over the position x.

2 Forces Depending only on Time (Symon Section 2.3)

In the case where the force depends only time and not the position or velocity of the particle, (0.2) becomes

$$m\ddot{x}(t) = F(t) \tag{2.1}$$

If we know the initial position $x(0)$ and velocity $\dot{x}(0)$, we can integrate twice to find the trajectory $x(t)$:

$$\dot{x}(t) = \dot{x}(0) + \int_0^t \frac{F(t')}{m}\,dt \tag{2.2}$$

and

$$\begin{aligned} x(t) &= x(0) + \int_0^t \dot{x}(t')\,dt' = x(0) + \int_0^t \left(\dot{x}(0) + \int_0^{t'} \frac{F(t'')}{m}\,dt'' \right) dt' \\ &= x(0) + \dot{x}(0)t + \int_0^t \int_0^{t'} \frac{F(t'')}{m}\,dt''\,dt' \end{aligned} \tag{2.3}$$

Of course, in a sufficiently complicated problem, we may not have been able to choose 0 as the initial time, which gives the form in Symon with the lower limits of the integrals as t_0.

We have used this method already in the special case where F is a constant. In the text it's applied to a sinusoidal driving force, which we sketch here.

2.1 Example: Sinusoidal Driving Force

Suppose the force is $F(t) = F_0 \cos(\omega t + \theta_0)$. Then Newton's Second Law tells us

$$\ddot{x}(t) = \frac{F(t)}{m} = \frac{F_0}{m} \cos(\omega t + \theta_0) \tag{2.4}$$

If the initial conditions are $\dot{x}(0) = v_0$ and $x(0) = x_0$, we integrate once to get

$$\begin{aligned}\dot{x}(t) &= \dot{x}(0) + \int_0^t \frac{F(t')}{m}\,dt' = v_0 + \frac{F_0}{m}\int_0^t \cos(\omega t' + \theta_0)\,dt' = v_0 + \frac{F_0}{m}\left[\frac{1}{\omega}\sin(\omega t' + \theta_0)\right]_0^t \\ &= v_0 + \frac{F_0}{m\omega}\left(\sin(\omega t + \theta_0) - \sin\theta_0\right)\end{aligned} \tag{2.5}$$

Integrating again gives

$$\begin{aligned}x(t) &= x(0) + \int_0^t \dot{x}(t')\,dt' = x_0 + \int_0^t \left(v_0 - \frac{F_0\sin\theta_0}{m\omega} + \frac{F_0}{m\omega}\sin(\omega t' + \theta_0)\right)\,dt' \\ &= x_0 + \left[\left(v_0 - \frac{F_0\sin\theta_0}{m\omega}\right)t' - \frac{F_0}{m\omega^2}\cos(\omega t' + \theta_0)\right]_0^t \\ &= x_0 + \left(v_0 - \frac{F_0\sin\theta_0}{m\omega}\right)t - \frac{F_0}{m\omega^2}[\cos(\omega t + \theta_0) - \cos\theta_0]\end{aligned} \tag{2.6}$$

3 Forces Depending only on Velocity (Symon Section 2.4)

The next simplest force is one which depends only on the position, but it's actually got a deeper meaning, so we postpone it for the moment and turn to a force depending only on the velocity. In that case, the key to integrating (0.2) is to write it as

$$m\frac{dv}{dt} = F(v) \tag{3.1}$$

To do the integral, we need to put everything with a v in it on one side of the equation and everything with a t on the other:

$$\frac{dv}{F(v)} = \frac{dt}{m} \tag{3.2}$$

so that when we integrate we get

$$\int_{v(0)}^{v(t)} \frac{dv'}{F(v')} = \int_0^t \frac{dt'}{m} = \frac{t}{m} \tag{3.3}$$

We then need to solve the resulting equation for v as a function of t so that we can ultimately find

$$x(t) = x(0) + \int_0^t v(t')\,dt' \tag{3.4}$$

The important velocity-dependent forces in mechanics are all damping forces, which are always counter to the direction of motion. In general, the velocity dependence is fairly complicated, but in some regimes, it can be reasonably approximated by a power law:

$$F = -b\,|v|^n\,\frac{v}{|v|} \tag{3.5}$$

Written this way, it's automatically in the opposite direction to the motion. Sliding friction, which has a constant magnitude, is just the $n = 0$ case of this. Note that for odd n, the absolute values are not necessary, for example if $n = 1$,

$$F(v) = -b\,|v|\,\frac{v}{|v|} = -bv \tag{3.6}$$

3.1 Example: Coasting to a Stop under Viscous Damping

As an example of how the method is applied, we consider the motion of an object which starts off with velocity v_0 and then is decelerated by the viscous damping force (3.6). Newton's Second Law is then

$$m\frac{dv}{dt} = -bv \tag{3.7}$$

which makes the integral

$$\int_{v_0}^{v} \frac{dv'}{v'} = -\frac{b}{m}\int_0^t dt' \tag{3.8}$$

i.e.,

$$\ln v' \bigg|_{v_0}^{v} = -\left.\frac{b}{m}\right|_0^t \tag{3.9}$$

or

$$\ln v - \ln v_0 = \ln\frac{v}{v_0} = -\frac{bt}{m} \tag{3.10}$$

Before we proceed to solving this equation for v, we note that this is an example of how logarithms can confuse dimensional analysis if we're not careful. The function $\ln x$, like e^x, is transcendental, and therefore we should only take logarithms of dimensionless quantities. And sure enough, v/v_0 is dimensionless. On the other hand, there are intermediate expressions like $\ln v$ and $\ln v_0$ where we seem to be breaking our own rules by taking transcendental functions of numbers with units of velocity. Of course, what makes this okay is that we combine the two of them to produce the log of something dimensionless. Because of the relationship

$$\ln a - \ln b = \ln\frac{a}{b} \tag{3.11}$$

we can't rule out an expression on dimensional grounds just because it involves the logarithm of a dimensionful number. If there is another logarithmic term which can be combined with the first to cancel out the units, it's okay. It's especially important to be aware of this when doing indefinite integrals, since the "arbitrary constant" might just contain the logarithm we need.

Okay, so having defined $v(t)$ implicitly in (3.10), we can invert the relationship by solving for v:

$$\frac{v}{v_0} = e^{-bt/m} \tag{3.12}$$

so

$$v(t) = v_0 e^{-bt/m} \tag{3.13}$$

Now we can integrate this to get

$$\begin{aligned} x(t) &= x(0) + \int_0^t v_0 e^{-bt'/m}\,dt' = x(0) + \frac{mv_0}{-b} e^{-bt'/m}\Big|_0^t = x(0) - \frac{mv_0}{b}\left(e^{-bt/m} - 1\right) \\ &= x(0) + \frac{mv_0}{b}\left(1 - e^{-bt/m}\right) \end{aligned} \tag{3.14}$$

Note that in this case the term arising from the lower limit $t = 0$ was *not* zero, so it was very important that we treated the limits of integration properly.

Further implications of this solution are explored in the text.

