Physics A300: Classical Mechanics I

Problem Set 6

Assigned 2002 October 21
Due 2002 October 28

Show your work on all problems!

1 Symon Chapter Three, Problem 12

Hint:
One possible parametrization of the path is $\vec{r}(s) = sx_0\hat{x} + sy_0\hat{y} + sz_0\hat{z}$, where s runs from 0 to 1.

2 Spherical Coördinates

Consider the unit vectors

$$\hat{r} = \sin\theta\cos\phi\;\hat{x} + \sin\theta\sin\phi\;\hat{y} + \cos\theta\;\hat{z} \tag{2.1a}$$

$$\hat{\theta} = \cos\theta\cos\phi\;\hat{x} + \cos\theta\sin\phi\;\hat{y} - \sin\theta\;\hat{z} \tag{2.1b}$$

$$\hat{\phi} = -\sin\phi\;\hat{x} + \cos\phi\;\hat{y} \tag{2.1c}$$

a) Using the orthonormality of the Cartesian basis vectors [Symon's equation (3.20)], calculate explicitly the six independent inner products $\hat{r}\cdot\hat{r}$, $\hat{r}\cdot\hat{\theta}$, $\hat{r}\cdot\hat{\phi}$, $\hat{\theta}\cdot\hat{\theta}$, $\hat{\theta}\cdot\hat{\phi}$ and $\hat{\phi}\cdot\hat{\phi}$, and thereby show that the unit vectors defined in (2.1) are themselves an orthonormal basis.

b) Using the cross products of the Cartesian basis vectors [Symon's (3.31)], calculate $\hat{r}\times\hat{\theta}$, $\hat{\theta}\times\hat{\phi}$, and $\hat{\phi}\times\hat{r}$.

c) By differentiating the form (2.1), calculate the nine partial derivatives $\frac{\partial\hat{r}}{\partial r}$, $\frac{\partial\hat{r}}{\partial\theta}$, $\frac{\partial\hat{r}}{\partial\phi}$, $\frac{\partial\hat{\theta}}{\partial r}$, $\frac{\partial\hat{\theta}}{\partial\theta}$, $\frac{\partial\hat{\theta}}{\partial\phi}$, $\frac{\partial\hat{\phi}}{\partial r}$, $\frac{\partial\hat{\phi}}{\partial\theta}$ and $\frac{\partial\hat{\phi}}{\partial\phi}$. First express your results in terms of the Cartesian basis vectors (with components written in terms of the spherical coördinates r, θ, and ϕ). Then use your results along with (2.1) to verify Symon's Equation (3.99) for the derivatives written purely in terms of the spherical coördinates and the corresponding basis.

3 Projectile Motion with Air Resistance

Consider a projectile of mass m fired into the air with initial speed $v_0 > 0$, at an angle of α above the horizontal. Assume the only forces acting on the projectile are gravity and air resistance. Let the gravitational force correspond to a constant downward acceleration of magnitude $g > 0$. Assume the force due to air resistance is in the opposite direction to the projectile's instantaneous velocity, with a magnitude of $b > 0$ times the projectile's instantaneous speed. Define a Cartesian coördinate system with its origin at the point from which the projectile is fired, oriented such that the z axis points straight up and the projectile's initial motion is in the x-z plane.

a) List the five parameters (constants) of the problem and specify their units. (Express this in terms of abstract units rather than a particular system, i.e., "length" rather than "meters" and "velocity" rather than "furlongs per fortnight".)

b) Write a vector expression for the force $\vec{F}_g$ due to gravity in terms of the parameters of the problem and (some or all of) the unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$.

c) Write a vector expression for the force $\vec{F}_f$ due to air resistance in terms of the parameters of the problem, the trajectory $\vec{r}(t)$, and its time derivatives.

d) Rewrite your answer to part c) in terms of the parameters of the problem, the components $x(t)$, $y(t)$ and $z(t)$ of the trajectory (and their derivatives), and the Cartesian basis vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$.

e) Use the results of parts b) and d) to write a vector expression for the total force on the projectile.

f) Use Newton's second law to write the vector equation of motion for $\ddot{\vec{r}}(t)$ in terms of the parameters, the trajectory and its first derivative, and the Cartesian basis vectors.

g) Use the initial conditions of the problem to construct vector expressions for $\vec{r}(0)$ and $\dot{\vec{r}}(0)$ in terms of the parameters and the Cartesian basis vectors.

h) From the vector expressions in the previous two parts, read off the component equations of motion for $\ddot{x}(t)$, $\ddot{y}(t)$, $\ddot{z}(t)$, and the initial conditions on $x(0)$, $y(0)$, $z(0)$, $\dot{x}(0)$, $\dot{y}(0)$, and $\dot{z}(0)$.

i) By solving the differential equation for $\ddot{y}(0)$ and using the initial conditions $\dot{y}(0)$ and $y(0)$, find $y(t)$. (The equations for $x(t)$ and $z(t)$ are more complicated, and we leave them for another time.)