

# Two- and Three-Dimensional Motion (Symon Chapter Three)

Physics A300\*

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# 1 Vector Analysis

## 1.1 Definitions

There are numerous definitions of a vector, including

- An arrow in space
- Something defined by its magnitude and direction
- An ordered list of numbers
- A directional derivative

We'll start from the physical/geometrical definition in terms of magnitude and direction.

### 1.1.1 Notation

When talking about a vector, we will write it with an arrow, like so:  $\vec{A}$ . Symon uses boldface, like so:  $\mathbf{A}$ . To avoid confusion, it's very important never to accidentally drop the vector symbol from a vector, since the same letter without a vector symbol may mean something entirely different.

Note that vectors give us additional grammatical rules. Just as you can't add a length to a force, or compare a mass to a time, it's completely meaningless to try to say that a vector is equal to something that's not a vector. So it's okay to say  $\vec{A} = \vec{B}$  or  $a = b$ , but something like " $\vec{A} = a$ " just makes no sense.

## 1.2 Basic Vector Arithmetic

### 1.2.1 Equality

We think of a vector as being defined solely by its magnitude and direction, so if you've got two different vectors on a diagram in different places, you're free to "slide" them around without changing their directions to compare them. So in the diagram below  $\vec{A} = \vec{B}$ :



### 1.2.2 Magnitude

The magnitude or "length" of a vector is a positive number, and is *not* a vector. We write it with an absolute value sign, like so:  $|\vec{A}|$ . Note that the magnitude of a vector will not in general have units of length. The force  $\vec{F}$ , momentum  $\vec{p}$ , and acceleration  $\vec{a}$  are all vectors.

### 1.2.3 Multiplication by a Scalar

The simplest thing you can do to a vector is multiply it by a number, which is also called a scalar. (There are more precise definitions of a scalar, specifically that it should be something which doesn't depend on the choice of coordinate system, but for the time being it's fine to think of it as a number.)

It's convenient to consider one special case first, namely multiplication by  $-1$ . In this case, we define the result to be a new vector with the same magnitude, pointing in the opposite direction:



Note that

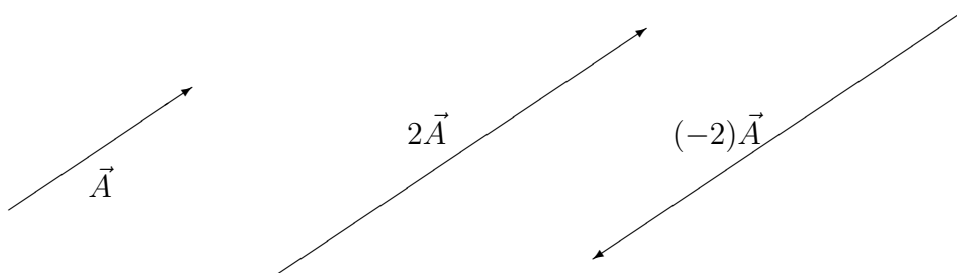
$$|-\vec{A}| = |\vec{A}| \quad (1.1)$$

Now, we can consider multiplication by a general scalar, which is defined as follows:  $c\vec{A}$  is a vector whose magnitude is

$$|c\vec{A}| = |c| |\vec{A}| \quad (1.2)$$

(note that we need the absolute value sign, to make sure the magnitude of  $c\vec{A}$  is not negative) and which points in the same direction as  $\vec{A}$  if  $c > 0$  and in the opposite direction if  $c < 0$ .

Here are some examples:

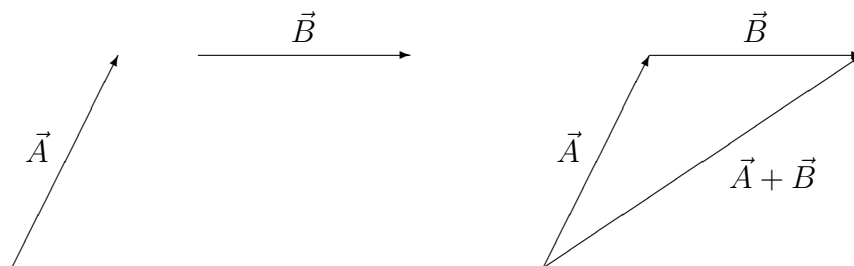


Multiplication of a scalar by a vector behaves in exactly the way you imagine it would; in particular it obeys the associative property:

$$b(c\vec{A}) = (bc)\vec{A} \quad (1.3)$$

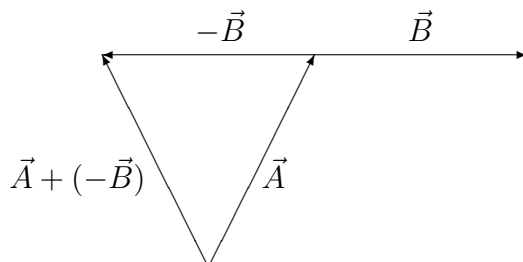
### 1.2.4 Addition of Vectors

The geometrical definition of the sum of two vectors follows from the idea of a vector as an arrow. To get the effect of  $\vec{A}$  plus  $\vec{B}$ , you just line the arrows up, putting the tail of  $\vec{B}$  at the head of  $\vec{A}$ , and their sum  $\vec{A} + \vec{B}$  goes from the tail of  $\vec{A}$  to the head of  $\vec{B}$ :



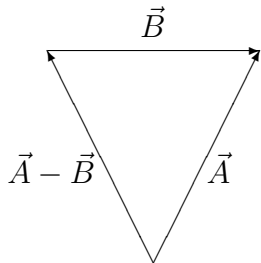
There are two reasonable ways to define subtraction of one vector from another. One is as multiplication by  $-1$  followed by addition:

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B}) \quad (1.4)$$



Alternatively, we can note that  $\vec{A} - \vec{B}$  is the vector to which you would add  $\vec{B}$  and be left with  $\vec{A}$ :

$$(\vec{A} - \vec{B}) + \vec{B} = \vec{A} \quad (1.5)$$



Both definitions are of course equivalent, and give the same result for  $\vec{A} - \vec{B}$ .

Note that with vector addition, as with ordinary addition, the quantities being added must have the same units. It makes no sense to write  $\vec{F} + \vec{p}$  if  $\vec{F}$  is a force and  $\vec{p}$  is a momentum.

Vector addition, together with multiplication by a scalar, obeys all of the properties you'd expect. These are collected in equations (3.5–3.8) of Symon.

Note that it makes no sense to try to add a vector to a scalar, just as it makes no sense to say that a vector equals a scalar. An expression like  $\vec{A} + a$  is meaningless.