3.2 Dimensional Considerations and "Small" Damping Forces

In Sections 2.4 and 2.6, Symon considers two different problems with velocity-dependent damping forces. They can be summarized as follows

1. A boat pushed with an initial velocity v_0 is decelerated by a damping force $F = -bv$
2. An object is dropped from rest, and falls under the influence of gravity and air resistance, experiencing a net force $F = -mg - bv$

Or, mathematically, in problem #1

$$\ddot{x}(t) = -\frac{b}{m}\dot{x}(t) \tag{3.15a}$$

$$\dot{x}(0) = v_0 \tag{3.15b}$$

$$x(0) = 0 \tag{3.15c}$$

while in problem #2

$$\ddot{x}(t) = -g - \frac{b}{m}\dot{x}(t) \tag{3.16a}$$

$$\dot{x}(0) = 0 \tag{3.16b}$$

$$x(0) = 0 \tag{3.16c}$$

In both cases, the problem can be solved exactly, and the solution approximated for "small t" and "large t". But what is meant by something with dimensions being large or small? We can't be talking about $t \ll 1$ or $t \gg 1$ because our lessons in dimensional analysis have told us that it's meaningless to compare two quantities with different dimensions. Instead, we have to compare t to something with dimensions. Each problem has three dimensionful parameters associated with it:

1. m, b, and v_0 in problem 1
2. m, b, and g in problem 2

In fact, in the equations of motion, m and b only appear in the combination b/m, so the list of parameters can actually be cut to two in each case:

1. b/m and v_0 in problem 1
2. b/m and g in problem 2

The combination b/m has units of

$$\frac{\text{(acceleration)}}{\text{(velocity)}} = \frac{1}{\text{(time)}} \tag{3.17}$$

so in each problem the *only* combination of parameters with units of time is m/b, and in each case we mean $\frac{bt}{m} \ll 1$ or $\frac{bt}{m} \gg 1$. So it's worth noting that in an absolute sense, we cannot say than b is small for either of these problems. Only the combination $\frac{bt}{m}$ can be large or small, which means whatever the value of b, t will eventually get large enough that the damping term will dominate. (This is the boat drifting to a near-stop in problem #1 and the falling object reaching terminal velocity in problem #2.)

Note that this would not necessarily be the situation in a problem with an additional dimensionful parameter. For example, if we fire a projectile into the air with initial speed v_0 subject to the equation of motion

$$\ddot{x}(t) = -g - \frac{b}{m}\dot{x}(t) \tag{3.18}$$

we now have three dimensionful parameters b/m, g, and v_0, and there are combinations of these which are dimensionless, allowing us to say "b is small" in a sense which applies for all times by constructing a dimensionless combination of b/m, g, and v_0. Similarly, we can construct a combination of t, g, and v_0 which is dimensionless and thus say "t is not too big" in a way which is independent of the size of b.

4 Forces Depending only on Position (Symon Section 2.5)

4.1 Potential Energy

If the force $F(x)$ experienced by a particle is a function only of the particle's location x and not the time it's there or how fast it's moving, the concept of work becomes even more useful. Recall that the change in kinetic energy $T = \frac{1}{2}mv^2$ between two instants in a particle's trajectory is given by

$$T_2 - T_1 = T(t_1) - T(t_2) = \int_{t_1}^{t_2} F(x)\frac{dx}{dt}\,dt = \int_{x_1}^{x_2} F(x)\,dx \tag{4.1}$$

This is even more simply written in terms of the *Potential Energy*

$$V(x) = -\int F(x)\,dx \tag{4.2}$$

Note that we've used an indefinite integral to define the potential energy. This means it's only defined up to a constant, and we need to fix that constant once per problem. An equivalent definition, which extends more easily to multiple dimensions, is

$$F(x) = -V'(x) \tag{4.3}$$

In terms of the potential energy, we have

$$T_2 - T_1 = -\left[V(x)\right]_{x_1}^{x_2} = -[V(x_2) - V(x_1)] = -(V_2 - V_1) \tag{4.4}$$

This means that

$$T_1 + V_1 = T_2 + V_2 \tag{4.5}$$

or that the total energy

$$E = T + V = \frac{1}{2}m\dot{x}^2 + V(x) \tag{4.6}$$

is a constant of the motion.

The potential energy can be used to gain a lot of insight into the motion of a particle, and in particular into the possible ways a particle can move in a given potential. An individual trajectory can be characterized, among other things, by its constant value of E. Solving for the velocity tells us

$$\dot{x}^2 = \frac{2[E - V(x)]}{m} \tag{4.7}$$

We can tell a number of things from this:

1. The particle can only move in regions where $V(x) < E$
2. If the particle reaches a point where $V(x) = E$, the velocity is instantaneously zero
3. If this is not a local maximum, the particle must turn around and go the other way, since it can't move into the forbidden region
4. A minimum is a stable equilibrium
5. A maximum is an unstable equilibrium

4.2 Interlude: Taylor Series and the Euler Relation

Math department graffiti: "$e^{i\pi} = -1$. Yeah, right." But it is true. Unfortunately, have to prove with Taylor Series:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \tag{4.8}$$

Apply this to three functions: $e^{i\theta}$, $\cos\theta$, and $\sin\theta$:

$f(\theta)$	$e^{i\theta}$	$\cos\theta$	$\sin\theta$
$f(0)$	1	1	0
$f'(\theta)$	$ie^{i\theta}$	$-\sin\theta$	$\cos\theta$
$f'(0)$	i	0	1
$f'''(\theta)$	$-e^{i\theta}$	$-\cos\theta$	$-\sin\theta$
$f'''(0)$	-1	-1	0
$f^{(4)}(\theta)$	$-ie^{i\theta}$	$\sin\theta$	$-\cos\theta$
$f^{(4)}(0)$	$-i$	0	-1
$f^{(n)}(0)(n \text{ even})$	$(-1)^{n/2} = i^n$	$(-1)^{n/2} = i^n$	0
$f^{(n)}(0)(n \text{ odd})$	$-1^{(n-1)/2} i = i^n$	0	$-1^{(n-1)/2} = i^{n-1}$

So the three Taylor series are

$$e^{i\theta} = 1 + i\theta - \frac{1}{2}\theta^2 - \frac{i}{3!}\theta^3 + \dots \tag{4.9a}$$

$$\cos\theta = 1 - \frac{1}{2}\theta^2 + \dots \tag{4.9b}$$

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \dots \tag{4.9c}$$

from which, along with

$$i\sin\theta = i\theta - \frac{i}{3!}\theta^3 + \dots \tag{4.10}$$

we see

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{4.11}$$

This is called the *Euler relation*.

4.3 Small Oscillations about a Stable Equilibrium

Recall that a local minimum is a stable equilibrium point. If the equilibrium is at $x = x_e$, equilibrium tells us $V'(x_e) = 0$ and stability tells us $V''(x_e) > 0$. Expand in powers of $x - x_e$ to get

$$V(x) = V(x_e) + V'(x_e)(x-x_e) + \frac{1}{2}V''(x_e)(x-x_e)^2 + \frac{1}{3!}V^{(3)}(x_e)(x-x_e)^3 + \mathcal{O}([x-x_e]^4) \tag{4.12}$$

To make life easy, we can choose the origin of the coördinate so that the equilibrium point of interest lies at $x = 0$. Also exploit the arbitrary freedom to add a constant to the potential, and set $V(0) = 0$. Then if we do a Taylor expansion of $V(x)$ about $x = 0$, we'll find

$$V(x) = \overbrace{V(0)}^{0 \text{ by def'n}} + x \overbrace{V'(0)}^{0 \text{ for equilibrium}} + \frac{x^2}{2} \overbrace{V''(0)}^{>0 \text{ by stability}} + \frac{x^3}{6} V'''(0) + \dots \tag{4.13}$$

So for small enough x, i.e.,

$$x \ll \frac{3V''(0)}{V'''(0)} \tag{4.14a}$$

$$x \ll \frac{12V''(0)}{V''''(0)} \tag{4.14b}$$

etc

the potential is well approximated by

$$V(x) \approx \frac{1}{2}kx^2 \tag{4.15}$$

where

$$k = V''(0) > 0 \tag{4.16}$$

This is called the *Harmonic Oscillator Potential*

Part II
The Harmonic Oscillator

So far we've looked at the special cases $F(t)$, $F(v)$ and $F(x)$. The rest of the chapter consists of building up step-by-step a classic problem involving a force which is a sum of all three:

- A linear position-dependent restoring force $F_{\text{restoring}}(x) = -kx$ arising from a potential
- A linear velocity-dependent damping force $F_{\text{damping}}(v) = -bv$
- A time-dependent external driving force $F_{\text{driving}}(t)$, both in general and of the specific sinusoidal form $F(t) = F_0 \cos(\omega t + \theta_0)$

We build this up by adding one force at a time.

