Systems of Particles
(Symon Chapter Four)

Physics A300*

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Contents

So far we've considered first one-dimensional and then multi-dimensional motion of a single particle moving under the influence of some outside forces. Now we consider the physically much more interesting situation where there are N particles exterting forces on each other and possibly also experiencing the influence of outside forces.

0 Notation

We label the individual particles with the index $k = 1 \ldots N$, so that $\vec{r}_k$ could be $\vec{r}_1$, $\vec{r}_2$, etc up to $\vec{r}_N$. It's important to note that this k does *not* label different components of a vector, but rather different vectors.

Particle k:

- has mass m_k
- is located at position $\vec{r}_k$
- has velocity $\vec{v}_k = \dot{\vec{r}}_k$
- experiences acceleration $\ddot{\vec{r}}_k$
- experiences total force $\vec{F}_k$, which is made up of the "internal force" $\vec{F}^i_k$ due to the other particles and the external force $\vec{F}^e_k$ applied on it. Note that $\vec{F}^e_k$ is generally different for different particles. The total force on particle k is, according to this definition,

$$\vec{F}_k = \vec{F}^e_k + \vec{F}^i_k \tag{0.1}$$

Additionally, we define the force exerted *on* particle k *by* particle ℓ by $\vec{F}^i_{\ell \to k}$ and note that by definition the total internal force on particle k is

$$\vec{F}^i_k = \sum_{\ell=1}^{N} \vec{F}^i_{\ell \to k} \tag{0.2}$$

As a reminder of the rules governing sums, the student is directed to the handout on mathematical grammar.

1 Conservation Laws (Symon Sections 4.1-4.4)

We saw in the case of a single particle how some quantities like momentum, energy and angular momentum are conserved under certain circumstances, and in general, how the rate at which they change is related to force, work, and torque. Now we consider what happens when we add up the momentum, energy, or angular momentum of all the particles in the system.

1.1 Momentum

We define the total momentum $\vec{P}$ of the system as the vector sum of the momenta of all the particles that make it up:

$$\vec{P} = \sum_{k=1}^{N} \vec{p}_k \tag{1.1}$$

where

$$\vec{p}_k = m_k \vec{v}_k = m_k \dot{\vec{r}}_k \tag{1.2}$$

The rate of change of this is, by the sum rule,

$$\frac{d\vec{P}}{dt} = \sum_{k=1}^{N} \frac{d\vec{p}_k}{dt} = \sum_{k=1}^{N} \vec{F}_k = \sum_{k=1}^{N} \vec{F}_k^e + \sum_{k=1}^{N} \vec{F}_k^i \tag{1.3}$$

The first term is the total external force on the system, and the second is the total internal force of all the particles on each other. We will now show that if the internal forces between the particles obey Newton's third law, the second term vanishes.

Newton's third law (in its so-called "weak form") says that the forces between any two particles k and ℓ are equal and opposite:

$$\vec{F}^i_{\ell \to k} = -\vec{F}^i_{k \to \ell} \tag{1.4}$$

In particular, the "self-force" of the particle on itself must vanish

$$\vec{F}^i_{k \to k} = \vec{0} \tag{1.5}$$

so we can rewrite (0.2) as

$$\vec{F}^i_k = \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N} \vec{F}^i_{\ell \to k} \tag{1.6}$$

We can write the second term in (1.3) as

$$\sum_{k=1}^{N} \vec{F}^i_k = \sum_{k=1}^{N} \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N} \vec{F}^i_{\ell \to k} = \sum_{k=1}^{N} \sum_{\ell=1}^{k-1} \vec{F}^i_{\ell \to k} + \sum_{k=1}^{N} \sum_{\ell=k+1}^{N} \vec{F}^i_{\ell \to k} \tag{1.7}$$

Each of the two sums includes half of the possible terms in which $k \neq \ell$; the first sum contains the terms with $\ell < k$, and the second all the terms with $\ell > k$. In (1.7), we sum over ℓ first, but we could also count all the relevant pairs of terms by summing first over all the possible k values for a given ℓ, then summing over the full range of ℓ values. Two different ways to collect

$$\begin{array}{ccccc}
(k=1,\ell=2) & (k=1,\ell=3) & (k=1,\ell=4) & \dots & (k=1,\ell=N) \\
 & (k=2,\ell=3) & (k=2,\ell=4) & \dots & (k=2,\ell=N) \\
 & & (k=3,\ell=4) & \dots & (k=3,\ell=N) \\
 & & & \dots & \dots \\
 & & & & (k=N-1,\ell=N)
\end{array}$$