### 1.3 Bases and Components

These geometrical definitions are all well and good, but they can only go so far. Eventually we want to be able to do vector calculations algebraically, without having to draw a picture every time. To do this, we start with a Cartesian  $(x, y, z)$  coordinate system, and define unit vectors  $\hat{x}$ ,  $\hat{y}$  and  $\hat{z}$  parallel to the three coordinate axes. We write a hat over them, rather than an arrow, to emphasize that they're unit vectors, i.e., of magnitude one:

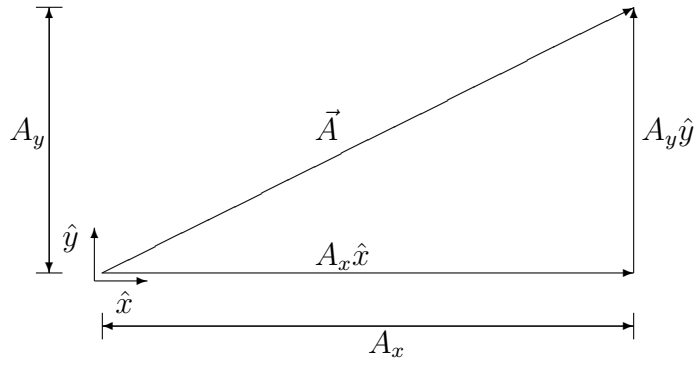
$$|\hat{x}| = |\hat{y}| = |\hat{z}| = 1 \quad (1.6)$$

Note that this means that the unit vectors  $\hat{x}$ ,  $\hat{y}$ , and  $\hat{z}$ , like all unit vectors, are dimensionless, and in particular that they do *not* have units of length like the coordinates  $x$ ,  $y$ , and  $z$ .

Given these unit vectors, one can project *any* vector  $\vec{A}$  onto the three coordinate axes, defining the projections as  $A_x$ ,  $A_y$ , and  $A_z$ . These are all numbers (*not* vectors) with units equal to those of the original vector  $\vec{A}$ . Geometrically, the law of vector addition makes it relatively easy to see that

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \quad (1.7)$$

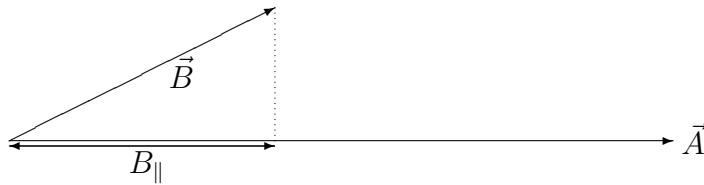
We illustrate this in two dimensions:



## 1.4 The Scalar (Dot, Inner) Product

We've defined how to add vectors and how to multiply a vector by a scalar. There turn out to be several different sensible ways to define the product of one vector with another, which have names corresponding to the object which results from the multiplication.

The simplest of these is the scalar product, also known as the dot product.



## 1.5 The Vector (Cross) Product

## 1.6 The Position Vector

## 1.7 Differentiation and Integration

# 2 Vectors in Mechanics and Kinematics

## 2.1 Force

## 2.2 Velocity, Momentum, and Acceleration

# 3 Non-Cartesian Coördinate Systems

## 3.1 Plane Polar Coördinates

## 3.2 Cylindrical Coördinates

## 3.3 Spherical Coördinates

# 4 Conservation Theorems in Two or More Dimensions

We continue our look at the analogies between one- and three-dimensional mechanics by looking at some of the conservation theorems.

## 4.1 Force and Momentum

In one dimension, we saw that we can integrate

$$F = \frac{dp}{dt} \quad (4.1)$$

to get

$$\int_{t_1}^{t_2} F(t) dt = p(t_2) - p(t_1) \quad (4.2)$$

and the same thing carries over perfectly well into two or more dimensions; we can integrate

$$\vec{F} = \frac{d\vec{p}}{dt} \quad (4.3)$$

to get

$$\int_{t_1}^{t_2} \vec{F}(t) dt = \vec{p}(t_2) - \vec{p}(t_1) \quad (4.4)$$

since the acts of differentiating and integrating the vectors  $\vec{F}$  and  $\vec{p}$  with respect to the scalar  $t$  are well-defined.

## 4.2 Work and Kinetic Energy

Recall that in one dimension, we defined the kinetic energy as

$$T = \frac{1}{2}mv^2 ; \quad (4.5)$$

this was useful because

$$\frac{dT}{dt} = mv \frac{dv}{dt} = Fv = F \frac{dx}{dt} \quad (4.6)$$

where we used Newton's second law to replace  $ma$  with  $F$ . We could then integrate this to see

$$T(t_2) - T(t_1) = \int_{t_1}^{t_2} F \frac{dx}{dt} dt = \int_{x(t_1)}^{x(t_2)} F dx = W_{1 \rightarrow 2} \quad (4.7)$$

i.e., the increase in kinetic energy is just the work done on the particle.

When generalizing the kinetic energy to two or more dimensions, we have to note that it involves the product of the velocity with itself. We know two ways to multiply vectors by vectors, and since  $\vec{v} \times \vec{v} = \vec{0}$  is not very interesting, we opt for the dot product and define

$$T = \frac{1}{2}m\vec{v} \cdot \vec{v} = \frac{1}{2}m|\vec{v}|^2 \quad (4.8)$$

This means the kinetic energy is still one-half the mass times the speed squared. Note that while force, position, velocity, momentum and acceleration all become vectors in more than one dimension, work and energy remain scalars.

The time derivative of the kinetic energy is

$$\frac{dT}{dt} = \frac{1}{2}m \left( \frac{d\vec{v}}{dt} \cdot \vec{v} + \vec{v} \cdot \frac{d\vec{v}}{dt} \right) = m \frac{d\vec{v}}{dt} \cdot \vec{v} = \vec{F} \cdot \vec{v} \quad (4.9)$$

which makes the increase in kinetic energy between two times  $t_1$  and  $t_2$

$$T(t_2) - T(t_1) = \int_{t_1}^{t_2} \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \oint_{\substack{\vec{r}(t_1) \rightarrow \vec{r}(t_2) \\ \text{along } \vec{r}(t)}} \vec{F} \cdot d\vec{r} = W_{1 \rightarrow 2} \quad (4.10)$$

where the contour integral is just what we need to define the work done on the particle as it moves along its trajectory  $\vec{r}(t)$  from time  $t_1$  to time  $t_2$ .

### 4.2.1 Complications Arising from Potential Energy

In one dimension, we were able to use an indefinite integral to define the potential energy

$$V(x) = - \int F dx \quad (4.11)$$

up to an overall additive constant. This was equivalent to the force being given by

$$F = - \frac{dV}{dx} \quad (4.12)$$



and meant that the negative of the work done on a particle was just the increase in its potential energy as it moved from position  $x_1$  to position  $x_2$ :

$$V(x_2) - V(x_1) = - \int_{x_1}^{x_2} F(x) dx \quad (4.13)$$

where the arbitrary constant in the definition of  $V(x)$  cancels out.

When we try to generalize this to three dimensions, we run into trouble. Now the work is defined by a contour integral, and we can't just define  $V(\vec{r})$  as something like

$$\text{“ } - \oint \vec{F} \cdot d\vec{r} \text{”} \quad (4.14)$$

because there's no such thing as an “indefinite contour integral”. The contour integral depends not only on the endpoints, but the path through space taken from one to the other. The implicit definition in terms of a derivative would be

$$\text{“ } \vec{F} = - \frac{dV}{d\vec{r}} \text{”} \quad (4.15)$$

which is likewise in terms of something which we never defined (a derivative with respect to a vector). We could try to fall back to the change in potential energy being related to the work done:

$$V(\vec{r}_2) - V(\vec{r}_1) = - \oint_{\vec{r}_1 \rightarrow \vec{r}_2} \vec{F} \cdot d\vec{r} \quad (4.16)$$

but in general, this will not be well-defined, since the right-hand side will depend not only on the positions  $\vec{r}_1$  and  $\vec{r}_2$ , but also on the path taken between them.

### 4.3 Conservative Forces

For some force fields, however, the right-hand side of (4.16) will be the same regardless of the path taken. We'll assume this is true and see what that tells us about the force field  $F(\vec{r})$  and how it can be derived from  $V(\vec{r})$ .