5 The Simple Harmonic Oscillator

As we showed in section 4.3, a quite generic potential can be approximated near a local minimum, which for convenience we take to be at $x = 0$, as

$$V(x) = \frac{1}{2}kx^2 \tag{5.1}$$

where $k = V''(x) > 0$. (There are some exceptions, e.g., if $V''(0) = 0 = V'''(0)$ and $V^{(4)}(0) > 0$.) The force associated with this potential is

$$F(x) = -V'(x) = -kx \tag{5.2}$$

Now, when you've learned about harmonic oscillators, you've probably started with something called Hooke's Law:

$$F_{\text{Hooke}}(x) = -kx \tag{5.3}$$

which was probably described as a special property of springs. But actually, Hooke's Law is an approximation to just about any force field sufficiently close to a stable equilibrium, thanks to the Taylor expansion (4.13).

Returning to the equation of motion

$$m\ddot{x} = F(x) = -kx \tag{5.4}$$

we can write this as

$$\ddot{x} + \omega_0^2 x = 0 \tag{5.5}$$

where we have defined

$$\omega_0 = \sqrt{k/m} \tag{5.6}$$

Note that this does indeed have units of frequency, which is easiest to see by looking at (5.5).

To obtain the general solution to (5.5), we note that it's a second order homogeneous linear ordinary differential equation.

- second order because the highest number of time derivatives is two
- homogeneous because there are no terms that don't depend on $x(t)$ or its derivatives
- linear because each term has only one power of $x(t)$ or one of its derivatives
- ordinary (as opposed to partial) because it has only one independent variable t.

This means it has two important properties.

- If $x_1(t)$ and $x_2(t)$ are solutions, then the superposition $c_1 x_1(t) + c_2 x_2(t)$ is also, for any constants c_1 and c_2.
- If we have *two* linearly independent solutions $x_1(t)$ and $x_2(t)$, then any solution can be written as a superposition of those two.

To find the needed two independent solutions, we try a solution of the form

$$x(t) = ce^{pt} \tag{5.7}$$

differentiating gives

$$\dot{x}(t) = cpe^{pt} \tag{5.8a}$$

$$\ddot{x}(t) = cp^2 e^{pt} \tag{5.8b}$$

so (5.5) becomes, for this candidate solution,

$$cp^2 e^{pt} + \omega_0^2 ce^{pt} = 0 \tag{5.9}$$

Dividing by ce^{pt}, we have

$$p^2 + \omega_0^2 = 0 \tag{5.10}$$

now, the two solutions to this are

$$p = \pm i\omega_0 \tag{5.11}$$

which would make the general solution

$$x(t) = c_+ e^{i\omega_0 t} + c_- e^{-i\omega_0 t} \tag{5.12}$$

Now, this looks funny, since it involves complex numbers, and our physical $x(t)$ will have to be real. But if we choose $c_\pm$ carefully, we can ensure that $x(t)$ is indeed real. The key is the Euler relation (4.11) which allows us to write

$$x(t) = c_+(\cos\omega_0 t + i\sin\omega_0 t) + c_-(\cos\omega_0 t - i\sin\omega_0 t) = (c_+ + c_-)\cos\omega_0 t + i(c_+ - c_-)\sin\omega_0 t \tag{5.13}$$

Now, if we define new constants

$$A_c = c_+ + c_- \tag{5.14a}$$

$$A_s = i(c_+ - c_-) \tag{5.14b}$$

and require that A_c and A_s be real, we can give the general real solution

$$x(t) = A_c \cos\omega_0 t + A_s \sin\omega_0 t \tag{5.15}$$

in terms of arbitrary real constants A_c and A_s. (Symon calls these B_1 and B_2, respectively.) A slightly more useful set of constants can be obtained by defining $A \geq 0$ and θ so that

$$A_c = A\cos\theta \tag{5.16a}$$

$$A_s = -A\sin\theta \tag{5.16b}$$

with this definition, we have

$$x(t) = A\cos\omega_0 t\cos\theta - A\sin\omega_0 t\sin\theta = A\cos(\omega_0 t + \theta) \tag{5.17}$$

Note that this is at a maximum when $\omega_0 t = -\theta$, and the maximum value is A. A is called the amplitude and θ the phase of the oscillation. (It's fairly common to use $\phi = -\theta$ as the phase instead.)

6 The Damped Harmonic Oscillator

The next level of complexity we introduce into the system is a retarding force. We considered a whole family of these in section 3, but the most convenient one is the viscous damping force (3.6), which we write as

$$F_{\text{damping}} = -b\dot{x} \tag{6.1}$$

This is the kind of resisting force you get when moving through a viscous medium like oil or honey, and the usual physical image is to have the mass on a spring attached to some sort of apparatus which moves an object through a sealed pot of oil (this was for some reason described as chicken fat when I was a student), generating the damping force described by (6.1).

This is not a terribly common sort of damping in what you'd think of as traditional mechanical systems, but it does come up in more complicated oscillations, and it's the natural resisting term in the analogous electric circuit. Of course, the real reason we talk about it here is that it's linear in the velocity, so it keeps the differential equation linear.

Putting together the restoring force and the damping force, the one-dimensional equation of motion becomes

$$m\ddot{x} = -b\dot{x} - kx \tag{6.2}$$

As before, we divide by m and define the natural frequency

$$\omega_0 = \sqrt{k/m} \tag{6.3}$$

We also define a damping parameter with units of inverse time

$$\gamma = \frac{b}{2m} \tag{6.4}$$

Note the factor of two, which will make things more convenient later. The ODE becomes

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \tag{6.5}$$

This is another second order homogeneous linear differential equation, so we apply the same strategy to find two independent solutions, guessing the form

$$x(t) = ce^{pt} \tag{6.6}$$

This makes the differential equation

$$cp^2e^{pt} + 2\gamma pce^{pt} + \omega_0^2 ce^{pt} = 0 \tag{6.7}$$

Again, we can divide by ce^{pt} to get

$$p^2 + 2\gamma p + \omega_0^2 = 0 \tag{6.8}$$

Applying the quadratic equation gives the solutions

$$p_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \tag{6.9}$$

If $\gamma \neq \omega_0$ this will give us our two independent solutions; we'll handle the $\gamma = \omega_0$ case later. Clearly, the nature of the two solutions will depend on the sign of $\gamma^2 - \omega_0^2$. We give names to the three situations as follows

$\gamma^2 - \omega_0^2 < 0$	Underdamped
$\gamma^2 - \omega_0^2 = 0$	Critically damped
$\gamma^2 - \omega_0^2 > 0$	Overdamped

6.1 Underdamped Oscillations

In this case, $\sqrt{\gamma^2 - \omega_0^2}$ is an imaginary number, so we define

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2} \tag{6.10}$$

so that

$$p_\pm = -\gamma \pm i\omega_1 \tag{6.11}$$

Now the general solution is

$$x(t) = c_+e^{p_+t} + c_-e^{p_-t} = c_+e^{-\gamma t}e^{i\omega_1 t} + c_-e^{-\gamma t}e^{-i\omega_1 t} \tag{6.12}$$