Thus we see

$$\sum_{k=1}^{N}\sum_{\ell=k+1}^{N}\vec{F}^i_{\ell\to k} = \sum_{\ell=1}^{N}\sum_{k=1}^{\ell-1}\vec{F}^i_{\ell\to k} \tag{1.8}$$

but since k and ℓ are just dummy indices in sums, we can rename them:

$$\sum_{k=1}^{N}\sum_{\ell=k+1}^{N}\vec{F}^i_{\ell\to k} = \sum_{\ell=1}^{N}\sum_{k=1}^{\ell-1}\vec{F}^i_{\ell\to k} = \sum_{k=1}^{N}\sum_{\ell=1}^{k-1}\vec{F}^i_{k\to \ell} \tag{1.9}$$

which we can substitute back into (1.7) to get

$$\sum_{k=1}^{N}\vec{F}^i_k = \sum_{k=1}^{N}\sum_{\ell=1}^{k-1}\vec{F}^i_{\ell\to k} + \sum_{k=1}^{N}\sum_{\ell=1}^{k-1}\vec{F}^i_{k\to\ell} = \sum_{k=1}^{N}\sum_{\ell=1}^{k-1}\underbrace{(\vec{F}^i_{\ell\to k}+\vec{F}^i_{k\to\ell})}_{\vec{0}\text{ by Newton's 3rd}} = \vec{0} \tag{1.10}$$

We thus find that the rate of change in the total momentum of the system is equal to the sum of the *external* forces for systems obeying Newton's third law:

$$\frac{d\vec{P}}{dt} = \sum_{k=1}^{N}\vec{F}^e_k \tag{1.11}$$

Symon calls this total external force $\vec{F}$.

A consequence of this is that when there is no external force, and the internal forces obey Newton's third law, the total momentum of the system is conserved (although it may be exchanged between particles due to the internal forces):

$$\frac{d\vec{P}}{dt} = \vec{0} \qquad \text{when } \vec{F}^e_k = \vec{0} \text{ and } \vec{F}^i_{\ell\to k} = -\vec{F}^i_{k\to\ell} \tag{1.12}$$

Symon makes the point that this result can also be derived when the total work done in moving all the particles in the system by the *same* infinitesimal displacement $d\vec{r}$ vanishes. This is the physically deep statement that conservation of momentum follows from isotropy of space.

1.2 Center of Mass

The total momentum of a system takes on a particularly interesting form in terms of the *center of mass* of the system, which is a point whose position vector is defined as the weighted average of all the position vectors (weighted by the corresponding particle masses):

$$\vec{R} = \frac{\sum_{k=1}^{N} m_k\vec{r}_k}{\sum_{k=1}^{N} m_k} = \frac{\sum_{k=1}^{N} m_k\vec{r}_k}{M} \tag{1.13}$$

if we multilply by mass to get

$$M\vec{R} = \sum_{k=1}^{N} m_k\vec{r}_k \tag{1.14}$$

and then take the time derivative, we get

$$M\dot{\vec{R}} = \sum_{k=1}^{N} m_k \dot{\vec{r}}_k = \sum_{k=1}^{N} \vec{p}_k = \vec{P} \tag{1.15}$$

so the total momentum of the system is the same as that of a particle with a mass equal to that of the whole system moving with the center of mass.

Similarly, we can write the result of section 1.1 as

$$\vec{F} = \sum_{k=1}^{N} \vec{F}_k^e = \frac{d\vec{P}}{dt} = M\ddot{\vec{R}} \tag{1.16}$$

so the center of mass of the system moves as if it were a particle of mass M (the total mass) being acted on by the total external force. This fact often makes it sensible to break up the motion of a system into the motion of the center of mass and the motions of the particles about that center of mass.

