Physics A300: Classical Mechanics I

Problem Set 4

Assigned 2002 September 23
Due 2002 September 30

Show your work on all problems!

1 Overdamped Oscillations

By analogy to the formulas

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

which can be derived from the Euler relation, one defines the *hyperbolic trigonometric functions*

$$\sinh\eta = \frac{e^{\eta} - e^{-\eta}}{2} \qquad \cosh\eta = \frac{e^{\eta} + e^{-\eta}}{2}$$

a) Using the definitions above, *derive* (i.e., do not just look up) expressions for $\sin i\eta$ and $\cos i\eta$ in terms of $\sinh\eta$ and $\cosh\eta$.

b) Show that the general overdamped solution given in equation (2.140) of Symon can be rewritten as

$$x(t) = e^{-\gamma t}(B_c \cosh\omega_h t + B_s \sinh\omega_h t) \qquad (1)$$

where $\omega_h = \sqrt{\gamma^2 - \omega_0^2}$. Find the values of B_c, B_s in terms of C_1, and C_2.

c) By extending the definition (2.129) to the case where $\gamma > \omega_0$ as $\omega_1 = i\omega_h$ (where again $\omega_h = \sqrt{\gamma^2 - \omega_0^2}$) show that the general underdamped solution (2.134) is equivalent to (1) above and determine the resulting B_c and B_s in terms of B_1 and B_2. What conditions must B_1 and B_2 satisfy for $x(t)$ to be real in this case?

2 Critically Damped Oscillations

a) Consider a damped harmonic oscillator with $x(0) = x_0$ and $\dot{x}(0) = v_0$. Find the B_1 and B_2 needed to make the general solution (2.134) in Symon satisfy these initial conditions, and use this to write $x(t)$ as a function of only x_0, v_0, t, γ, and ω_1. Simplify your answer as much as possible.

b) Take the limit $\gamma \to \omega_0$, i.e., $\omega_1 \to 0$, of your result to the previous part, and thereby obtain an expression for $x(t)$ in terms of x_0, v_0, and γ in the critically-damped limit which does not contain ω_1. Your solution should agree with equation (2.146) of Symon for a suitable choice of C_1 and C_2.

3 Driven Oscillations

Consider the steady-state solution

$$x(t) = A_s \cos(\omega t + \theta_s)$$

to a forced, damped harmonic oscillator.

a) Calculate the work done on the oscillator *per unit time per unit mass* (because $W = \int F\,dx$, work per unit time–i.e., power–is Fv) by the three forces in the problem:

   i) $F_{\text{Hooke}} = -m\omega_0^2 x$

   ii) $F_{\text{damping}} = -2m\gamma\dot{x}$

   iii) $F_{\text{driving}} = F_0 \cos(\omega t + \theta_0)$

   This is the net power per unit mass deposited into the oscillator by each force.

b) Calculate the net work per unit mass done by each individual force over a complete cycle of the oscillator. Show *explicitly* that no net work is done by the restoring force F_{Hooke} and that the amount of energy dissipated by the retarding force F_{damping} is equal to the amount of work done by the driving force F_{driving}. Calculate the frequency ω_{max} at which this energy transfer per cycle is greatest, for a fixed driving amplitude F_0.

4 Initial Conditions and Transients

Symon Chapter Two, Problem 50