Rigid Bodies and Mass Distributions (Symon Chapter Five)

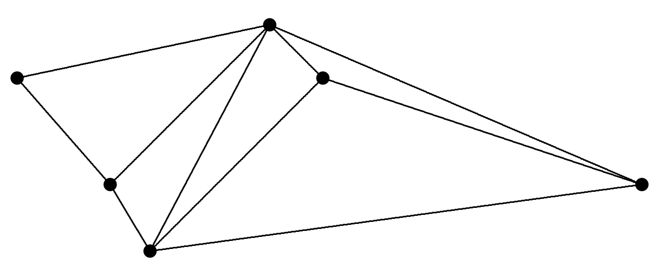
Physics A300*

Fall 2003

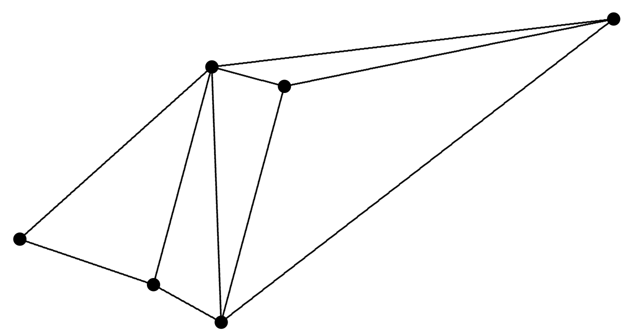
Contents

1 Definition of a Rigid Body

A rigid body is a system of particles which all move together, like a bunch of point particles connected by massless rigid rods:



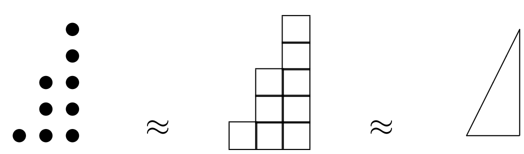
The rigidity is reflected by the fact that while each particle can move, and therefore change the position vectors, all the distances $|\vec{r}_k - \vec{r}_\ell|$ are constant. Only the orientation and overall location can change; there is no flexing or twisting:



Next semester, we will consider the general case; for now we will just consider rotation about a single axis.

2 Correspondence Between Collections of Point Masses and Continuous Mass Distributions

It is often useful to think about a continuous distribution of mass rather than a collection of discrete points. For example, one might want to know the properties of a slab of rock of a given density without having to count up all the individual atoms in the slab. We can still carry over our standard formulas by thinking of the mass distribution as a collection of infinitesimal little volume elements; the sum over particles is replaced by a sum over volume elements, which becomes a triple integral over the volume in the limit that the individual elements become small:



To make the correspondence, one replaces the particle of mass m_k at position $\vec{r}_k$ with a little block of volume d^3V at position $\vec{r}$. The solid has some density $\rho(\vec{r})$ at that position, in terms of which the mass of the little block, which we can call d^3M to emphasize its triply-infinitesimal nature, is[1]

$$d^3M = \rho(\vec{r})\, d^3V \tag{2.1}$$

To translate any of our expressions from chapter four into the realm of continuous mass distributions, we just need to replace sums over k with triple integrals, and the factor of

[1]Warning: Symon writes this relationship as $\rho = \frac{dM}{dV}$ which looks deceptively like a derivative, but doesn't make much sense if we try to interpret it as one.

m_k which always appears in those sums is replaced with the infinitesimal volume d^3M from (2.1). This is easiest to see from the example of the total mass and the center of mass vector:

Point Masses	Mass Distribution
$M = \sum_{k=1}^{N} m_k$	$M = \iiint \rho(\vec{r})\, d^3V$
$\vec{R} = \frac{\sum_{k=1}^{N} m_k \vec{r}_k}{M}$	$\vec{R} = \frac{\iiint \rho(\vec{r})\, \vec{r}\, d^3V}{M}$

2.1 Notational Aside

Introducing the traditional notation of ρ for density has put is in a quandry, since we were already using ρ as a cylindrical coördinate equal to the distance from the z axis. Oue notation has been (x, y, z) for Cartesian, (ρ, φ, z) for cylindrical and (r, θ, ϕ) for spherical coördinates. What should we use instead of ρ as our cylindrical radial coördinate? Most of the traditional choices have problems:[2]

- r (used by Symon) can cause confusion when comparing cylindrical and spherical coördinates; we'd always like r to be equal to the length of the position vector in three (or two) dimensions.
- R is no good, since we're already using it for the length of the center of mass vector: $R = |R|$
- r_2, as in the two-dimensional radial coördinate becomes confusing if there is a position vector $\vec{r}_2$ of the 2nd particle, where we'd expect $r_2 = |\vec{r}_2|$.
- s (used by Griffiths in the latest edition of *Introduction to Electrodynamics*) isn't quite ideal in our case, since we already use that as an arbitrary parameter when parametrizing curves.

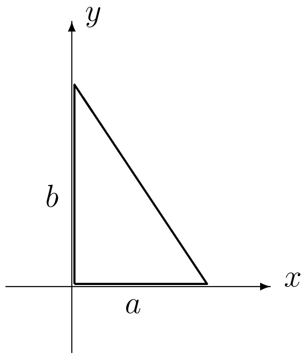
So we will henceforth use what is probably a new notation, q. The cylindrical coördinates of a point are thus now (q, ϕ, z),

2.2 Example: Mass of a Prism

We'll worry about motion later; for now let's look at an example of how to calculate the mass for a solid body, This body is a right triangular prism of height h, which for simplicity we take to have uniform density ρ. The triangular faces have legs of length a and b, and

[2] Another approach would be to use a different letter for density, but since the Greek letter rho is so universal there, we'd end up using something like ϱ, which would be hard to write.

look like so:



The prism is defined by

$$x \geq 0 \tag{2.2a}$$

$$y \geq 0 \tag{2.2b}$$

$$-\frac{h}{2} \leq z \leq \frac{h}{2} \tag{2.2c}$$

$$xb + ay \leq ab \tag{2.2d}$$

To calculate the total mass, we have to integrate $\rho\, d^3V$ over the entire prism:

$$M = \iiint\limits_{\text{prism}} \rho\, d^3V = \int_{-h/2}^{h/2} \iint\limits_{\text{triangle}} \rho\, \overbrace{dx\, dy\, dz}^{d^3V} = \rho\, h \underbrace{\iint\limits_{\text{triangle}} dx\, dy}_{\text{area}} \tag{2.3}$$

where the double integral over x and y is the area of the triangle.