Again, we use the Euler relation to write

$$\begin{aligned} x(t) =& c_+e^{-\gamma t}(\cos\omega_1 t + i\sin\omega_1 t) + c_-e^{-\gamma t}(\cos\omega_1 t - i\sin\omega_1 t) \\ =& (c_+ + c_-)e^{-\gamma t}\cos\omega_1 t + i(c_+ - c_-)e^{-\gamma t}\sin\omega_1 t \end{aligned} \tag{6.13}$$

and define new real constants

$$A_c = c_+ + c_- \tag{6.14a}$$

$$A_s = i(c_+ - c_-) \tag{6.14b}$$

(again, Symon calls these B_1 and B_2) which gives us a solution

$$x(t) = A_c e^{-\gamma t}\cos\omega_1 t + A_s e^{-\gamma t}\sin\omega_1 t \tag{6.15}$$

And as before we can define A and θ by

$$A_c = A\cos\theta \tag{6.16a}$$

$$A_s = -A\sin\theta \tag{6.16b}$$

which gives us a general solution

$$x(t) = Ae^{-\gamma t}\cos(\omega_1 t + \theta) \tag{6.17}$$

Note that this differs from the solution from the undamped oscillator only in that the oscillations are multiplied by the decaying exponential $e^{-\gamma t}$, and in that the oscillation frequency is

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2} \tag{6.18}$$

In the limit $\gamma \to 0$, $\omega_1 \to \omega_0$ and we get back the undamped solution, as we must.

6.1.1 Example: Choosing a Particular Solution to Match Initial Conditions

To give an example of how to set the values of A and θ if we're given initial conditions for the problem, suppose we're told to consider an underdamped oscillator with spring constant $m\omega_0^2$ and damping parameter $2m\gamma$, which is released from rest at a position x_0 away from equilibrium. We know the general solution is given by (6.17), and its derivative is

$$\dot{x}(t) = -\gamma Ae^{-\gamma t}\cos(\omega_1 t + \theta) - \omega_1 Ae^{-\gamma t}\sin(\omega_1 t + \theta) \tag{6.19}$$

That means that the general solution has

$$x(0) = A\cos(\theta) \tag{6.20a}$$

$$\dot{x}(0) = -\gamma A\cos(\theta) - \omega_1 A\sin(\theta) = -A(\omega_1\sin\theta + \gamma\cos\theta) \tag{6.20b}$$

so we need to determine A and θ from

$$x_0 = A\cos\theta \tag{6.21a}$$

$$0 = -A(\omega_1\sin\theta + \gamma\cos\theta) \tag{6.21b}$$

Equation (6.21b) tells us

$$\omega_1 \sin\theta = -\gamma \cos\theta \tag{6.22}$$

or

$$\tan\theta = -\frac{\gamma}{\omega_1} \tag{6.23}$$

Now, to substutite this back into (6.21a), we need to define $\cos\theta$ in terms of $\tan\theta$. We can do this by using one of the only three trig identities we need to memorize:

$$\cos^2\theta + \sin^2\theta = 1 \tag{6.24}$$

We divide both sides by $\cos^2\theta$ to get

$$1 + \tan^2\theta = \frac{1}{\cos^2\theta} \tag{6.25}$$

or

$$\cos\theta = \frac{1}{\pm\sqrt{1+\tan^2\theta}} \tag{6.26}$$

we're justified in taking the positive square root in this case, because we can take A to have the same sign as x_0 in (6.21a).

Now, consider

$$\sqrt{1+\tan^2\theta} = \sqrt{1+\frac{\gamma^2}{\omega_1^2}}\ . \tag{6.27}$$

For the most part, life is simplified by not writing out $\omega_1 = \sqrt{\omega_0^2 - \gamma^2}$ explicitly, but here it's squared, so it actually helps a little:

$$\begin{aligned}\frac{1}{\cos\theta} &= \sqrt{1+\tan^2\theta} = \sqrt{1+\frac{\gamma^2}{\omega_1^2}} = \sqrt{1+\frac{\gamma^2}{\omega_0^2-\gamma^2}} = \sqrt{\frac{\omega_0^2-\gamma^2}{\omega_0^2-\gamma^2}+\frac{\gamma^2}{\omega_0^2-\gamma^2}} = \sqrt{\frac{\omega_0^2}{\omega_0^2-\gamma^2}} \\ &= \frac{\omega_0}{\omega_1}\ .\end{aligned} \tag{6.28}$$

So this means

$$\cos\theta = \frac{\omega_1}{\omega_0} \tag{6.29}$$

(As a sanity check, note that by definition $\omega_1 \le \omega_0$.) It's then easy to see that

$$\sin\theta = \tan\theta\cos\theta = -\frac{\gamma}{\omega_0} \tag{6.30}$$

We can now substitute into (6.21a) to get

$$x_0 = A\cos\theta = A\frac{\omega_0}{\omega_1} \tag{6.31}$$

or

$$A = \frac{\omega_0}{\omega_1}x_0 \tag{6.32}$$

Now, we know A and θ, so we can substitute them into (6.17), but the answer will be a little awkward in the form

$$x(t) = x_0 \frac{\omega_0}{\omega_1} e^{-\gamma t} \cos\left(\omega_1 t - \tan^{-1}\left[\frac{\gamma}{\omega_1}\right]\right) \ ; \tag{6.33}$$

since we know $\cos\theta$ and $\sin\theta$, it's nicer to write

$$\begin{aligned} x(t) =& A\cos(\omega_0 t + \theta) = A(\cos\omega_1 t \cos\theta - \sin\omega_1 t \sin\theta) = x_0 \frac{\omega_0}{\omega_1} e^{-\gamma t}\left(\frac{\omega_1}{\omega_0}\cos\omega_1 t + \frac{\gamma}{\omega_0}\sin\omega_1 t\right) \\ =& x_0 e^{-\gamma t}\left(\cos\omega_1 t + \frac{\gamma}{\omega_1}\sin\omega_1 t\right) \ . \end{aligned} \tag{6.34}$$

Of course, this makes us see in retrospect that it would have been easier to start from the form (6.15), but we did learn some interesting things along the way about the phase angle in this case.

Finally, since the statement of the problem talked about ω_0 and γ, but not ω_1, we should be sure to define what we mean by ω_1, when presenting the answer, so we say:

$$x(t) = x_0 e^{-\gamma t}\left(\cos\omega_1 t + \frac{\gamma}{\omega_1}\sin\omega_1 t\right) \qquad \text{where } \omega_1 = \sqrt{\omega_0^2 - \gamma^2} \tag{6.35}$$

6.2 Overdamped Oscillations

Turning to another general class of solution, consider the case when $\gamma^2 - \omega_0^2 > 0$. In this case the two roots

$$p_{\pm} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \tag{6.36}$$

are real. This makes the general solution

$$x(t) = c_1 e^{-\gamma_1 t} + c_2 e^{-\gamma_2 t} \tag{6.37}$$

where

$$\gamma_1 = \gamma + \sqrt{\gamma^2 - \omega_0^2} \tag{6.38a}$$

$$\gamma_2 = \gamma - \sqrt{\gamma^2 - \omega_0^2} \tag{6.38b}$$

This does not oscillate, but has two terms which go to zero at different rates. (It can, however, change sign once or twice if c_+ and c_- have different signs.)

An overdamped oscillator has "too much" damping in the sense that if it starts off out of equilibrium, the damping force can resist the motion so much that it takes a long time to get back to equilibrium. An example would be a door that takes forever to close because the pneumatic cylinder provides too much resistance.

