Physics A300: Classical Mechanics I

Problem Set 3
Corrected corrected version

Assigned 2002 September 11
Due 2002 September 18

Show your work on all problems!

1 Motion in a Potential

Throughout this problem, you should limit your attention to $x > 0$.

Consider the potential energy

$$V(x) = -\frac{a}{x^2} + \frac{b}{x^3}$$

where a and b are positive constants.

a) Sketch $V(x)$.

b) Find the equilibrium point(s) associated with this potential and state whether they're stable or unstable.

c) Find the frequency of small oscillations about any stable equilibrium points.

d) If a particle of mass m has an initial position $x_0 = 2b/a$ and initial velocity v_0, what is the energy of its trajectory?

e) What is the smallest velocity v_{esc} such that if a particle starts off at $x_0 = 2b/a$ and $v_0 > v_{\text{esc}}$, it will never turn around and move back towards the origin? (You should be able to determine this from the potential energy without calculating any forces.)

f) For a particle with initial $x_0 = 2b/a$ and $v_0 = -v_{\text{esc}}$, at what value of x will the velocity vanish? What will the force be at that point?

2 Angle Sum Formulas and the Euler Relation

Use the Euler relation $e^{i\theta} = \cos\theta + i\sin\theta$ to expand out

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$$

in terms of sines and cosines. By requiring the real and imaginary parts of the resulting complex equation to hold separately, derive the angle sum formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$.

3 Energy in the Simple Harmonic Oscillator

Consider a particle of mass m moving in the potential $V(x) = \frac{1}{2}m\omega_0^2 x^2$, whose trajectory is

$$x(t) = A\cos(\omega_0 t + \phi) \tag{3.1}$$

a) Using the explicit solution (3.1), find

   i) The potential energy $V(t)$ as a function of time in terms of A, ω_0, and ϕ.

   ii) The kinetic energy $T(t)$ as a function of time in terms of A, ω_0, and ϕ.

   iii) The total energy $E(t) = V(t) + T(t)$ as a function of time in terms of A, ω_0, and ϕ; show (by simplifying the resulting expression) that it is actually constant.

b) The notation $\langle\cdot\rangle$ indicates the time average of a quantity; since everything to do with the simple harmonic oscillator is periodic, it is reasonable to define this average over one period:

$$\langle f(t)\rangle = \frac{\omega_0}{2\pi}\int_{\phi/\omega_0}^{(\phi+2\pi)/\omega_0} f(t)\,dt$$

Calculate the time average of each of your results from part a) and show explicitly that $\langle V(t)\rangle + \langle T(t)\rangle = \langle E(t)\rangle$.