1.3 Angular Momentum

In Chapter Three we motivated a definition of the angular momentum $\vec{L}$ of a single particle about the origin as

$$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \dot{\vec{r}} \tag{1.17}$$

We now generalize this to be the angular momentum of the kth particle defined with respect to an arbitrary point $\mathcal{Q}$ which may or may not be moving, and replace $\vec{r}$ with $\vec{r} - \vec{r}_{\mathcal{Q}}$:

$$\vec{L}_{k\mathcal{Q}} = m_k(\vec{r}_k - \vec{r}_{\mathcal{Q}}) \times (\dot{\vec{r}}_k - \dot{\vec{r}}_{\mathcal{Q}}) \tag{1.18}$$

If we take the time derivative of this, we get (by the product rule)

$$\begin{aligned} \frac{d\vec{L}_{k\mathcal{Q}}}{dt} &= m_k \underbrace{(\dot{\vec{r}}_k - \dot{\vec{r}}_{\mathcal{Q}}) \times (\dot{\vec{r}}_k - \dot{\vec{r}}_{\mathcal{Q}})}_{\vec{0}} + m_k(\vec{r}_k - \vec{r}_{\mathcal{Q}}) \times (\ddot{\vec{r}}_k - \ddot{\vec{r}}_{\mathcal{Q}}) \\ &= (\vec{r}_k - \vec{r}_{\mathcal{Q}}) \times m_k\ddot{\vec{r}}_k - m_k(\vec{r}_k - \vec{r}_{\mathcal{Q}}) \times \ddot{\vec{r}}_{\mathcal{Q}} \end{aligned} \tag{1.19}$$

We define the total angular momentum about the point $\mathcal{Q}$ by

$$\vec{L}_{\mathcal{Q}} = \sum_{k=1}^{N} \vec{L}_{k\mathcal{Q}} \tag{1.20}$$

You will work out on the homework some of the properties of this sum, but for the time being, let's use (1.19) to calculate its time derivative

$$\begin{aligned} \frac{d\vec{L}_{\mathcal{Q}}}{dt} = \sum_{k=1}^{N} \frac{d\vec{L}_{k\mathcal{Q}}}{dt} &= \sum_{k=1}^{N} \left[(\vec{r}_k - \vec{r}_{\mathcal{Q}}) \times m_k\ddot{\vec{r}}_k - m_k(\vec{r}_k - \vec{r}_{\mathcal{Q}}) \times \ddot{\vec{r}}_{\mathcal{Q}} \right] \\ &= \sum_{k=1}^{N} (\vec{r}_k - \vec{r}_{\mathcal{Q}}) \times (\vec{F}_k^e + \vec{F}_k^i) - M(\vec{R} - \vec{r}_{\mathcal{Q}}) \times \ddot{\vec{r}}_{\mathcal{Q}} \end{aligned} \tag{1.21}$$

The last term vanishes in a lot of problems of interest. In particular, $\vec{R}-\vec{r}_{\mathcal{Q}}=\vec{0}$ if the point $\mathcal{Q}$ is the center of mass, while $\ddot{\vec{r}}_{\mathcal{Q}}=\vec{0}$ if $\mathcal{Q}$ is moving with a constant velocity. We assume we have one of those two cases, so that

$$\frac{d\vec{L}_{\mathcal{Q}}}{dt}=\sum_{k=1}^{N}(\vec{r}_k-\vec{r}_{\mathcal{Q}})\times\vec{F}_k^e+\sum_{k=1}^{N}(\vec{r}_k-\vec{r}_{\mathcal{Q}})\times\vec{F}_k^i \tag{1.22}$$

As in the case of linear momentum, the latter term vanishes as a consequence of Newton's third law, but here we need not only that the internal forces between any pair of particles are equal and opposite

$$\vec{F}^i_{k\to\ell}=-\vec{F}^i_{\ell\to k} \tag{1.23}$$

but also that they are directed along the line between the particles, so that

$$(\vec{r}_k-\vec{r}_\ell)\times\vec{F}^i_{k\to\ell}=\vec{0} \tag{1.24}$$

It is left as an exercise to the student to show that (1.23) and (1.24) imply that the second sum in (1.22) vanishes, leaving

$$\frac{d\vec{L}_{\mathcal{Q}}}{dt}=\sum_{k=1}^{N}(\vec{r}_k-\vec{r}_{\mathcal{Q}})\times\vec{F}_k^e=\vec{N}_{\mathcal{Q}} \tag{1.25}$$

where we have defined the total torque $\vec{N}_{\mathcal{Q}}$ about the point $\mathcal{Q}$ due to all the external forces.