First, we consider the special case where the endpoints  $\vec{r}_1$  and  $\vec{r}_2$  differ only in their  $x$  coordinate:

$$y_1 = y_2 = y_0 \quad (4.17a)$$

$$z_1 = z_2 = z_0 \quad (4.17b)$$

so that

$$\vec{r}_1 = x_1 \hat{x} + y_0 \hat{y} + z_0 \hat{z} \quad (4.18a)$$

$$\vec{r}_2 = x_2 \hat{x} + y_0 \hat{y} + z_0 \hat{z} \quad (4.18b)$$

$$(4.18c)$$

We're assuming the work done by the force field when moving between these two points is independent of the path, so let's choose the simplest one, on which only the  $x$  coordinate is changing, and parametrize it by that  $x$  coordinate:

$$\vec{r}(s) = s \hat{x} + y_0 \hat{y} + z_0 \hat{z} \quad s : x_1 \rightarrow x_2 \quad (4.19)$$

This then means

$$\frac{d\vec{r}}{ds} = \hat{x} \quad (4.20)$$

and

$$\vec{F} \cdot \frac{d\vec{r}}{ds} = \vec{F} \cdot \hat{x} = F_x \quad (4.21)$$

The difference (4.16) in the potential energies then becomes

$$V(\vec{r}_2) - V(\vec{r}_1) = - \int_{x_1}^{x_2} F_x(\vec{r}(s)) ds = V(x_2, y_0, z_0) - V(x_1, y_0, z_0) = - \int_{x_1}^{x_2} F_x(x, y_0, z_0) dx \quad (4.22)$$

Now we know that in one dimension (4.13) is equivalent to (4.12); this is the same construction, with  $y_0$  and  $z_0$  going along for the ride, so we get the same result, with the derivative with respect to  $x$  now a partial derivative at constant  $y$  and  $z$ :

$$F_x(\vec{r}) = - \frac{\partial V}{\partial x} \quad (4.23a)$$

But there is nothing special about the  $x$  direction; we could make the same argument holding  $x$  and  $z$  constant and varying  $y$ , or holding  $x$  and  $y$  constant and varying  $z$ , which means

$$F_y(\vec{r}) = - \frac{\partial V}{\partial y} \quad (4.23b)$$

$$F_z(\vec{r}) = - \frac{\partial V}{\partial z} \quad (4.23c)$$

so if the force field  $\vec{F}(\vec{r})$  is associated with the potential  $V(\vec{r})$  according to (4.16), we can write it as

$$\vec{F} = - \left( \hat{x} \frac{\partial V}{\partial x} + \hat{y} \frac{\partial V}{\partial y} + \hat{z} \frac{\partial V}{\partial z} \right) \quad (4.24)$$

Not every force field can be written in the form (4.24); for example, the force field

$$\vec{F} = A(x\hat{y} - y\hat{x}) \quad (4.25)$$

(where  $A$  is a constant) cannot. The easiest way to see this is to recall that you get the same result no matter in which order you take a mixed partial derivative:

$$\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x} \quad (4.26)$$

but if  $\vec{F}$  is defined by (4.24), that means

$$\frac{\partial F_y}{\partial x} = \frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x} = \frac{\partial F_x}{\partial y} \quad (4.27)$$

On the other hand, the force (4.25) has

$$\frac{\partial F_y}{\partial x} = \frac{\partial(Ax)}{\partial x} = A \quad (4.28a)$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial(-Ay)}{\partial y} = -A \neq \frac{\partial F_y}{\partial x} \quad (4.28b)$$

Similar conditions can be worked out for the other mixed partial derivatives of  $V(x, y, z)$ , resulting in the conditions

$$\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y} \quad (4.29a)$$

$$\frac{\partial F_z}{\partial y} = \frac{\partial F_y}{\partial z} \quad (4.29b)$$

$$\frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x} \quad (4.29c)$$

in order for  $\vec{F}(\vec{r})$  to be associated with a potential. We've shown that they are necessary [no force field can be associated with a potential unless it satisfies (4.29)]; it turns out that they are also sufficient [and force field which satisfies (4.29) can be associated with a potential by (4.24)].

## 5 Fundamentals of Vector Calculus

### 5.1 The Gradient

The expression (4.24) for generating a vector field  $\vec{F}(\vec{r})$  given a scalar field  $U(\vec{r})$  allows us to give a well defined meaning to the “differentiation by a vector” we were trying to define in (4.15). It is a useful enough operation that it has its own name, the *gradient*

$$\vec{\nabla}V = \hat{x} \frac{\partial V}{\partial x} + \hat{y} \frac{\partial V}{\partial y} + \hat{z} \frac{\partial V}{\partial z} \quad (5.1)$$

The symbol  $\vec{\nabla}$  (pronounced “del”) can be thought of as a *vector differential operator*

$$\vec{\nabla} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \quad (5.2)$$

We could also express this operator in terms of non-Cartesian basis vectors in a brute force manner as follows:

1. Invert the relationship

$$\begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \quad (5.3)$$

to get  $\hat{x}$ ,  $\hat{y}$ , and  $\hat{z}$  in terms of the spherical coordinate basis vectors.

2. Calculate the partial derivatives of the spherical coordinates with respect to the Cartesian ones and construct

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi} \quad (5.4a)$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y} \frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y} \frac{\partial}{\partial \phi} \quad (5.4b)$$

$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z} \frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z} \frac{\partial}{\partial \phi} \quad (5.4c)$$

3. Substitute, simplify, and rearrange.

Fortunately, there's a much easier, clever way to do things.

Consider the the infinitesimal change in the value of the scalar field  $V(\vec{r})$  induced by an infinitesimal change

$$d\vec{r} = \hat{x}dx + \hat{y}dy + \hat{z}dz \quad (5.5)$$

position vector  $\vec{r}$ . We know from the chain rule of multi-variable calculus that

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = (\vec{\nabla}V) \cdot (d\vec{r}) \quad (5.6)$$

But we also know that in spherical coördinates,

$$d\vec{r} = \hat{r}dr + \hat{\theta}r d\theta + \hat{\phi}r \cos\theta d\phi \quad (5.7)$$

which means

$$dV = (\vec{\nabla}V) \cdot (d\vec{r}) = (\hat{r} \cdot \vec{\nabla}V)dr + (\hat{\theta} \cdot \vec{\nabla}V)r d\theta + (\hat{\phi} \cdot \vec{\nabla}V)r \sin\theta d\phi \quad (5.8)$$

On the other hand, the chain rule tells us

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial \theta}d\theta + \frac{\partial V}{\partial \phi}d\phi \quad (5.9)$$

Since the spherical coördinate basis is orthonormal, this gives us the form of the gradient in spherical coördinates as

$$\vec{\nabla}V = \hat{r}\frac{\partial V}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial V}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} \quad (5.10)$$

## 5.2 Divergence, Curl, and All That

Thinking of  $\vec{\nabla}$  as a vector operator, we can also define how it acts on vector fields by analogy with the dot and cross products:

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad (5.11)$$

and

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = (\partial_y A_z - \partial_z A_y)\hat{x} + (\partial_z A_x - \partial_x A_z)\hat{y} + (\partial_x A_y - \partial_y A_x)\hat{z} \quad (5.12)$$

where we have defined the phenomenally useful notation

$$\partial_x = \frac{\partial}{\partial x} \quad (5.13)$$

There's a whole rich subject of vector calculus, and the relationship between derivatives and integrals, for which I refer you to section 3.6 of Symon, your Calc III text, and last year's lecture notes.