6.3 Critically Damped Oscillations

We turn at last to the special case $\gamma = \omega_0$. In this case our attempt to find two independent solutions of the form

$$x(t) = ce^{pt} \tag{6.39}$$

fails, because p solves the quadratic equation

$$0 = p^2 + 2\gamma p + \gamma^2 = (p + \gamma)^2 \tag{6.40}$$

which has only one solution, $p = -\gamma$

It's thus necessary to cast a wider net in looking for a pair of independent solutions, and it turns out that what works is

$$x(t) = (C_1 + C_2 t)e^{-\gamma t} \tag{6.41}$$

We can verify that this is a solution for all C_1 and C_2; differentiating gives

$$\dot{x}(t) = (-\gamma C_1 + C_2 - C_2\gamma t)e^{-\gamma t} \tag{6.42}$$

and differentiating again gives

$$\ddot{x}(t) = (\gamma^2 C_1 - \gamma C_2 - C_2\gamma + C_2\gamma^2 t)e^{-\gamma t} \tag{6.43}$$

So that

$$\ddot{x} + 2\gamma\dot{x} + \gamma^2 = \big(C_1(\gamma^2 - 2\gamma^2 + \gamma^2) + C_2(\gamma^2 t + 2\gamma - 2\gamma^2 t - 2\gamma + \gamma^2 t)\big) = 0 \tag{6.44}$$

Note that in all three cases, the general solution dies off exponentially as $t \to 0$, so long as $\gamma > 0$.

7 The Forced, Damped Harmonic Oscillator

Finally, we combine all three forces:

- A restoring force $F_{\text{restoring}}(x) = -kx = V'(x)$
- A damping force $F_{\text{damping}}(v) = -b\dot{x}$
- A driving force $F_{\text{driving}}(t) = F(t)$

(We write the restoring force as simply $F(t)$, even though it's not the net force by itself, because it will save us some writing.)

Newton's second law tells us

$$m\ddot{x} = -kx - b\dot{x} + F(t) \tag{7.1}$$

which gives us the differential equation

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F(t)}{m} \tag{7.2}$$

This is a slightly different sort of differential equation we've seen before; it's a second order inhomogeneous linear ordinary differential equation:

- second order because the highest number of time derivatives is two
- inhomogeneous because of the driving force $F(t)$
- linear because each term has only one power of $x(t)$ or one of its derivatives
- ordinary (as opposed to partial) because it has only one independent variable t.

It's conventional physically to think of the oscillator itself as defined by the damping and restoring forces, and then apply various different external driving forces to the same oscillator. This also has a mathematical parallel, in that we can write (7.2) as

$$\mathcal{L}x = \frac{F(t)}{m} \tag{7.3}$$

where $\mathcal{L}$ is the *linear differential operator*

$$\mathcal{L} = \frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2 \tag{7.4}$$

The linearity of the differential operator $\mathcal{L}$ tells us two very useful things:

1. If $x_h(t)$ is a solution to the homogenous equation $\mathcal{L}x_h = 0$ and $x_p(t)$ is a solution to the inhomogeneous equation $\mathcal{L}x_p = F/m$, then

$$\mathcal{L}(x_h + x_p) = \mathcal{L}x_h + \mathcal{L}x_p = 0 + \frac{F(t)}{m} \tag{7.5}$$

so $x_h + x_p$ is also a solution of the inhomogeneous equation. But this makes it easy to find the general solution to the inhomogeneous equation, which needs to have two arbitrary constants. We can simply add together the *general* solution $x_h(t)$ to the homogeneous equation and *any* solution $x_p(t)$ to the inhomogeneous equation, and the sum $x_h(t) + x_p(t)$ will have two arbitrary constants and solve the inhomogeneous equation. This means we've already done much of the work of solving the forced damped harmonic oscillator problem by solving the unforced problem for the same oscillator.

2. The other useful fact is that if we have two different forces $F_1(t)$ and $F_2(t)$, and $x_1(t)$ and $x_2(t)$ are the solutions to the respective differential equations:

$$\mathcal{L}x_1 = F_1/m \tag{7.6a}$$
$$\mathcal{L}x_2 = F_2/m \tag{7.6b}$$

(which means they are the responses of the oscillator to the respective forces) then $a_1x_1(t) + a_2x_2(t)$ is the oscillator's response to the combined force $aF_1(t) + bF_2(t)$:

$$\mathcal{L}(a_1x_1 + a_2x_2) = \frac{aF_1 + bF_2}{m} \tag{7.7}$$

7.1 Sinusoidal Driving Forces

One very useful driving force is a general sinusoid

$$F(t) = F_0 \cos(\omega t + \theta_0) \tag{7.8}$$

where F_0, ω, and θ_0 are all parameters of the force. Note that in general, ω, ω_0, and γ are all different.

In the usual tradition of Physicists solving differential equations, we guess an answer and see if it works. A reasonable thing to try is an oscillating solution; we know that if the driving force keeps going forever, the oscillator will never settle down to zero displacement, so we should try a solution which just oscillates and doesn't decay; and since the driving force is what's keeping it going, let's assume it oscillates at that frequency, but not necessarily in phase. The solution we try is thus

$$x(t) = A_s \cos(\omega t + \theta_s) \tag{7.9}$$

The output amplitude A_s and phase θ_s are *not* arbitrary; we need to figure out which values are needed for (7.9) to be a Solution to (7.2). Now, we could calculate $\dot{x}$ and $\ddot{x}$ and substitute those into (7.2) to find out θ_s and A_s, but it turns out the math is easier if we us the superposition property, noting that

$$\cos(\omega t + \theta_0) = \frac{e^{i(\omega t+\theta_0)} + e^{-i(\omega t+\theta_0)}}{2} \tag{7.10}$$

If we define

$$F_\pm = F_0 e^{\pm i(\omega t+\theta_0)} \tag{7.11}$$

and

$$x_\pm = A_s e^{\pm i(\omega t+\theta_s)} \tag{7.12}$$

then the actual force is

$$F(t) = \frac{1}{2}F_+(t) + \frac{1}{2}F_-(t) \tag{7.13}$$

which means the oscillator response will be

$$x(t) = \frac{1}{2}x_+(t) + \frac{1}{2}x_-(t) \tag{7.14}$$

We pause to make contact with Symon's notation, which uses boldfaced letters to represent complex quantities. There are a couple of reasons why this is a bad idea:

1. It's easy to confuse boldfaced quantities with non-boldfaced ones, and to make matters worse, the relationship between, for example, $\mathbf{x}(t)$ and $x(t)$ is not the same as that between $\mathbf{F}_0$ and F_0.

2. Symon will use boldface to indicate vectors in the next chapter (we will use arrows instead).

But at any rate, here's the correspondence:

$$F_+(t) = \mathbf{F}(t) \tag{7.15a}$$
$$F_-(t) = \mathbf{F}^*(t) \tag{7.15b}$$
$$x_+(t) = \mathbf{x}(t) \tag{7.15c}$$
$$x_-(t) = \mathbf{x}^*(t) \tag{7.15d}$$
$$F_0 e^{i\theta_0} = \mathbf{F}_0 \tag{7.15e}$$
$$A_s e^{i\theta_s} = \mathbf{x}_0 \tag{7.15f}$$

Note that the parameters F_0, ω, θ_0, A_s, and θ_s in the original expressions (7.8) and (7.9) are real. This didn't seem worth mentioning when we defined them (of course we were only working with real numbers) but now that we've introduced complex quantities like $x_+(t)$ it's important. If we consider that

$$F_\pm = F_0 e^{\pm i(\omega t+\theta_0)} = F_0 \cos(\omega t + \theta_0) \pm i F_0 \sin(\omega t + \theta_0) \tag{7.16}$$

we'll see that taking the complex conjugate, changing the sign of the imaginary part, tells us that

$$F_+(t)^* = F_-(t) \tag{7.17}$$

and likewise

$$x_+(t)^* = x_-(t) \tag{7.18}$$

and thus

$$F(t) = \frac{F_+(t) + F_-(t)}{2} = \frac{F_+(t) + F_+^*(t)}{2} = \operatorname{Re} F_+(t) \tag{7.19}$$

and

$$x(t) = \frac{x_+(t) + x_-(t)}{2} = \frac{x_+(t) + x_+^*(t)}{2} = \operatorname{Re} x_+(t) \tag{7.20}$$

This means that if we can choose A_s and θ_s so that x_+ solves $\mathcal{L}x_+ = F_+/m$, then x_- will automatically satisfy $\mathcal{L}x_- = F_-/m$ (since the differential equations are just complex conjugates of each other).