1.4 Energy

Recall the one-particle work-energy theorem. There we defined the kinetic energy

$$T=\frac{1}{2}m\vec{v}\cdot\vec{v} \tag{1.26}$$

and showed via Newton's second law that

$$\frac{dT}{dt}=\vec{F}\cdot\vec{v} \tag{1.27}$$

If $\vec{F}$ depended only on the position of the particle (e.g., expressed in the position vector $\vec{r}$), then we could define the work done in going from one point to another along a particular path. For the actual trajectory followed, the total work done was found to be equal to the change in kinetic energy.

$$W_{1\to 2}=\oint_{1\to 2}\vec{F}\cdot d\vec{r}=T_2-T_1 \tag{1.28}$$

Furthermore, if the force field happened to obey

$$\vec{\nabla}\times\vec{F}=\vec{0} \tag{1.29}$$

we could define a scalar field $V(\vec{r})$ such that

$$\vec{F}=-\vec{\nabla}V \tag{1.30}$$

and then the total energy (kinetic plus potential) would be conserved:

$$\frac{d}{dt}(T+V)=0 \tag{1.31}$$

For a system of particles, we can define the total kinetic energy

$$T=\sum_{k=1}^{N}T_k=\sum_{k=1}^{N}\frac{1}{2}m\vec{v}_k\cdot\vec{v}_k \tag{1.32}$$

and the sum rule along with Newton's second law tells us

$$\frac{dT}{dt}=\sum_{k=1}^{N}\frac{dT_k}{dt}=\sum_{k=1}^{N}\vec{F}_k\cdot\vec{v}_k \tag{1.33}$$

For forces depending only on position, we assume the external force on each particle depends only on that particle's location:

$$\vec{F}_k=\vec{F}^e_k(\vec{r}_k)+\vec{F}^i_k(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N,) \tag{1.34}$$

We could define a general work done in moving all the particles to new positions, but we'd need to worry about not only the path each particle took through 3-dimensional space, but where each other particle was on its path at each point along the way. Basically, we'd have to integrate along a path in $3N$-dimensional *configuration space*.

Instead, let's specialize to cases where a potential energy can be defined up front. To work with a potential energy which can depend on the coördinates of many particles, we'll define the gradient operator for the kth particle

$$\vec{\nabla}_k:=\hat{x}\frac{\partial}{\partial x_k}+\hat{y}\frac{\partial}{\partial y_k}+\hat{z}\frac{\partial}{\partial z_k} \tag{1.35}$$

which is just analogous to the position vector for the kth particle

$$\vec{r}_k=\hat{x}\,x_k+\hat{y}\,y_k+\hat{z}\,z_k \tag{1.36}$$

Consider now two special cases in which potential energy can be defined:

1) All the forces, internal and external combined, can be derived from a single potential $V(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N)$ (which has the $3N$ coördinates of the N particles as its arguments) as

$$\vec{F}_k=-\vec{\nabla}_kV \tag{1.37}$$

Then, using (1.33), we find

$$\begin{aligned}\frac{dT}{dt}&=\sum_{k=1}^{N}\left(-\vec{\nabla}_kV\right)\cdot\frac{d\vec{r}_k}{dt}=-\underbrace{\sum_{k=1}^{N}\left(\frac{\partial V}{\partial x_k}\frac{dx_k}{dt}+\frac{\partial V}{\partial y_k}\frac{dy_k}{dt}+\frac{\partial V}{\partial z_k}\frac{dz_k}{dt}\right)}_{3N\text{ terms}}\\&=-\left(\frac{\partial V}{\partial x_1}\frac{dx_1}{dt}+\frac{\partial V}{\partial y_2}\frac{dy_2}{dt}+\ldots+\frac{\partial V}{\partial z_N}\frac{dz_N}{dt}\right)=-\frac{dV}{dt}\end{aligned} \tag{1.38}$$

where the last step follows by the chain rule.

Thus

$$\frac{dT}{dt} + \frac{dV}{dt} = \frac{d}{dt}(T+V) = 0 \tag{1.39}$$

and the total energy $\sum_{k=1}^{N} T_k + V$ is conserved.