The one thing that's of interest to us right now is the observation that the conditions (4.29) for a force field  $\vec{F}(\vec{r})$  to be conservative are just equivalent to

$$\vec{\nabla} \times \vec{F} = \vec{0} \quad (5.14)$$

This is associated with the vector calculus identity

$$\vec{\nabla} \times \vec{\nabla} V = \vec{0} \quad (5.15)$$

which is usually stated as a consequence of the fact that a vector crossed with itself is zero  $\vec{A} \times \vec{A}$ , but note that we have to be a little careful because the vector in question here is an operator  $\vec{\nabla}$ . For instance, if the operator were  $r\vec{\nabla}$ , the result would be

$$r\vec{\nabla} \times (r\vec{\nabla} V) = r(\vec{\nabla} r) \times (\vec{\nabla} V) \neq \vec{0} \quad (5.16)$$

The real reason for this identity is once again the fact that partial derivatives “commute”, e.g., (4.26).

### 5.3 Practical Calculation of Potential Energy

We end our consideration of conservative forces with an illustration of a technique to determine the potential from which a conservative force field was generated. One method, by integrating from the origin along a conveniently chosen path to an arbitrary point, is illustrated in Section 3.12 of Symon. We illustrate here an alternative method, using the same force.

The force field in the example is

$$\vec{F} = ay(y^2 - 3z^2)\hat{x} + 3ax(y^2 - z^2)\hat{y} - 6axyz\hat{z} \quad (5.17)$$

which means the potential energy  $V(x, y, z)$  has the derivatives

$$\frac{\partial V}{\partial x} = ay(3z^2 - y^2) \quad (5.18a)$$

$$\frac{\partial V}{\partial y} = 3ax(z^2 - y^2) \quad (5.18b)$$

$$\frac{\partial V}{\partial z} = 6axyz \quad (5.18c)$$

If we knew the derivative of a one-dimensional function, we could take the indefinite integral and obtain the function up to an additive integration constant. We do basically the same thing here, integrating with respect to one variable at a time and holding the other ones constant.

First, we treat  $y$  and  $z$  as constants and integrate the  $x$  derivative:

$$V(x, y, z) = \int \frac{\partial V(x, y, z)}{\partial x} dx = \int ay(3z^2 - y^2) dx = axy(3z^2 - y^2) + A(y, z) \quad (5.19)$$

When we do the indefinite integral we get an arbitrary integration constant, but since  $y$  and  $z$  are constants for the purposes of the partial derivative, the integration constant can in principle depend on  $y$  and  $z$ .

We can use (5.18b) to determine a condition the  $A(y, z)$  introduced in (5.19):

$$3ax(z^2 - y^2) = \frac{\partial V}{\partial y} = 3axz^2 - 3xy^2 + \frac{\partial A}{\partial y} \quad (5.20)$$

we solve this to find

$$\frac{\partial A}{\partial y} = 0 \quad (5.21)$$

Now, ordinarily we'd integrate this, and get an integration "constant" which depends only on  $z$ , but here life is easier because the partial derivative vanishes, so there is no  $y$  dependence in  $A(y, z)$ :

$$A(y, z) = B(z) \quad (5.22)$$

and thus

$$V(x, y, z) = axy(3z^2 - y^2) + B(z) \quad (5.23)$$

Now we do the same thing as before, applying (5.18c) to (5.23) to find

$$6axyz = \frac{\partial V}{\partial z} = 6axyz + \frac{dB}{dz} \quad (5.24)$$

or

$$\frac{dB}{dz} = 0 \quad (5.25)$$

Again, we'd normally have to do an integral here, but this time, it's just telling us that  $B(z)$  is a constant

$$B(z) = C \quad (5.26)$$

i.e.,

$$V(x, y, z) = axy(3z^2 - y^2) + C \quad (5.27)$$

Note that in this case, the constant is just the value of the potential at the origin:

$$V(0, 0, 0) = C \quad (5.28)$$

so that

$$V(x, y, z) = V(0, 0, 0) + axy(3z^2 - y^2) \quad (5.29)$$

Typically in a physical problem, one chooses the constant in a potential so that the potential is zero at some convenient point, usually the origin or "infinity".

The same principle applies in non-Cartesian coördinates, except that you have to be careful about the definition of the gradient. So for example, in spherical coördinates,

$$\vec{F} = F_r \hat{r} + F_\theta \hat{\theta} + F_\phi \hat{\phi} = -\vec{\nabla}V = -\left(\frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}\right) V(r, \theta, \phi) \quad (5.30)$$

so

$$\frac{\partial V}{\partial r} = -F_r(r, \theta, \phi) \quad (5.31a)$$

$$\frac{\partial V}{\partial \theta} = -rF_\theta(r, \theta, \phi) \quad (5.31b)$$

$$\frac{\partial V}{\partial \phi} = -r \sin \theta F_\phi(r, \theta, \phi) \quad (5.31c)$$

and we can then use the same multi-variable calculus technique to find  $V(r, \theta, \phi)$  up to an additive constant, given its partial derivatives.

A good thing to keep in mind is that vector calculus treats Cartesian and non-Cartesian coordinate systems differently, but multi-variable calculus does not.

## 5.4 Application of Potential Energy

If we are dealing with a conservative force field, one which is associated with a potential energy:

$$\vec{F} = -\vec{\nabla}V \quad (5.32)$$

then conservation of energy holds just as in one dimension:

$$T_2 - T_1 = W_{1 \rightarrow 2} = \int_{1 \rightarrow 2} \vec{F} \cdot d\vec{r} = -[V(\vec{r}_2) - V(\vec{r}_1)] \quad (5.33)$$

and therefore

$$T_1 + V_1 = T_2 + V_2 \quad (5.34)$$

At this point, we've more or less covered Symon sections 3.1-3.7 (except for the part about torque in section 3.3), 3.9, and 3.12. Sections 3.10 and 3.11 illustrate some simple and non-simple applications of the methods (the not-so-simple one also appearing on a homework), and you should read about these on your own. We will now consider rotational motion (Symon sections 3.3 and 3.8) which will prepare us for a major sub-topic, central force motion, which is described in sections 3.13 through 3.16.

# 6 Rotational Motion

In introductory mechanics, one usually learns about rotation around an axis or rotation in a plane. In general, rotational motion can be described with vector quantities which deal with rotations about all possible axes through a given origin. Our goal here is to bridge the gap between these two descriptions.

## 6.1 Torque

There are a few different ways to define torque about an axis, but a convenient one for our purposes is

$$(\text{Torque about axis}) = (\text{Distance from axis})(\text{Perpendicular component of force}) \quad (6.1)$$

where “perpendicular” means “perpendicular to the direction from the point of application to the closest point on the axis” and also “perpendicular to the axis”. We'll eventually want a vector expression for this, but let's start by looking at it in a convenient coordinate system. The natural one to use is cylindrical coordinates with the  $z$  axis along the axis of rotation. For a given point, the distance to the axis is the  $\rho$  coordinate of that point. The “perpendicular” direction at that point is the  $\phi$  direction, so the perpendicular component

of the force is just  $F_\phi$ . This means that if we call the torque about the  $z$  axis  $N_z$  (using  $N$  to stand for torque for reasons which completely escape me),

$$N_z = \rho F_\phi \quad (6.2)$$

Notice that this also has the sign we'd like; if  $F_\phi > 0$ , we are applying positive torque around the  $z$  axis as defined by the right-hand rule.