To do this, we note that since

$$x_+ = A_s e^{i\theta_s} e^{i\omega t} \tag{7.21}$$

we have

$$\dot{x}_+ = i\omega x_+ \tag{7.22}$$

and

$$\ddot{x}_+ = -\omega^2 x_+ \tag{7.23}$$

so

$$\mathcal{L}x_+ = (-\omega^2 + 2i\gamma\omega + \omega_0^2) A_s e^{i\theta_s} e^{i\omega t} = \frac{F_0 e^{i\theta_0} e^{i\omega t}}{m} \tag{7.24}$$

or, cancelling out the $e^{i\omega t}$ and rearranging,

$$\frac{F_0}{m} = A_s e^{i(\theta_s - \theta_0)} \left[(\omega_0^2 - \omega^2) + 2i\gamma\omega \right] \tag{7.25}$$

Now, (7.25) is a complex equation, which states that one complex number is equal to another. Recall that if $z = x + iy$ and $w = u + iv$ are two complex numbers, the complex equation

$$z = w \tag{7.26}$$

is equivalent to the two real equations

$$x = u \tag{7.27a}$$

$$y = v \tag{7.27b}$$

i.e., if two complex expressions are equal, then the real parts are equal and the imaginary parts are equal. Another real equation, which is not independent of the other two, is that the magnitude-squared of the two equations is equal:

$$z^* z = x^2 + y^2 = u^2 + v^2 = w^* w \tag{7.28}$$

In this case, the most useful pair of equations is the equality of the imaginary parts (which A_s drops out of) and of the squared magnitudes (which doesn't involve θ_s).

7.1.1 Determination of the Amplitude

The most direct way to find the amplitude A_s is to take the square of the magnitude of each side of (7.25) by multiplying it by its complex conjugate:

$$\left\{ \left[(\omega_0^2 - \omega^2) + 2i\gamma\omega \right] A_s e^{i\theta_s} \right\} \left\{ \left[(\omega_0^2 - \omega^2) - 2i\gamma\omega \right] A_s e^{-i\theta_s} \right\} = \frac{F_0}{m} \frac{F_0}{m} \tag{7.29}$$

or

$$\left[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2 \right] {A_s}^2 = \frac{F_0^2}{m^2} \tag{7.30}$$

which tells us

$$A_s = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \tag{7.31}$$

We have chosen to take the positive square root, but this is a reasonable thing to do, since it just means we choose A_s to have the same sign as F_0. We'll see that there is always a choice of θ_s which makes this work.

7.1.2 Determination of the Phase

To find θ_s, we concentrate on the imaginary part of (7.25), which can be extracted by writing it as

$$\begin{aligned} \frac{F_0}{m} =& A_s [\cos(\theta_s - \theta_0) + i \sin(\theta_s - \theta_0)] \left[(\omega_0^2 - \omega^2) + 2i\gamma\omega \right] \\ =& A_s \left[(\omega_0^2 - \omega^2) \cos(\theta_s - \theta_0) - 2\gamma\omega \sin(\theta_s - \theta_0) \right. \\ & \left. + i \left(2\gamma\omega \cos(\theta_s - \theta_0) + (\omega_0^2 - \omega^2) \sin(\theta_s - \theta_0) \right) \right] \end{aligned} \tag{7.32}$$

Now, we could set the imaginary part to zero to find out the tangent of $\theta_s - \theta_0$, but despite having initially defined θ_s in his equation (2.149), Symon proceeds to find the solution in terms of a different phase angle β, a wacky convention which he says comes from electrical engineering and is related to θ_s by

$$\beta = \theta_s - \theta_0 + \frac{\pi}{2} \tag{7.33}$$

so that

$$\cos(\theta_s - \theta_0) = \sin\beta \tag{7.34a}$$

$$\sin(\theta_s - \theta_0) = -\cos\beta \tag{7.34b}$$

and the candidate solution (7.9) is

$$x_p(t) = A_s \sin(\omega t + \theta_0 + \beta) \tag{7.35}$$

In terms of this new angle, (7.32) becomes

$$\begin{aligned}\frac{F_0}{m} =& A_s\big[(\omega_0^2 - \omega^2)\sin\beta + 2\gamma\omega\cos\beta \\ &+ i\left(2\gamma\omega\sin\beta - (\omega_0^2 - \omega^2)\cos\beta\right)\big]\end{aligned} \tag{7.36}$$

The imaginary part of the left-hand side vanishes, so setting the imaginary part of the right-hand side to zero gives

$$\tan\beta = \frac{\omega_0^2 - \omega^2}{2\gamma\omega} \tag{7.37}$$

Now, knowing the tangent of an angle only tells us that angle modulo π (because $\tan(\theta + \pi) = \cos(\theta+\pi)/\sin(\theta+\pi) = (-\cos\theta)/(-\sin\theta) = \tan\theta$) so we should pause and make sure we know which branch of the arctangent we want to take when we calculate β. Put another way, given the tangent, we can calculate

$$\sin\beta = \pm\frac{\tan\beta}{1+\tan^2\beta} = \pm\frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}} \tag{7.38a}$$

$$\cos\beta = \pm\frac{1}{1+\tan^2\beta} = \pm\frac{2\gamma\omega}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}} \tag{7.38b}$$

where all of the $\pm$ signs are the same, but we need to decide whether they're all plus or all minus. Fortunately, once we figure out the value of β for $\omega = 0$, we can vary ω smoothly, and since we can see from (7.31) that the amplitude A_s doesn't go through zero for any ω, we don't have to worry about β suddenly jumping by π.

At $\omega = 0$, the real part of (7.36) tells us that

$$\left.\frac{F_0}{m}\right|_{\omega=0} = A_s\omega_0^2\sin\beta \tag{7.39}$$

but we've chosen A_s so that it has the same sign as F_0. This means $\sin\beta > 0$ at $\omega = 0$ and we want to take the plus sign to get

$$\sin\beta = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}} \tag{7.40a}$$

$$\cos\beta = \frac{2\gamma\omega}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}} \tag{7.40b}$$

Armed with this knowledge, we can follow the behavior of β as a function of ω:

$\omega = 0$	$\sin\beta = 1$	$\cos\beta = 0$	$\beta = \pi/2$
$0 < \omega < \omega_0$	$0 < \sin\beta < 1$	$0 < \cos\beta < 1$	$0 < \beta < \pi/2$
$\omega = \omega_0$	$\sin\beta = 0$	$\cos\beta = 1$	$\beta = 0$
$\omega > \omega_0$	$-1 < \sin\beta < 0$	$0 < \cos\beta < 1$	$-\pi/2 < \beta < 0$
$\omega \to \infty$	$\sin\beta \to -1$	$\cos\beta \to 0$	$\beta \to -\pi/2$

Note that this behavior is consistent with Figure 2.6 of Symon, but his discussion in the paragraph after equation (2.159) is completely wrong. (He must have been thinking of a different angle.)