2) The internal forces $\vec{F}_k^i(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$ are conservative and derivable from a potential $V_k^i(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$ (note that Symon leaves off the superscript i, but it's clearer if we make explicit that this is the internal potential energy) but the internal ones are not. This means

$$\vec{F}_k = \vec{F}_k^e(\vec{r}_k) - \vec{\nabla}_k V^i \tag{1.40}$$

Now

$$\frac{dT}{dt} = \sum_{k=1}^{N} \left[\vec{F}_k^e \cdot \vec{v}_k - \left(\vec{\nabla}_k V^i\right) \cdot \vec{v}_k\right] \tag{1.41}$$

so the rate of change of the *internal* energy is

$$\frac{\partial}{\partial t}\left(\underbrace{T + V^i}_{E_i}\right) = \sum_{k=1}^{N} \vec{F}_k^e \cdot \frac{d\vec{r}_k}{dt} \tag{1.42}$$

Note that now we can say

$$E_F^i - E_I^i = \sum_{k=1}^{N} \oint_{\mathcal{C}_k} \vec{F}_k^e(\vec{r}_k) \cdot d\vec{r}_k \tag{1.43}$$

The work done on the system (the change in total internal energy from initial state I to final state F) is the sum of N independent 3-dimensional line integrals, The kth integral is along $\mathcal{C}_k$, the path followed by the kth particle from its initial position $\vec{r}_{kI}$ to its final position $\vec{r}_{kF}$. This is a sum of N intependent 3-dimensional line integrals rather than a single integral through $3N$-dimensional configuration space because the external force on one particle doesn't depend on the positions of the others.

1.5 Comments on Symon Section 4.4 (originally sent via email)

Symon makes a few statements in his "Critique of the Conservation Laws" (Section 4.4) which ought to be updated in light of developments in Physics which have occurred in the 32 years since the book was published.

On page 169, he says "The theory of relativity predicts a few slight deviations from the classically predicted motion, but these are too small to be observed except in the case of the orbit of Mercury..." In the 1970s, Hulse and Taylor discovered the binary pulsar 1913+16, two very dense neutron stars orbiting fast enough and close enough to one another that relativistic gravitational effects become important. Among the predictions of general relativity is that the system should lose energy due to the emission of gravitational radiation, causing the orbits to slowly evolve. Sure enough, decades of observation has shown a change in their orbital parameters consistent with that predicted by general relativity. Hulse and Taylor received the 1993 Nobel Prize in Physics for this discovery. More details can be found at `http://www.nobel.se/physics/laureates/1993/`

Also on page 169, he says "Even quantum mechanics fails to describe such phenomena correctly, and physics is now struggling to produce a new theory which will describe this class of phenomena." In fact, most of the interactions among elementary particles such as electrons and the quarks which make up protons and neutrons are now well understood and described in terms of quantum field theory and the so-called standard model of particle Physics, which has had a number of Nobel Prizes associated with it, most notably that shared by Weinberg, Salam and Glashow in 1979 (`http://www.nobel.se/physics/laureates/1979/`). It is interesting to note that the fundamental description of this theory is in the form of a Lagrangian. Next semester we will learn about Lagrangian mechanics (see chapter 9), which will initially be introduced as an equivalent formulation of Newtonian mechanics. Interestingly, it turns out to be much more widely applicable than the Newtonian mechanics which it reproduces.

On page 170, Symon says "If the particles of which the larger body is composed are taken as atoms and molecules ... we should apply quantum mechanics, not classical mechanics, to their motion." The fact that classical mechanics is a good approximation for the behavior of everyday objects, and the successes of quantum mechanics as a fundamental theory, indicate that classical mechanics should emerge as an approximation to quantum mechanics, in the limit of large systems. This expectation is analogous to the way that Newtonian mechanics and gravitation emerge as approximations to Einstein's special and general theories of relativity. However, no one has produced a comprehensive description of this "quantum to classical transition", in part because the most practical interpretation of quantum mechanics requires an artificial division into a quantum-mechanical system and a classical observer.

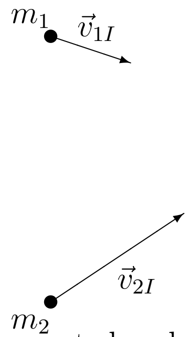
On page 171, Symon says "The gravitational forces acting between astronomical bodies are conservative, so that the principle of conservation of mechanical energy holds very accurately in astronomy." The one exception we now have to this is the aforementioned binary pulsar (and systems like it). In that case, the relativistic effects make the gravitational effects behave more like electromagnetism than Newtonian gravity, which means we have to assign energy and angular momentum to the gravitational radiation being given off by the binary in order to apply the conservation laws.

We will now skip section 4.5 ("Rockets, Conveyor Belts, and Planets") but you should read the last paragraph on pages 174–175 [which includes Equations (4.57–4.59)] about the evolution of the Earth-Moon system.