Now, if we think for a minute about the fact that  $\rho$  and  $F_\phi$  are components of vectors

$$\vec{F} = F_\rho \hat{\rho} + F_\phi \hat{\phi} + F_z \hat{z} \quad (6.3a)$$

$$\vec{r} = \rho \hat{\rho} + z \hat{z} \quad (6.3b)$$

we'll realize that the combination  $\rho F_\phi$  is just the  $z$  component of the cross product  $\vec{r} \times \vec{F}$ :

$$\hat{z} \cdot (\vec{r} \times \vec{F}) = \rho F_\phi \quad (6.4)$$

[Ordinarily there would be another term because  $\hat{z} \cdot (\vec{A} \times \vec{B}) = A_\rho B_\phi - A_\phi B_\rho$ , but the position vector has no  $\phi$  component:  $\hat{\phi} \cdot \vec{r} = 0$ .]

So this means the torque about the  $z$  axis is

$$N_z = \hat{z} \cdot (\vec{r} \times \vec{F}) \quad (6.5)$$

But now the only reference to the coördinate system is the unit vector  $\hat{z}$  along the  $z$  axis (and the origin implicit in the position vector  $\vec{r}$ ). So in general we can say that the torque  $N_n$  around some axis through the origin parallel to a constant unit vector  $\hat{n}$  is

$$N_n = \hat{n} \cdot (\vec{r} \times \vec{F}) \quad (6.6)$$

Of course, the notation is supposed to be suggestive, and we can encode the torque about all possible axes through the origin if we define the vector

$$\vec{N} = \vec{r} \times \vec{F} \quad (6.7)$$

which is the torque.

## 6.2 Angular Momentum

The geometry of the definition of angular momentum is the same. The angular momentum about an axis is the distance from that axis times the perpendicular component of the momentum, so working once more in cylindrical coördinates

$$L_z = \rho p_\phi \quad (6.8)$$

Now, since we know that the velocity in cylindrical coördinates is

$$\vec{v} = \dot{\rho} \hat{\rho} + \rho \dot{\phi} \hat{\phi} + \dot{z} \hat{z} \quad (6.9)$$

we can write

$$p_\phi = mv_\phi = m\rho \dot{\phi} \quad (6.10)$$



to get the perhaps more familiar

$$L_z = m\rho^2\dot{\phi} \quad (6.11)$$

By the same vector argument as we used for torque, we can see that

$$L_z = \hat{z} \cdot (\vec{r} \times \vec{p}) \quad (6.12)$$

and define a vector angular momentum

$$\vec{L} = \vec{r} \times \vec{p} \quad (6.13)$$

### 6.3 Relationship of Angular Momentum and Torque

We can use Newton's second law to show that angular momentum and torque are related in the same way as linear momentum and force.

$$\frac{d\vec{L}}{dt} = \frac{d}{dt} (\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \quad (6.14)$$

Now, the first term is

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times (m\vec{v}) = m\vec{v} \times \vec{v} = \vec{0} \quad (6.15)$$

while we can use Newton's second law to make the replacement

$$\frac{d\vec{p}}{dt} = \vec{F} \quad (6.16)$$

and thus obtain

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{N} \quad (6.17)$$

An important special case of this is when the torque vanishes, i.e.,  $\vec{r} \times \vec{F} = \vec{0}$ . Then the vector angular momentum  $\vec{L}$  is a constant of the motion.

## 7 Central Force Motion

An important physical problem is that of a particle moving in an fixed force field, where the magnitude of the force is depends only on the distance from a specified point, and the direction of the force is always either towards or away from the that point. Mathematically, if we define a spherical coördinate system with the origin at the point in question, this means

$$\vec{F}(\vec{r}) = F(r)\hat{r} \quad (7.1)$$

Note that  $F(r)$  is the  $r$ -component of the force  $\vec{F}$  and is *not* the magnitude. In particular, it can be positive or negative.

The most obvious physical example is that of a planet orbiting a star. It's a decent approximation to assume that the star doesn't move much, and therefore the planet is moving in the fixed gravitational field of the star:

$$\vec{F} = -\frac{GMm}{r^2}\hat{r} \quad (7.2)$$

We'll see in chapter four that even if  $m \ll M$  we can still describe the two-body problem exactly in terms of an equivalent one-body problem with a particle orbiting a fixed center of force.

## 7.1 Conservation of Energy

It's easy to show that (7.1) is a conservative force. We could calculate the curl  $\vec{\nabla} \times \vec{F}$ , but it's actually more straightforward just to construct the potential explicitly. Since the force is determined by one function of one variable, we can take its indefinite integral with respect to  $r$ :

$$V(r) = - \int F(r) dr \quad (7.3)$$

We can show explicitly that this is indeed the potential energy by taking its gradient:

$$-\vec{\nabla}V = -\frac{\partial V}{\partial r}\hat{r} = F(r)\hat{r} = \vec{F} \quad (7.4)$$

This means that the total energy

$$E = T + V = \frac{1}{2}m\vec{v} \cdot \vec{v} + V(r) \quad (7.5)$$

is a constant of the motion.

## 7.2 Conservation of Angular Momentum

We can also show that the torque (about the origin) associated with this force is zero:

$$\vec{N} = \vec{r} \times \vec{F} = (r\hat{r}) \times (F(r)\hat{r}) = rF(r)(\hat{r} \times \hat{r}) = \vec{0} \quad (7.6)$$

This means that the vector angular momentum

$$\vec{L} = \vec{r} \times m\vec{v} \quad (7.7)$$

is a constant. As a consequence (which we will show):

1. The motion is confined to a plane.
2. There is an additional non-vector constant of the motion

To see that the motion remains confined to a plane, align the coordinate axes so that initially (at  $t = 0$ ), both the position and the velocity lie in the  $xy$  plane:

$$z(0) = \hat{z} \cdot \vec{r}(0) = 0 \quad (7.8a)$$

$$v_z(0) = \hat{z} \cdot \vec{v}(0) = 0 \quad (7.8b)$$

Now define a cylindrical coordinate system and calculate the components of the angular momentum (at any time) in that system:

$$\vec{L} = m\vec{r} \times \vec{v} = m \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \rho & 0 & z \\ v_\rho & v_\phi & v_z \end{vmatrix} = -mzv_\phi\hat{\rho} + m(zv_\rho - \rho v_z)\hat{\phi} + m\rho v_\phi\hat{z} \quad (7.9)$$

We evaluate this at the initial time and see

$$\vec{L}(0) = m\rho(0)v_\phi(0)\hat{z} \quad (7.10)$$

where we have used (7.8a) to show that the other components of the angular momentum vanish. But  $\vec{L}(0)$  is a constant of the motion, which means

$$\vec{L}(t) = \vec{L}(0) = m\rho(0)v_\phi(0)\hat{z} \quad (7.11)$$

Because we're working in cylindrical coördinates, we need to be a little careful about just setting the components of  $\vec{L}(0)$  and  $\vec{L}(t)$  equal to each other, because the basis vectors  $\hat{\rho}$  and  $\hat{\phi}$  depend on the position, which in turn depends on time. But fortunately the only basis vector appearing in (7.11) is  $\hat{z}$ , which is a constant, and  $\hat{\rho}$  and  $\hat{\phi}$ , while they will depend on  $t$ , are always perpendicular to  $z$ . This means we can say that  $\vec{L}(t)$  has no  $\rho$  or  $\phi$  component, and that its  $z$  component equals that of

$$-mz(t)v_\phi(t) = 0 \quad (7.12a)$$

$$mz(t)v_\rho(t) - m\rho(t)v_z(t) = 0 \quad (7.12b)$$

$$m\rho(t)v_\phi(t) = m\rho(0)v_\phi(0) \quad (7.12c)$$

Now, unless the initial angular momentum vanishes (which would be a special case, where the particle was initially moving straight towards or away from the origin), the product  $\rho v_\phi$ , which is a constant, would remain non-zero, which means neither  $\rho$  nor  $v_\phi$  can be zero. This means that (7.12a) tells us  $z(t) = 0$  for all time, which in turn tells us, along with (7.12b), that  $v_z(t) = 0$  for all time. In other words, if the motion starts out in the  $xy$  plane and the angular momentum is a constant (which it is for a central force) the motion remains in the  $xy$  plane forever.