So when all is said and done, we can recover the desired solution in the presence of the sinusoidal force (7.8). Since

$$x_+ = A_s e^{i(\omega t+\theta_0+\beta-\pi/2)} = \frac{A_s}{i} e^{i(\omega t+\theta_0+\beta)} \tag{7.41}$$

and

$$x_- = -\frac{A_s}{i} e^{-i(\omega t+\theta_0+\beta)} \tag{7.42}$$

then

$$x_p(t) = \frac{x_+ + x_-}{2} = A_s \frac{e^{i(\omega t+\theta_0+\beta)} - e^{-i(\omega t+\theta_0+\beta)}}{2i} = A_s \sin(\omega t + \theta_0 + \beta) \tag{7.43}$$

where

This means that $\sin\beta$ always has the same sign as ω and $\cos\beta$ always has the same sign as $\omega_0^2 - \omega^2$, which allows us to use (7.37) and the fact that $\sin^2\delta + \cos^2\delta = 1$ to write

$$\sin\delta = \frac{2\gamma\omega}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}} \tag{7.44a}$$

$$\cos\delta = \frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}} \tag{7.44b}$$

where A_s is given by (7.31) and β is given by (7.40).

Exercise: Consider A_s as a function of ω and follow it as ω goes from 0 to ∞, noting any maxima or minima. Compare to figure 2.6 of Symon.

7.2 Linear Superposition Methods for the Forced, Damped Harmonic Oscillator

We've solved the problem of a forced damped harmonic oscillator where the driving force is a sinusoid of a fixed frequency. You might worry that this is not terribly general, and we'd have to solve the equation

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F(t)/m \tag{7.45}$$

over and over again for different external driving forces.

Fortunately, the linearity of the differential operator means we only have to solve the problem for a few basic types of forces, then build up the solution for a more general force as a superposition of these solutions.

We will consider two different ways to build up general forces out of component functions, and thus to build up the oscillator response to those forces out of the responses to the component functions.

1. We will find the response of a damped harmonic oscillator to an impulsive force, i.e., one which is very large for a very small stretch of time. An arbitrary force can be built up as a series of impulses, allowing the solution to a general problem by what's known as a Green's function method.

2. We've already found the response of an oscillator to a sinusoidal driving force; we'll show how an arbitrary periodic force can be written as a sum of such sinusoidal forces.

7.2.1 Impulse Response and Green's Function Methods

Impulse Response Recall that we defined the change in momentum as a result of a force

$$\Delta p = \int_{t_1}^{t_2} F(t) dt \tag{7.46}$$

as the *impulse* delivered by that force. We'd like to consider the response of a harmonic oscillator to a force that operates over a very short period of time, but delivers a finite impulse. This is a force which obeys

$$F(t) \begin{cases} = 0 & t < t_0 \\ \neq 0 & t_0 < t < t + \delta t \\ = 0 & t > t_0 + \delta t \end{cases} \tag{7.47}$$

and

$$\int_{t_0}^{t_0+\delta t} F(t)\, dt = p_0 \tag{7.48}$$

We'd like to consider the behavior of such a force in the limit $\delta t \to 0$ with p_0 held constant. For example, we could think about a family of such forces with different values of δt and then ask about the limit as $\delta t \to 0$.

For cultural reference, I'd like to mention that there is a pseudo-mathematical notation we Physicists like to use to describe such a force, namely

$$F(t) = p_0 \delta(t - t_0) \tag{7.49}$$

where $\delta(t - t_0)$ is the *Dirac delta function* which is defined by the properties

$$\delta(t - t_0) = \begin{cases} 0 & t \neq t_0 \\ \infty & t = t_0 \end{cases} \tag{7.50}$$

and

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)\, dt = f(t_0) \tag{7.51}$$

where $f(t)$ is any function which is sufficiently well-behaved at $t = t_0$.

We could actually derive the solution in the presence of this force by mathematical manipulation of the differential equation

$$\mathcal{L}x = p_0 \delta(t - t_0) \tag{7.52}$$

but we will instead use a more physically-motivated approach.

For simplicity, we will consider initial conditions of $x(t < t_0) = 0$ and $\dot{x}(t < t_0) = 0$. Physically, this means we have left the oscillator alone for long enough for all the transients to die out, then given it a sharp whack. In the limit that $\delta t \to 0$, we don't really care what's happening during the whack (and we can't really talk about it sensibly anyway); what we care about is the condition after the whack (i.e., $x(t_0 + \delta t)$ and $\dot{x}(t_0 + \delta t)$), which we can use as the initial conditions for subsequent unforced evolution.

First we argue that the only force that's relevant during the whack is the impulse itself, and the damping and restoring forces are irrelevant. We can see this by noting that ω_0 and γ describe the strengths of these forces, and that for given values of the parameters, we can always choose δt small enough that $\omega_0 \ll (\delta t)^{-1}$ and $\gamma \ll (\delta t)^{-1}$, meaning they are so small as to be irrelevant.

Next we argue that while $\dot{x}(t_0 + \delta t)$ is finite in the $\delta t \to 0$ limit, $\lim_{\delta t \to 0} x(t_0 + \delta t) = 0$. This is basically because

$$F = m \frac{dv}{dt} = m \frac{d^2 x}{dt^2} \tag{7.53}$$

and so if $F \propto (\delta t)^{-1}$, then $\delta v \propto F\, \delta t$ is finite, while $\delta x \propto F\, (\delta t)^2 \propto \delta t$. We can see this explicitly for the case where $F(t)$ is constant during the whack:

$$F(t) = \begin{cases} 0 & t < t_0 \\ \frac{p_0}{\delta t} & t_0 < t < t + \delta t \\ 0 & t > t_0 + \delta t \end{cases} \tag{7.54}$$

The motion under uniform acceleration is described by

$$\dot{x}(t) = \frac{p_0}{m\delta t}(t - t_0) \tag{7.55}$$

and

$$x(t) = \frac{p_0}{m\delta t} \frac{(t - t_0)^2}{2} \tag{7.56}$$

for $t_0 < t < t_0 + \delta t$. This means for this case

$$x(t_0 + \delta t) = \frac{p_0}{m\delta t} \frac{(\delta t)^2}{2} = \frac{p_0}{2m} \delta t \to 0 \tag{7.57a}$$

$$\dot{x}(t_0 + \delta t) = \frac{p_0}{m\delta t} (\delta t) = \frac{p_0}{m} \tag{7.57b}$$

And sure enough $x(t_0 + \delta t)$ vanishes, while $\dot{x}(t_0 + \delta t)$ is finite.

In fact, since we know that whatever the family of forces, the impulse is

$$m[\dot{x}(t_0 + \delta t) - \dot{x}(t_0)] = \int_{t_0}^{t_0 + \delta t} F(t)\, dt = p_0 \tag{7.58}$$

we can say that in general for sufficiently small δt,

$$x(t_0^+) = 0 \tag{7.59a}$$

$$\dot{x}(t_0^+) = \frac{p_0}{m} \tag{7.59b}$$

we then use these as initial conditions for future evolution of the undriven damped oscillator. (We've defined $t_0^+ = t_0 + \delta t$ for convenience.) It's easiest to fit them if we're a little clever, and write the solution in the form

$$x(t) = \begin{cases} 0 & t < t_0 \\ Ae^{-\gamma(t-t_0^+)} \sin\omega_1(t - t_0^+) & t > t_0^+ \end{cases} \tag{7.60}$$

which ensures $x(t_0^+) = 0$ as desired. Taking the time derivative

$$\dot{x}(t) = \begin{cases} 0 & t < t_0 \\ Ae^{-\gamma(t-t_0^+)}[-\gamma \sin\omega_1(t - t_0^+) + \omega_1 \cos\omega_1(t - t_0^+)] & t > t_0^+ \end{cases} \tag{7.61}$$

we can find A by requiring

$$\frac{p_0}{m} = \dot{x}(t_0^+) = A\omega_1 \tag{7.62}$$

which then gives us

$$x(t) = \begin{cases} 0 & t \le t_0 \\ \frac{p_0}{m\omega_1} e^{-\gamma(t-t_0)} \sin\omega_1(t - t_0) & t \ge t_0 \end{cases} \tag{7.63}$$

where we have taken the $\delta t \to 0$ limit and replaced t_0^+ with t_0. This is the response of a damped harmonic oscillator to an impulsive force delivering an impulse p_0.