2 Collision Problems (Symon Section 4.6)

This is a special, rather useful, case of a multi-particle interaction, with $N = 2$. Analysis of collisions relies purely on conservation of energy and momentum. Here is the basic picture (note that there are no external forces ($\vec{F}_k^e = \vec{0}$)

- **Before the collision**:



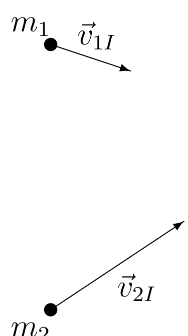
The particles are widely separated and there are no or negligible internal forces, which means no net force on either particle:

$$\vec{F}_k = m_k \dot{\vec{v}}_k = \vec{0} \tag{2.1}$$

This means that each particle's velocity vector is constant before the collision:

$$\vec{v}_k = \text{constant} = \vec{v}_{kI} \tag{2.2}$$

- **During the collision**: The particles are interacting (so the internal forces are non-zero). The interaction is often not well modelled. All we know is
  - Newton's third law holds, so the total momentum $\vec{P} = \vec{p}_1 + \vec{p}_2$ is conserved.
  - *Sometimes*, we also know the forces to be conservative. (This is called an *elastic collision*.) In those cases, total energy $E = E_1 + E_2$ is also conserved.
- **Before the collision**:



Just as before the collision, the particles are widely separated and there are no or negligible internal forces, which again means that each particle's velocity vector is constant after the collision:

$$\vec{v}_k = \text{constant} = \vec{v}_{kF} \tag{2.3}$$

In general, though, $\vec{v}_{kI} \neq \vec{v}_{kF}$; the velocity of each particle changes as a result of the collision.

Applying conservation of momentum, $\vec{P}$ is unchanged as a result of the collision, so

$$m_1\vec{v}_{1I} + m_2\vec{v}_{2I} = \vec{P}_I = \vec{P}_F = m_1\vec{v}_{1F} + m_2\vec{v}_{2F} \tag{2.4}$$

This single vector equation gives us three equations relating the components of velocity before and after the collision.

IF the collision is elastic, we also have a fourth condition from conservation of total energy. The total energy before the collision is

$$E_I = T_I = \frac{1}{2}m_1\vec{v}_{1I}\cdot\vec{v}_{1I} + \frac{1}{2}m_2\vec{v}_{2I}\cdot\vec{v}_{2I} \tag{2.5}$$

If we define the speed e.g., of particle 1 before the collision as

$$v_{1I} = |\vec{v}_{1I}| \tag{2.6}$$

then

$$\vec{v}_{1I}\cdot\vec{v}_{1I} = v_{1I}^2 \tag{2.7}$$

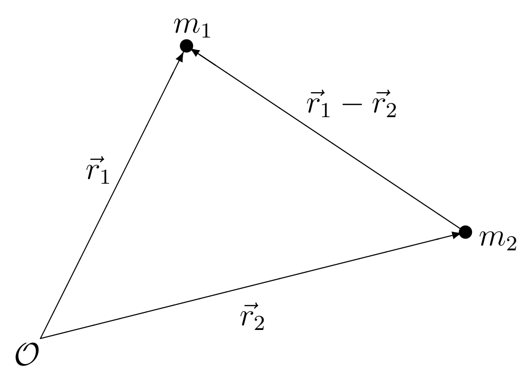
which makes the additional equation, valid for elastic collisions,

$$\frac{1}{2}m_1v_{1I}^2 + \frac{1}{2}m_2v_{2I}^2 = E_I = E_F = \frac{1}{2}m_1v_{1F}^2 + \frac{1}{2}m_2v_{2F}^2 \tag{2.8}$$

Note that if we know m_1, m_2, $\vec{v}_{1I}$, and $\vec{v}_{2I}$, this analysis is not enough to uniquely determine $\vec{v}_{1F}$ and $\vec{v}_{2F}$, even for an elastic collision. This is because $\vec{v}_{1F}$ and $\vec{v}_{2F}$ between them have $2\times 3 = 6$ components, but there are only $3+1=4$ equations. This is not surprising, since we did not specify all the details of the collision.

3 The Two-Body Problem (Symon Section 4.7)

Another useful application of the "systems of particles" formalism, again with $N=2$, is the two-body problem, where the motion of two particles is divided into overall motion of the center of mass and motion of the two particles relative to each other.