We needed two of the components of the conserved angular momentum to tell us this; the third tells us that

$$L_z = m\rho v_\phi = \text{constant} \quad (7.13)$$

As a consequence of the fact that  $z \equiv 0$  in this coördinate system,

$$r = \sqrt{\rho^2 + z^2} \equiv \rho \quad (7.14)$$

and this is basically a two-dimensional problem described in plane polar coördinates  $(r, \phi)$ .

### 7.3 The Nature of the Differential Equations

In plane polar coördinates, thanks to the time derivatives

$$\frac{d\hat{r}}{dt} = \dot{\phi}\hat{\phi} \quad (7.15a)$$

$$\frac{d\hat{\phi}}{dt} = -\dot{\phi}\hat{r} \quad (7.15b)$$

of the basis vectors, we know the position, velocity and acceleration are

$$\vec{r} = r\hat{r} \quad (7.16a)$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} \quad (7.16b)$$

$$\vec{a} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (r\ddot{\phi} - 2\dot{r}\dot{\phi})\hat{\phi} \quad (7.16c)$$

and so using Newton's second law

$$\vec{F} = m\vec{a} \quad (7.17)$$

we could derive the equations of motion

$$m(\ddot{r} - r\dot{\phi}^2) = F(r) \quad (7.18a)$$

$$m(r\ddot{\phi} - 2\dot{r}\dot{\phi}) = 0 \quad (7.18b)$$

These are two second-order ODEs for the two functions  $r(t)$  and  $\phi(t)$ , so we could in principle solve them subject to four initial conditions, e.g., specified values of  $r(0)$ ,  $\dot{r}(0)$ ,  $\phi(0)$  and  $\dot{\phi}(0)$ .

However, conservation of energy and angular momentum allows us to cast the problem more simply. A given trajectory has a given value of  $E$  and  $L$ , which are related to  $r(t)$  and  $\phi(t)$  by

$$E = \frac{1}{2}m\vec{v} \cdot \vec{v} + V(r) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + V(r) \quad (7.19a)$$

$$L = mrv_{\phi} = mr^2\dot{\phi} \quad (7.19b)$$

These are then, for given values of  $E$  and  $L$ , two *first-order* ODES for  $r(t)$  and  $\phi(t)$ . Given  $E$  and  $L$  plus two initial conditions such as values of  $r(0)$ ,  $\dot{r}(0)$ , and  $\phi(0)$ , we can solve them for  $r(t)$  and  $\phi(t)$ .

The explicit formal solution is as follows:

First, we note that since (7.19b) contains only  $r$  and  $\dot{\phi}$ , we can use it to solve for  $\dot{\phi}$  and then remove it from (7.19a):

$$\dot{\phi} = \frac{L}{mr^2} \quad (7.20)$$

and therefore

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2 \left( \frac{L}{mr^2} \right)^2 + V(r) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2} \frac{L^2}{2mr^2} + V(r) \quad (7.21)$$

Equation (7.21) is then a single first-order ODE for  $r(t)$ . We can solve for

$$\dot{r} = \pm \sqrt{\frac{2}{m}} \sqrt{E - \left[ V(r) + \frac{L^2}{2mr^2} \right]} = \frac{dr}{dt} \quad (7.22)$$

which can then be solved by integration

$$\int_{r(0)}^{r(t)} \frac{dr}{\sqrt{E - \left[ V(r) + \frac{L^2}{2mr^2} \right]}} = \int_0^t \sqrt{\frac{2}{m}} dt' = t \sqrt{\frac{2}{m}} \quad (7.23)$$

Once we've got  $r(t)$ , we can then plug it into (7.20) and integrate to get

$$\phi(t) = \phi(0) + \int_0^t \dot{\phi}(t') dt' = \phi(0) + \int_0^t \frac{L}{mr^2(t')} dt' \quad (7.24)$$

## 7.4 Effective Potential

We've just described formally how one could solve a central force motion problem. It may often turn out, though, that the integrals (7.23) and (7.24) can't be evaluated in closed form. It is thus useful to be able to gain more insight into the problem by other methods of analysis.

Look at the conserved total energy (7.21). Although we started out with a three-dimensional problem and thus among other things a different expression for the kinetic energy, the end product actually looks a lot like the energy in a one-dimensional problem. Recall that this is

$$E = \frac{1}{2}m\dot{x}^2 + V(x) \quad (7.25)$$

while our total energy can be written in the form

$$E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r) \quad (7.26)$$

where

$$V_{\text{eff}}(r) = V(r) + \frac{L^2}{2mr^2} \quad (7.27)$$

is the so-called “effective potential”. We've reduced our three-dimensional problem to an equivalent one-dimensional problem where the only complication is the addition to the potential energy of a term, dependent on the angular momentum  $L$ , which accounts for the rotational motion. This is often referred to as the “centrifugal barrier”, since it blows up at small  $r$ .

Equation (7.26) allows us to use all the same methods as we did back in one-dimensional potential problems, but with  $V_{\text{eff}}$  as the potential. For example, at a given energy  $E$ , the turning points  $r_t$  are the solutions to  $V_{\text{eff}}(r_t) = E$ , and a stable equilibrium occurs at a minimum of  $V_{\text{eff}}$ . Now, of course, equilibrium means constant  $r$ . The angular coordinate  $\phi$ , meanwhile, is always changing at a rate

$$\dot{\phi} = \frac{L}{mr^2} \quad (7.28)$$

So the “equilibrium” corresponds to a circular orbit, around which the particle moves at a constant angular speed. In general, for a given value of  $L$ , which fixes a given  $V_{\text{eff}}(r)$ , this can only occur at particular values of  $r$ .

As in the one-dimensional case, if we are near the equilibrium radius  $r_e$  for a given  $V_{\text{eff}}(r)$ , the  $r$  value of the particle will oscillate about  $r_e$  with a frequency

$$\omega_R \approx \frac{V_{\text{eff}}''(r_e)}{m} \quad (7.29)$$

On the other hand, the angular frequency of the azimuthal motion will be

$$\omega_\Phi \approx \frac{L}{mr_e^2} \quad (7.30)$$

and in general these two will not be the same, so the orbit will not close, leading to something like Symon's figure 3.34.

