Physics A300: Classical Mechanics I

Problem Set 8

Assigned 2002 November 11
Due 2002 November 18

1 Central Force with Quadratic Potential

Consider a potential $V(r) = \frac{1}{2}kr^2$.

a) For a particle of mass m moving in this potential, with angular momentum L, construct the effective potential $V_{\text{eff}}(r)$ and sketch a plot of $V_{\text{eff}}(r)$ versus r.

b) For what values of total energy are there two turning points r_{min} and r_{max}? Find r_{min} and r_{max} in terms of the energy E.

c) Use the function $V_{\text{eff}}(r)$ to find the radius r_{circ} of a circular orbit with angular momentum L. What is the total energy E_{circ} of this orbit?

d) For an energy only slightly larger than E_{circ}, calculate the frequency ω_R of the small radial oscillations about r_{circ}. Calculate the angular frequency ω_Φ of the angular oscillations when $r \approx r_{\text{circ}}$ and compare the two frequencies *quantitatively*. (Both frequencies should be expressed in terms of the parameters k, m, and L, and not in terms of e.g., r_{circ} or E_{circ}.)

2 Conic Sections

Demonstrate that the orbit

$$r(1 + \varepsilon \cos\phi) = \alpha \tag{2.1}$$

with constants $\alpha > 0$ and $\varepsilon \geq 0$ is indeed a conic section with eccentricity ε, semimajor axis $\alpha/(1-\varepsilon^2)$, and one focus at $r = 0$ as follows:

a) Consider the points $\mathcal{P} \equiv (x, y)$, $\mathcal{O} \equiv (0, 0)$, $\mathcal{F}_\pm \equiv (\pm 2c, 0)$, (where $c > 0$) and the line $\mathcal{L} \equiv x = 2p > 0$. Calculate the following distances in Cartesian coördinates, then convert your results into the standard polar coördinates using $x = r\cos\phi$ and $y = r\sin\phi$, simplifying as much as possible.

   i) the length $d_{\mathcal{OP}}$ of the straight line segment from $\mathcal{O}$ to $\mathcal{P}$
   ii) the length $d_{\mathcal{F}_\pm\mathcal{P}}$ of the straight line segment from $\mathcal{F}_\pm$ to $\mathcal{P}$
   iii) the distance $d_{\mathcal{LP}}$ between the point $\mathcal{P}$ and the line $\mathcal{L}$

b) A circle of radius a centered at $\mathcal{O}$ is the set of all points a distance a from $\mathcal{O}$:

$$d_{\mathcal{OP}} = a \tag{2.2}$$

Show that when $\varepsilon = 0$, (2.1) is equivalent to (2.2) for a suitable choice of a, and find this a in terms of α.

c) An ellipse of semimajor axis $a > 0$ with foci at $\mathcal{F}_-$ and $\mathcal{O}$ is the set of all points such that the sum of their distances from the two foci is $2a$:

$$d_{\mathcal{F}_-\mathcal{P}} + d_{\mathcal{O}\mathcal{P}} = 2a \tag{2.3}$$

Show that when $0 < \varepsilon < 1$, (2.1) is equivalent to (2.3) for a suitable choice of a and c, and find these values in terms of α and ε. (Hint: this is easiest if you solve (2.3) for $d_{\mathcal{F}_-\mathcal{P}}$, square it, and set it equal to the square of the result from part a), using (2.1) to eliminate $\cos\phi$, and requiring equality for any value of r.)

d) A parabola with focus $\mathcal{O}$ and directrix $\mathcal{L}$ is the set of all points equidistant from $\mathcal{O}$ and $\mathcal{L}$:

$$d_{\mathcal{L}\mathcal{P}} = d_{\mathcal{O}\mathcal{P}} \tag{2.4}$$

Show that when $\varepsilon = 1$, (2.1) is equivalent to (2.4) for a suitable choice of p, and find this p in terms of α.

e) The left branch of a hyperbola of semimajor axis $a < 0$ with foci at $\mathcal{O}$ and $\mathcal{F}_+$ is the set of all points such that the difference of their distances from the two foci is $-2a > 0$:

$$d_{\mathcal{F}_+\mathcal{P}} - d_{\mathcal{O}\mathcal{P}} = -2a \tag{2.5}$$

Show that when $\varepsilon > 1$, (2.1) is equivalent to (2.5) for a suitable choice of a and c, and find these values in terms of α and ε. (Hint: this is easiest if you solve (2.5) for $d_{\mathcal{F}_+\mathcal{P}}$, square it, and set it equal to the square of the result from part a), using (2.1) to eliminate $\cos\phi$, and requiring equality for any value of r.)

3 Circular Orbits in a Gravitational Field

Note: None of your answers to this problem should involve the constant K; you should use the relationship $K = -GMm$ to express them in terms of the masses of the attracting body and the test particle.

Consider a test particle of mass m moving in a circular orbit of radius R under the gravitational attraction of a body of mass M fixed at the center of the circle.

a) Use Kepler's third law to calculate the orbital speed v as a function of R.

b) Express the total energy E and angular momentum L as functions of the radius R of the orbit (and not of each other or v).

c) Use the result of part a) to find the kinetic energy K as a function of R.

d) Write the potential energy $V(R)$ and verify that $T + V = E$.

e) Suppose we reduce the orbital energy from a satellite in such a way that it changes from one circular orbit to another. Do the following quantities increase or decrease?

i) orbital radius; ii) orbital speed; iii) orbital period

iv) kinetic energy; v) potential energy; vi) orbital angular momentum