In the case $N=2$, each particle feels an internal force due to the other, which assume obeys (the weak form of) Newton's Second Law. Thus

$$\vec{F}_2^i = \vec{F}_{1\to2}^i = -\vec{F}_{2\to1}^i = -\vec{F}_1^i \tag{3.1}$$

so the equations of motion (from Newton's second law) are

$$m_1\ddot{\vec{r}}_1 = \vec{F}_1^e + \vec{F}_1^i \tag{3.2a}$$

$$m_2\ddot{\vec{r}}_2 = \vec{F}_2^e - \vec{F}_1^i \tag{3.2b}$$

Recalling the definitions of the total mass

$$M = m_1 + m_2 \tag{3.3}$$

and the center-of-mass vector

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M} \tag{3.4}$$

we see that adding (3.2a) and (3.2b) tells us something we already know, namely that

$$M\ddot{\vec{R}} = \vec{F}_1^e + \vec{F}_2^e = \vec{F} \tag{3.5}$$

We'd now like to look at the relative motion, as described by the time dependence of the relative position vector

$$\vec{r} := \vec{r}_1 - \vec{r}_2 \tag{3.6}$$

Multiplying (3.2a) by m_1 and (3.2b) by m_2 and subtracting gives us an equation of motion for $\vec{r}$:

$$m_1 m_2\ddot{\vec{r}} = m_2(m_1\ddot{\vec{r}}_1) - m_1(m_2\ddot{\vec{r}}_2) = m_2\vec{F}_1^e - m_1\vec{F}_2^e + (m_1 + m_2)\vec{F}_1^i \tag{3.7}$$

if we divide by $m_1 + m_2$ we get

$$\frac{m_1 m_2}{m_1 + m_2}\ddot{\vec{r}} = \vec{F}_1^i + \frac{m_1 m_2}{m_1 + m_2}\left(\frac{\vec{F}_1^e}{m_1} - \frac{\vec{F}_2^e}{m_2}\right) \tag{3.8}$$

The combination of masses that appears in two of the three terms in (3.8) is so useful that it has its own name: the *reduced mass*

$$\mu := \frac{m_1 m_2}{m_1 + m_2} \tag{3.9}$$

(Note that $m_1 m_2 = M\mu$.)

Now, let's specialize to cases where the last term in (3.8) vanishes, e.g., when there are no external forces *or* a constant external gravitational field, in which $\vec{F}_k^e = m_k\vec{g}$ with *constant* $\vec{g}$, so that $\frac{\vec{F}_1^e}{m_1} = \vec{g} = \frac{\vec{F}_2^e}{m_2}$. (Note also that if $\vec{F}_k^e = m_k\vec{g}$, $\vec{F} = \vec{F}_1^e + \vec{F}_2^e = m_1\vec{g} + m_w\vec{g} = M\vec{g}$.) In these cases,

$$\mu\ddot{\vec{r}} = \vec{F}_1^i \tag{3.10}$$

This is the equation of motion for a body of mass μ with position vector $\vec{r}$ moving under the influence of a force $\vec{F}_1^i$; for this reason this is often called the *effective one-body formalism* for the two-body problem.

This formalism is even more manifest in the forces between the particles can be derived from a potential $V(\vec{r}_1 - \vec{r}_2)$ which depends only on the relative position of the two particles:

$$\vec{F}_k^i = -\vec{\nabla}_k V(\vec{r}_1 - \vec{r}_2) \tag{3.11}$$

To be explicit,

$$V(\vec{r}) = V(x, y, z) = V(x_1 - x_2, y_1 - y_2, z_1 - z_2) \tag{3.12}$$

so

$$\frac{\partial V}{\partial x_1} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial x_1} = \frac{\partial V}{\partial x} \tag{3.13a}$$

$$\frac{\partial V}{\partial x_2} = \frac{\partial V}{\partial x}\underbrace{\frac{\partial x}{\partial x_2}}_{\frac{\partial(x_1-x_2)}{\partial x_2}=-1} = \frac{\partial V}{\partial x} \tag{3.13b}$$

and likewise for the y_k and z_k derivatives. This means that Newton's third law is satisfied:

$$\vec{\nabla}_2 V = \vec{\nabla}_1 V \tag{3.14}$$

but also

$$\vec{F}^i_1 = -\vec{\nabla}_1 V = -\vec{\nabla} V \tag{3.15}$$

So this really does look like a one-body problem:

$$\mu\ddot{\vec{r}} = -\vec{\nabla} V \tag{3.16}$$

The other piece of the overall motion is the motion of the center of mass, which we know in a couple of specific cases:

$$M\ddot{\vec{R}} = \begin{cases} \vec{0} & \text{if no external forces} \\ M\vec{g} & \text{if constant external gravitational field} \end{cases} \tag{3.17}$$