## 7.5 Finding the Trajectory

As mentioned above, the integrals (7.23) and (7.24) for  $r(t)$  and  $\phi(t)$  often can't be evaluated in closed form. Sometimes it's easier to find the shape of the orbit  $r(\phi)$  and not worry about when the particle reaches which position. For given  $E$  and  $L$ , we get a first-order equation

$$\frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}} = \frac{\pm\sqrt{2/m}\sqrt{E - V(r) - L^2/2mr^2}}{L/2mr^2} \quad (7.31)$$

In practice, it often turns out to be even easier to work in terms of  $u(\phi) = 1/r(\phi)$ . This is because

$$\frac{du}{d\phi} = -\frac{1}{r^2} \frac{dr}{d\phi} = -\frac{\dot{r}}{r^2 \dot{\phi}} = \mp \frac{\sqrt{2[E - V(r)]/m - (L/mr)^2}}{L/m} = \mp \sqrt{\frac{2m[E - V(1/u)]}{L^2} - u^2} \quad (7.32)$$

We'll see this in detail when we consider the potential energy which is meaningful for gravity.

## 7.6 Kepler's Second Law

In the 17th century, Johannes Kepler discovered three empirical properties of planetary orbits, which are known as Kepler's Laws of Planetary Motion. They are:

1. Planetary orbits are in the shape of ellipses with the Sun at one focus.
2. A planet moves around its orbit at such a rate that the line from the planet to the Sun sweeps out equal areas in equal intervals of time.
3. The cube of the semimajor axis of a planet's orbit is proportional to the square of its orbital period.

Isaac Newton subsequently used these laws to deduce the nature of the gravitational interaction. The first and third laws turn out to be consequences of the particular spatial dependence of the gravitational interaction, but the second holds for any central force, being a consequence of the conservation of angular momentum.

This can be seen by considering the wedge swept out in a time  $dt$ . For sufficiently small time, the change in radius is negligible, and the area is approximately that of a circular wedge of radius  $r$  and angular extent  $d\phi = \dot{\phi} dt$ . This is approximately a triangle height  $r$  and base  $r d\phi$ , which has area

$$dA = \frac{1}{2}r(r d\phi) = \frac{1}{2}r^2 \dot{\phi} dt = \frac{L}{2m} dt \quad (7.33)$$

and since  $L$  is a constant for a central force, so is

$$\frac{dA}{dt} = \frac{L}{2m} \quad (7.34)$$

## 8 Inverse-Square-Law Forces

A very useful special case of a central force is one inversely proportional to the square of the distance:

$$F(r) = \frac{K}{r^2} \quad (8.1)$$

For example, the gravitational force on a particle of mass  $m$  due to a point source of mass  $M$  fixed at the origin is of this form with  $K = -GMm$ .

The potential energy corresponding to this force is

$$V(r) = - \int \frac{K}{r^2} dr = \frac{K}{r} + \text{const} \quad (8.2)$$

The conventional choice of the zero of the potential sets the constant to zero:

$$V(r) = \frac{K}{r} \quad (8.3)$$

this has the property that  $V(\infty) = 0$ .

### 8.1 Finding the Turning Points

We can examine the behavior by looking at the effective potential

$$V_{\text{eff}}(r) = \frac{K}{r} - \frac{L^2}{2mr^2} \quad (8.4)$$

At small  $r$ , as long as  $L \neq 0$ , the centrifugal term blows up faster and dominates:

$$V_{\text{eff}}(r) \xrightarrow{r \rightarrow 0} -\frac{L^2}{2mr^2} \quad (8.5)$$

At large  $r$ , the other term goes to zero more slowly, and eventually dominates:

$$V_{\text{eff}}(r) \xrightarrow{r \rightarrow \infty} \frac{K}{r} \quad (8.6)$$

Qualitatively, then we see the behavior:

- If  $K > 0$ , the effective potential is positive and repulsive for all  $r$ :  $V_{\text{eff}}(r) > 0$ ;  $V_{\text{eff}}'(r) < 0$ .
- If  $K < 0$ , there is a local minimum  $r_e$  for which  $V(r_e) < 0$ .

If  $K > 0$ , the only orbits possible are those with  $E > 0$ , which have one inner turning point  $r_t = r_{\min}$  at which  $V_{\text{eff}}(r_t) = E$ . (The value of  $r_t$  will of course depend on  $E$  and  $L$ .)

If  $K < 0$ , there are similarly unbound orbits with  $E > 0$  (actually now  $E \geq 0$ ) and a single turning point, but also orbits with  $E < 0$ , which have two turning points  $r_{\min} < r_e < r_{\max}$  at which  $V_{\text{eff}}(r_t) = E$ .

We write the equation for the turning points as

$$E = \frac{K}{r_t} + \frac{L^2}{2mr_t^2} \quad (8.7)$$

Again, it turns out to be easier to work in terms of  $u = 1/r$ , in which case the turning points are given by

$$\frac{L^2}{2m}u_t^2 + Ku_t - E = 0 \quad (8.8)$$

or

$$u_t^2 + \frac{2mK}{L^2}u_t - \frac{2mE}{L^2} = 0 \quad (8.9)$$

The solutions to this quadratic equation are

$$u_t = -\frac{2mK}{L^2} \pm \sqrt{\left(\frac{2mK}{L^2}\right)^2 + \frac{2mE}{L^2}} = -\frac{2mK}{L^2} \left(1 \pm \sqrt{1 + \frac{2EL^2}{mK^2}}\right) \quad (8.10)$$

Note that if

$$E < -\frac{mK^2}{2L^2} \quad (8.11)$$

both roots are imaginary, which corresponds to an energy below the minimum of the effective potential.

While we're at it, we should check that the dimensions work out:

$$\left[\frac{2EL^2}{mK^2}\right] = \frac{E(MVR)^2}{M[K]^2} \quad (8.12)$$

The units of the coupling constant can be deduced from

$$\left[\frac{K}{r}\right] = E \quad (8.13)$$

so

$$\left[\frac{2EL^2}{mK^2}\right] = \frac{E(MVR)^2}{M(ER)^2} = \frac{MV^2}{E} = 1 \quad (8.14)$$

which is what we expect, and

$$\left[\frac{2mK}{L^2}\right] = \frac{MER}{(MVR)^2} = \frac{E}{MV^2R} = \frac{1}{R} \quad (8.15)$$

which also what we need as the units of  $u$ .

To make life easier, we'll concentrate on the case of an attractive force  $K < 0$  and define the positive constants

$$\alpha = -\frac{L^2}{2mK} \quad (8.16a)$$

$$\varepsilon = \sqrt{1 + \frac{2EL^2}{mK^2}} \quad (8.16b)$$

so that

$$u_t = \alpha^{-1}(1 \pm \varepsilon) \quad (8.17)$$

Now, note that



- If  $E < 0$ ,  $\varepsilon < 1$  and both roots are positive, meaning there are two turning points, as we expect:

