Physics A300: Classical Mechanics I

Problem Set 7

Assigned 2002 October 28
Due 2002 November 4

Show your work on all problems!

1 Projectile Motion with Air Resistance (conclusion)

Recall that in problem 3 on Problem Set 6, you found that $y(t) = 0$ and that $x(t)$ and $z(t)$ obeyed the equations of motion

$$\ddot{x}(t) = -\frac{b}{m}\dot{x}(t) \tag{1.1a}$$

$$\ddot{z}(t) = -g - \frac{b}{m}\dot{z}(t) \tag{1.1b}$$

with initial conditions

$$x(0) = 0 \tag{1.2a}$$

$$z(0) = 0 \tag{1.2b}$$

and

$$\dot{x}(0) = v_0 \cos\alpha \tag{1.3a}$$

$$\dot{z}(0) = v_0 \sin\alpha \tag{1.3b}$$

a) Solve the equations of motion (1.1) with initial conditions (1.3) to find $\dot{x}(t)$ and $\dot{z}(t)$ in terms of the time t, the coördinates $x(t)$ and $z(t)$, and the parameters of the problem.

b) Solve the equations you got in part a) subject to the initial conditions (1.2) to find $x(t)$ and $z(t)$ and write the solution $\vec{r}(t)$ in terms of the time, the parameters of the problem, and the basis vectors $\hat{x}$, $\hat{y}$, $\hat{z}$.

c) If T is the time when the projectile lands on level ground, write the equation which implicitly defines T in terms of the parameters of the problem.

d) The result of part c) is called a *transcendental equation* because T appears both within and outside a transcendental function. It cannot be solved exactly, but when the air resistance is small, we can find an approximate solution perturbatively. As a first step in this process, define the dimensionless quantities $\beta = bv_0/mg$ and $\mathcal{T} = gT/v_0$ and use these definitions to rewrite your implicit equation for T as an equation relating only the dimensionless quantities $\mathcal{T}$, β, and α.

2 The Curl

a) If $a(\vec{r})$ is a scalar field and $\vec{B}(\vec{r})$ is a vector field, show, by explicit evaluation of the left- and right-hand sides in Cartesian coördinates, that

$$\vec{\nabla} \times (a\vec{B}) = (\vec{\nabla}a) \times \vec{B} + a(\vec{\nabla} \times \vec{B}) \ . \tag{2.1}$$

b) Writing the "del operator" in spherical coördinates according to Symon's equation (3.124) allows us to write the curl of a vector as

$$\vec{\nabla} \times \vec{A} = \hat{r} \times \frac{\partial \vec{A}}{\partial r} + \frac{\hat{\theta}}{r} \times \frac{\partial \vec{A}}{\partial \theta} + \frac{\hat{\phi}}{r \sin\theta} \times \frac{\partial \vec{A}}{\partial \phi} \ . \tag{2.2}$$

Use this, along with Symon's equation (3.99), to calculate i) $\vec{\nabla} \times \hat{r}$; ii) $\vec{\nabla} \times \hat{\theta}$; iii) $\vec{\nabla} \times \hat{\phi}$.

c) Using the results of parts a) and b), and writing a vector field $\vec{A}(\vec{r})$ as

$$\vec{A}(\vec{r}) = A_r(r,\theta,\phi)\ \hat{r} + A_\theta(r,\theta,\phi)\ \hat{\theta} + A_\phi(r,\theta,\phi)\ \hat{\phi} \tag{2.3}$$

show that the curl in spherical coördinates is

$$\begin{aligned} \vec{\nabla} \times \vec{A} = &\left(\frac{1}{r}\partial_\theta A_\phi - \frac{1}{r\sin\theta}\partial_\phi A_\theta + \frac{\cos\theta}{r\sin\theta}A_\phi\right)\ \hat{r} + \left(\frac{1}{r\sin\theta}\partial_\phi A_r - \partial_r A_\phi - \frac{1}{r}A_\phi\right)\ \hat{\theta} \\ &+ \left(\partial_r A_\theta - \frac{1}{r}\partial_\theta A_r + \frac{1}{r}A_\theta\right)\ \hat{\phi} \end{aligned} \tag{2.4}$$

3 Force, Potential and Torque

Consider the force field

$$\vec{F}(\vec{r}) = V_0\ \frac{x\,\hat{x} + y\,\hat{y}}{x^2 + y^2} \tag{3.1}$$

a) By explicitly calculating the (three-dimensional) curl $\vec{\nabla} \times \vec{F}$, verify that this is a conservative force.

b) Invert Symon's equation (3.89) to obtain expressions for $\hat{x}$, $\hat{y}$ and $\hat{z}$ in terms of the cylindrical coördinates ρ, ϕ and z and the basis vectors $\hat{\rho}$, $\hat{\phi}$, and $\hat{z}$. Simplify your answer as much as possible.

c) Use Symon's equation (3.87) and the results of part b) to write $\vec{F}$ above entirely in terms of the cylindrical coördinates ρ, ϕ and z and the basis vectors $\hat{\rho}$, $\hat{\phi}$, and $\hat{z}$ (and the constant V_0). Simplify your answer as much as possible.

d) Working in cylindrical coördinates, find the potential energy $V(\rho,\phi,z)$ such that $\vec{F} = -\vec{\nabla}V$. Include in your result an arbitrary constant (so that you capture the entire family of possible potentials) and indicate its units.

e) Calculate the vector torque $\vec{N}$ due to this force (in either Cartesian or cylindrical coördinates), and verify that the torque about the z axis vanishes.