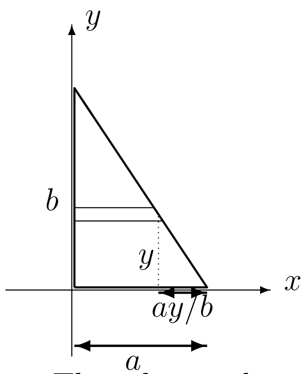
This integral has to cover every point in the triangle once. As a review of how one does area integrals (which extends naturally to volume integrals), we follow through in detail how one does the integral in two different ways, either one of which is correct, and both of which (naturally) give the right answer.

2.2.1 x integral inside y integral

Here we integrate over x first, then over y. We need to choose the limits of those two integrals to cover each point in the triangle once. Taking the integrals from the outside in, the range of y values included in the prism is

$$0 \leq y \leq b \tag{2.4}$$

Now, the x integral is inside the y integral, which means that the limits need to cover the possible x values only for *that* given y. We can look at the triangle to see that the lower limit of that integral is still $x = 0$, but the upper limit is cut off by the edge of the triangle at $x = a\left(1 - \frac{y}{b}\right)$:



That then makes the integral for the area of the triangle[3]

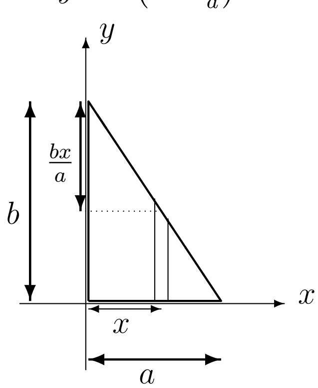
$$\begin{aligned} \text{Area} = \frac{M}{\rho h} &= \int_0^b \left(\int_0^{a\left(1-\frac{y}{b}\right)} dx \right) dy = \frac{M}{\rho h} = \int_0^b a\left(1 - \frac{y}{b}\right)\, dy \\ &= a\left(y - \frac{y^2}{2b}\right)\bigg|_0^b = a\left(b - \frac{b}{2}\right) = \frac{ab}{2} \end{aligned} \tag{2.5}$$

2.2.2 y integral inside x integral

Here we integrate over y first, then over x. We need to choose the limits of those two integrals to cover each point in the triangle once. Taking the integrals from the outside in, the range of x values included in the prism is

$$0 \le x \le a \tag{2.6}$$

Now, the y integral is inside the x integral, which means that the limits need to cover the possible y values only for *that* given x. We can look at the triangle to see that the lower limit of that integral is still $y = 0$, but the upper limit is cut off by the edge of the triangle at $y = b\left(1 - \frac{y}{a}\right)$:



[3]Note that the limit of the y integral does not depend on x, as it must not, since it's outside the x integral, while the limit of the x integral does depend on y, which it may, since it's inside the y integral.

That then makes the integral for the area of the triangle[4]

$$\begin{aligned}\text{Area} = \frac{M}{\rho h} &= \int_0^a \left(\int_0^{b\left(1-\frac{x}{a}\right)} dy \right) dx = \frac{M}{\rho h} = \int_0^a b\left(1 - \frac{x}{a}\right)\,dx \\ &= b\left(x - \frac{x^2}{2a}\right)\Big|_0^a = b\left(a - \frac{a}{2}\right) = \frac{ab}{2}\end{aligned} \tag{2.7}$$

which is the same answer as before

2.3 Excercise: Center-of-Mass Coördinates

As an exercise, you should do the corresponding integrals for the center of mass coördinates. These are the Cartesian compoments of the center-of-mass vector

$$\vec{R} = \frac{\iiint_{\text{prism}} \vec{r}\,d^3V}{M} = X\hat{x} + Y\hat{y} + Z\hat{z} \tag{2.8}$$

and the limits on the integrals over the prism are just the same ones we found when calculating the mass. This means, explicitly,

$$X = \frac{1}{M}\int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^b \int_0^{a\left(1-\frac{y}{b}\right)} \rho\,x\,dx\,dy\,dz \tag{2.9a}$$

$$Y = \frac{1}{M}\int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^b \int_0^{a\left(1-\frac{y}{b}\right)} \rho\,y\,dx\,dy\,dz \tag{2.9b}$$

$$Z = \frac{1}{M}\int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^b \int_0^{a\left(1-\frac{y}{b}\right)} \rho\,z\,dx\,dy\,dz \tag{2.9c}$$

$$\tag{2.9d}$$

Note that this only works because the Cartesian basis vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ are constants, and can be pulled out of the triple integrals.

3 Rotation about an Axis

A Appendix: Correspondence to Class Lectures

Date	Sections	Pages
2003 November 25	1–2.1	1–3
2003 December 2	2.2–3	3–6
2003 December 4	(Review)	

[4]Note that the limit of the x integral does not depend on y, as it must not, since it's outside the y integral, while the limit of the y integral does depend on x, which it may, since it's inside the x integral.