Finally, we return to the case of two pointlike bodies interacting under their mutual gravitational influences (which we looked at before in the approximation where the more massive body didn't move). The gravitational force on particle number one has a magnitude of

$$\left|\vec{F}^i_1\right| = \frac{Gm_1m_2}{\left|\vec{r}_1 - \vec{r}_2\right|^2} \tag{3.18}$$

(proportional to both masses and inversely proportional to the square of the distance between them). It is directed from particle one towards particle two, so it is parallel to the unit vector

$$-\frac{\vec{r}_1 - \vec{r}_2}{\left|\vec{r}_1 - \vec{r}_2\right|} \tag{3.19}$$

and thus

$$\left|\vec{F}^i_1\right| = -\frac{Gm_1m_2}{\left|\vec{r}_1 - \vec{r}_2\right|^2}\frac{\vec{r}_1 - \vec{r}_2}{\left|\vec{r}_1 - \vec{r}_2\right|} = -Gm_1m_2\frac{\vec{r}}{\left|\vec{r}\right|^3} \tag{3.20}$$

or, defining r to be the distance between the two bodies,

$$\left|\vec{F}^i_1\right| = -Gm_1m_2\frac{\vec{r}}{r^3}\ . \tag{3.21}$$

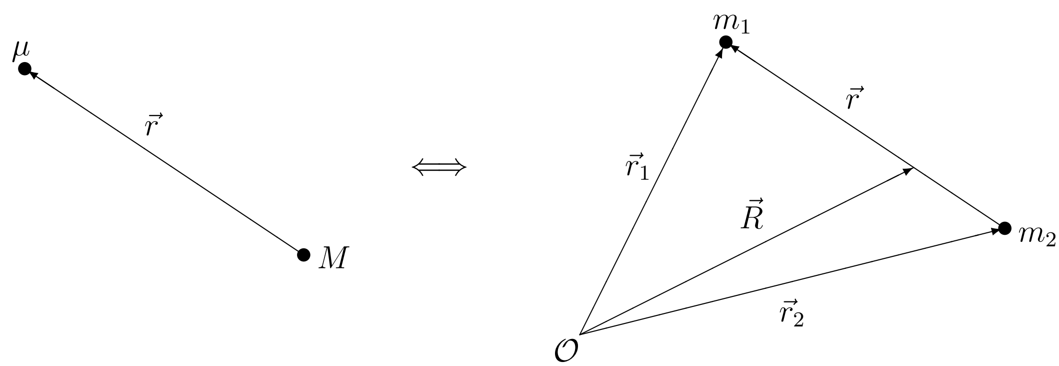
It's not too hard to show that this force is associating with a potential energy

$$V = -\frac{Gm_1m_2}{r} \tag{3.22}$$

Now, the nice thing is that, since $m_1 m_2 = M\mu$,

$$V = -\frac{GM\mu}{r} \tag{3.23}$$

and this two-body problem is completely equivalent to the one-body problem of a particle of mass μ moving in the gravitational field of a mass M fixed at the origin:



We can recover the two-body description of the motion (which corresponds, after all, to physical reality), by inverting

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \tag{3.24a}$$

$$M\vec{R} = m_1\vec{r}_1 - m_2\vec{r}_2 \tag{3.24b}$$

to find

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r} \tag{3.25a}$$

$$\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r} \tag{3.25b}$$

Note that if $m_2 \gg m_1$, then $m_2 \approx M$, $\mu \approx m_1$, $\vec{r}_1 \approx \vec{R} + \vec{r}$, $\vec{r}_2 \approx \vec{R}$, and the effective one-boty picture reduces approximately to the actual two-body picture. But the effective one-body solutions generate the *exact* two-body solution for any mass ratio via (3.25).

4 Further applications

The student is referred to Symon sections 4.7 and 4.8 for further applications of the formalism of systems of particles. We're going to skip the rest of the chapter (which is devoted to coupled oscillations) and jump to chapter five, specifically the introductory parts about rigid bodies and mass distributions.

A Appendix: Correspondence to Class Lectures

Date	Sections	Pages
2003 November 18	0–1.3	2–6
2003 November 20	1.4	6–11
2003 November 25	3–4	11–14