- The maximum value of  $u$ , which corresponds to the minimum value of  $r$ , the closest approach of the particle to the origin, which is known as *periapse*, is

$$u_p = \alpha^{-1}(1 + \varepsilon) \quad (8.18)$$

- The minimum value of  $u$ , which corresponds to the maximum value of  $r$ , the farthest displacement of the particle from the origin, which is known as *apoapse*, is

$$u_a = \alpha^{-1}(1 - \varepsilon) \quad (8.19)$$

- If  $E = 0$ ,  $\varepsilon = 1$  and one root is zero, meaning that the minimum value of  $u$  is zero and the maximum value of  $r$  is infinite, so there is no apoapse, and the periapse is still

$$u_p = \alpha^{-1}(1 + \varepsilon) \quad (8.20)$$

- If  $E > 0$ ,  $\varepsilon > 1$  and one root is negative, meaning that the minimum *allowed* value of  $u$  is still zero and the maximum allowed value of  $r$  is still infinite, so as in the marginal  $E = 0$  case, there is no apoapse and the periapse is located at

$$u_p = \alpha^{-1}(1 + \varepsilon) \quad (8.21)$$

## 8.2 Finding the Trajectory

Now we'd like to do the integral to find  $u(\phi)$  and thus  $r(\phi)$ . In all three cases,  $\frac{du}{d\phi}$  vanishes at the periapse  $u_p = \alpha^{-1}(1 + \varepsilon)$ , so let's use that as our lower limit of integration. Let's take the upper limit of integration to be some  $\phi > \phi_p$  between the two turning points, so that  $\frac{du}{d\phi} \leq 0$  (since  $u_p$  is the maximum value of  $u$ ) and

$$\phi - \phi_p = \int_{\phi_p}^{\phi} d\phi' = \int_{u_p}^{u(\phi)} \frac{du}{du/d\phi} = - \int_{(1+\varepsilon)/\alpha}^{u(\phi)} \frac{du}{\sqrt{\frac{2m(E-Ku)}{L^2} - u^2}} \quad (8.22)$$

where we have used  $V(r) = \frac{K}{r} = Ku$ .

Now, the integral will look a lot simpler if we substitute for  $E$  and  $L$  in terms of  $\alpha^{-1}$  and  $\varepsilon$ . We can do the algebra using the definitions (8.16), but in fact we should be able to see that the argument in the square root must be

$$\frac{2m(E - Ku)}{L^2} - u^2 = (u_+ - u)(u - u_-) \quad (8.23)$$

where

$$u_{\pm} = \alpha^{-1}(1 \pm \varepsilon) \quad (8.24)$$

This is because we got  $u_{\pm}$  in the first place by setting the quadratic expression appearing inside the square root to zero, so it has to vanish if  $u = u_+$  or  $u = u_-$ . After that, it's just a

question of getting the right coefficient (i.e.,  $-1$ ) for the  $u^2$  term. (If you don't believe me, you can do the algebra and check.)

Anyway,

$$(u_+ - u)(u - u_-) = -u^2 + (u_+ + u_-)u - u_+u_- = -u^2 + 2\alpha^{-1}u - \alpha^{-2}(1 - \varepsilon^2) = \alpha^{-2}\varepsilon^2 - (\alpha^{-1} - u)^2 \quad (8.25)$$

so

$$\phi - \phi_p = - \int_{\alpha^{-1}(1+\varepsilon)}^{u(\phi)} \frac{du}{\sqrt{\alpha^{-2}\varepsilon^2 - (\alpha^{-1} - u)^2}} \quad (8.26)$$

This is a doable integral; making the identification  $x = \alpha^{-1} - u$ , it's of the form

$$\int \frac{dx}{a^2 - x^2} \quad (8.27)$$

which calls for the trigonometric substitution

$$x = a \sin \theta \quad (8.28)$$

so in our case, we want to define a new integration variable  $\psi$  by

$$\alpha^{-1} - u = \alpha^{-1}\varepsilon \sin \psi \quad (8.29)$$

In that case,

$$-du = \alpha^{-1}\varepsilon \cos \psi d\psi \quad (8.30)$$

and

$$\sqrt{\alpha^{-2}\varepsilon^2 - (\alpha^{-1} - u)^2} = \alpha^{-1}\varepsilon \sqrt{1 - \sin^2 \psi} = \alpha^{-1}\varepsilon \cos \psi \quad (8.31)$$

We'll see we're justified in calling this  $\cos \psi$  and not  $-\cos \psi$  when we consider the limits of integration. The lower limit  $\psi_p$  obeys

$$\sin \psi_p = \frac{\alpha^{-1} - \alpha^{-1}(1 + \varepsilon)}{\alpha^{-1}\varepsilon} = 1 \quad (8.32)$$

so if we use

$$\psi_p = \frac{\pi}{2} \leq \psi \leq -\frac{\pi}{2} \quad (8.33)$$

we're justified in assuming that  $\cos \psi \geq 0$ .

Putting it all together, the integral is

$$\phi - \phi_p = \int_{\pi/2}^{\sin^{-1}\left(\frac{\alpha^{-1}-u(\phi)}{\alpha^{-1}\varepsilon}\right)} d\psi = \sin^{-1}\left(\frac{\alpha^{-1} - u(\phi)}{\alpha^{-1}\varepsilon}\right) - \frac{\pi}{2} \quad (8.34)$$

or, to solve for  $u(\phi)$ ,

$$\frac{\alpha^{-1} - u(\phi)}{\alpha^{-1}\varepsilon} = \sin\left(\frac{\pi}{2} + \phi - \phi_p\right) = \cos(\phi - \phi_p) \quad (8.35)$$

which can be solved to give

$$u(\phi) = \frac{1 + \varepsilon \cos(\phi - \phi_p)}{\alpha} \quad (8.36)$$

or, in terms of the radial coördinate,

$$r(\phi) = \frac{\alpha}{1 + \varepsilon \cos(\phi - \phi_p)} \quad (8.37)$$

Note that while our definition made some simplifying assumptions about where in the orbit we were, the solution actually holds in general.

### 8.3 Orbits in an Inverse-Square Force Field

Equation (8.37) is the equation, in polar coördinates, of a *conic section of eccentricity*  $\varepsilon$  and *latus rectum*  $\alpha$ . Specifically

- If  $\varepsilon > 1$ , this is a hyperbola with one focus at the origin
- If  $\varepsilon = 1$ , this is a parabola with its focus at the origin
- If  $\varepsilon < 1$ , this is an ellipse with one focus at the origin

This is basically Kepler's first law. Kepler only looked at planetary orbits, so he only saw the ellipses. Long-period comets have approximately parabolic orbits, while a piece of interstellar debris on a ballistic trajectory around the Sun would have a hyperbolic orbit.

If we focus on the  $\varepsilon < 1$  case, it's useful to define the *semimajor axis*

$$a = \frac{\alpha}{1 - \varepsilon^2} \quad (8.38)$$

so that the equation of the orbit is

$$r(\phi) = \frac{a(1 - \varepsilon^2)}{1 + \varepsilon \cos(\phi - \phi_p)} \quad (8.39)$$

The distances from the origin of periapse and apoapse are

$$r_p = \frac{a(1 - \varepsilon^2)}{1 + \varepsilon} = a(1 - \varepsilon) \quad (8.40a)$$

$$r_a = \frac{a(1 - \varepsilon^2)}{1 - \varepsilon} = a(1 + \varepsilon) \quad (8.40b)$$

So that

$$r_p + r_a = 2a \quad (8.41)$$

and the semimajor axis is just half the width of the ellipse at its widest point.

## A Appendix: Correspondence to Class Lectures

Date	Sections	Pages
2003 October 9	1–1.4	2–6
2003 October 16	1.5–1.7	7–7
2003 October 21	2–3.1	7–7
2003 October 23	3.2–4.2.1	7–9
2003 October 28	4.3–5.2	9–13
2003 October 29	5.3–6.3	13–17
2003 October 30	7–7.4	17–21
2003 November 4	7.3–8	20–23
2003 November 6	8.1–8.3	23–27