7.2.2 Green's Function Method

7.3 Periodic Forces and Fourier Methods

Fortunately, there are two facts which mean that we can reuse our answer from the sinusoidal case in the presence of a wide variety of driving forces.

1. Linear superposition. The differential equation (7.45) contains a linear differential operator

   $$\mathcal{L} = \frac{d^2}{dt^2} + 2\gamma\frac{d}{dx} + \omega_0^2 \tag{7.64}$$

   The linearity means

   $$\mathcal{L}(x_1(t) + x_2(t)) = \mathcal{L}x_1(t) + \mathcal{L}x_2(t) \tag{7.65}$$

   This is just the principle that we used to allow us to write the general solution of the inhomogeneous equation (7.2) as a superposition of the general solution to the inhomogeneous equation (6.5) and any solution to the inhomogeneous equation. That means that if we break up the driving force $A^{\text{in}}(t)$ into two terms:

   $$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \omega_0^2(A_1^{\text{in}}(t) + A_2^{\text{in}}(t)) \tag{7.66}$$

then the general solution will be a sum of the complementary function $x_c(t)$ and steady-state solutions corresponding to the two components of the driving force:

$$x(t) = x_c(t) + x_1(t) + x_2(t) \tag{7.67}$$

where

$$\ddot{x}_{1,2} + 2\gamma\dot{x}_{1,2} + \omega_0^2 x_{1,2} = \omega_0^2 A^{\text{in}}_{1,2}(t) \tag{7.68}$$

So if the driving force is a sum of sines and cosines, we can write down the solution. For instance, if

$$A^{\text{in}}(t) = x_0 \cos\omega t + 4x_0 \sin 3\omega t \tag{7.69}$$

then the steady-state solution is

$$x_p(t) = \frac{\omega_0^2 x_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \cos(\omega t - \delta_1) + \frac{4\omega_0^2 x_0}{\sqrt{(\omega_0^2 - 9\omega^2)^2 + 36\gamma^2\omega^2}} \sin(3\omega t - \delta_2) \tag{7.70}$$

where

$$\delta_1 = \tan^{-1}\frac{2\gamma\omega}{\omega_0^2 - \omega^2} \tag{7.71a}$$

$$\delta_2 = \tan^{-1}\frac{6\gamma\omega}{\omega_0^2 - 36\omega^2} \tag{7.71b}$$

$$\tag{7.71c}$$

2. By using Fourier series and Fourier transforms, *any* driving force can be written as an infinite sum of sines and cosines.

7.4 Periodic Driving Forces (Fourier Series)

Consider first the case where the driving force has some periodicity $A^{\text{in}}(t) = A^{\text{in}}(t+T)$. Not every sine or cosine will have this periodicity, just those with angular frequencies

$$\omega_n = \frac{2\pi n}{T} \tag{7.72}$$

The methods of Fourier series (see supplemental exercises) show us that any periodic function can be written as an infinite series of sines and cosines:

$$A^{\text{in}}(t) = \frac{a_0^{\text{in}}}{2} + \sum_{n=0}^{\infty} a_n^{\text{in}} \cos\omega_n t + \sum_{n=0}^{\infty} b_n^{\text{in}} \sin\omega_n t \tag{7.73}$$

with the coëfficients given by

$$a_n^{\text{in}} = \frac{2}{T}\int_{-T/2}^{T/2} A^{\text{in}}(t)\cos\omega_n t\,dt \tag{7.74a}$$

$$b_n^{\text{in}} = \frac{2}{T}\int_{-T/2}^{T/2} A^{\text{in}}(t)\sin\omega_n t\,dt \tag{7.74b}$$

By superposition, the steady-state solution to (7.45) is thus

$$\begin{aligned} x_c(t) =& \frac{1}{2}a_0^{\text{in}} \sum_{n=1}^{\infty} \frac{\omega_0^2 a_n^{\text{in}}}{\sqrt{(\omega_0^2-\omega_n^2)^2+4\gamma^2\omega_n^2}} \cos(\omega_n t - \delta_n) \\ &+ \sum_{n=1}^{\infty} b_n^{\text{out}} \frac{\omega_0^2 b_n^{\text{in}}}{\sqrt{(\omega_0^2-\omega_n^2)^2+4\gamma^2\omega_n^2}} \sin(\omega_n t - \delta_n) \end{aligned} \tag{7.75}$$

where

$$\delta_n = \tan^{-1} \frac{2\gamma\omega_n}{\omega_0^2 - \omega_n^2} \tag{7.76}$$

Note that

- We have used the zero-frequency behavior to write $\delta_0 = 0$ and $a_0^{\text{out}} = a_0^{\text{in}}$; in fact this means the constant term just effectively shifts the equilibrium position.
- To write (7.75) as a traditional Fourier series, we need to use the angle difference formulas to write e.g.,

$$\cos(\omega_n t - \delta_n) = \cos\delta_n \cos\omega_n t + \sin\delta_n \sin\omega_n t \tag{7.77}$$

and then reorganize the sums to give

$$x_c(t) = \frac{a_0^{\text{out}}}{2} + \sum_{n=0}^{\infty} a_n^{\text{out}} \cos\omega_n t + \sum_{n=0}^{\infty} b_n^{\text{out}} \sin\omega_n t \tag{7.78}$$

Excercise: work out the general formula for a^{out} and b^{out}.

7.5 Fourier Transforms: Expressing a General Function as a Superposition of Periodic Terms

$$x(t) = \sum_{n=-\infty}^{\infty} \hat{x}_n \, e^{-i\omega_n t} \tag{7.79}$$

$$\hat{x}_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \, e^{i\omega_n t} \, dt \tag{7.80}$$

frequency spacing

$$\delta\omega = \frac{2\pi}{T} \tag{7.81}$$

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{T}{2\pi} \hat{x}_n \, e^{-in\,\delta\omega\,t} \, \delta\omega \tag{7.82}$$

If we define

$$\tilde{x}(\omega_n) = \frac{T}{\sqrt{2\pi}} \hat{x}_n \tag{7.83}$$

then

$$x(t) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \tilde{x}(\omega)\, e^{-in\,\delta\omega\,t}\, \delta\omega \tag{7.84}$$

and

$$\tilde{x}(\omega_n) = \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} x(t)\, e^{i\omega_n t}\, dt \tag{7.85}$$

In the limit $T \to \infty$, $\delta\omega \to 0$ and the sum becomes an integral:

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{x}(\omega)\, e^{-i\omega t}\, d\omega \tag{7.86}$$

and

$$\tilde{x}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)\, e^{i\omega t}\, dt \tag{7.87}$$

$\tilde{x}(\omega)$ is called the Fourier transform of $x(t)$.

A Appendix: Correspondence to Class Lectures

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2003 September 25	7.1–7.1.2	17–24
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