Physics A300: Classical Mechanics I

Problem Set 10

Assigned 2002 November 25
Due 2002 December 4

1 Internal and External Potential Gravitational Forces

The gravitational potential energy of two point masses m_1 and m_2 moving in the external gravitational field of a point mass m_0 fixed at the origin is

$$V(\vec{r}_1, \vec{r}_2) = \underbrace{-\frac{Gm_0m_1}{r_1}}_{V_1^e(\vec{r}_1)} + \underbrace{-\frac{Gm_0m_2}{r_2}}_{V_2^e(\vec{r}_2)} + \underbrace{-\frac{Gm_1m_2}{r}}_{V^i(\vec{r}_1,\vec{r}_2)} \tag{1.1}$$

where

$$r_1 = |\vec{r}_1| = \sqrt{x_1^2 + y_1^2 + z_1^2} \tag{1.2a}$$

$$r_2 = |\vec{r}_2| = \sqrt{x_2^2 + y_2^2 + z_2^2} \tag{1.2b}$$

$$r = |\vec{r}_1 - \vec{r}_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \tag{1.2c}$$

a) Make a sketch of the locations of the three masses and the vectors $\vec{r}_1$, $\vec{r}_2$, and $\vec{r} = \vec{r}_1 - \vec{r}_2$. Indicate the distances r_1, r_2, r.

b) Calculate the gradients $\vec{\nabla}_1 r_1$, $\vec{\nabla}_1 r_2$, $\vec{\nabla}_1 r$, $\vec{\nabla}_2 r_1$, $\vec{\nabla}_2 r_2$, and $\vec{\nabla}_2 r$. Express your answers both in Cartesian coördinates and then in terms of the vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}$ and the magnitudes r_1, r_2, r.

c) Find the internal forces $\vec{F}_1^i = -\vec{\nabla}_1 V^i$ and $\vec{F}_2^i = -\vec{\nabla}_2 V^i$ and verify that the strong form of Newton's third law holds, i.e., that the vectors $\vec{F}_1^i$ and $\vec{F}_2^i$ are equal in magnitude, opposite in direction, and directed along the line connecting the locations of masses 1 and 2.

d) Find the external forces $\vec{F}_1^e = -\vec{\nabla}_1 V_1^e$ and $\vec{F}_2^e = -\vec{\nabla}_2 V_2^e$.

e) Using the formalism of the two-body problem, find an exact expression for $M\ddot{\vec{R}}$ in terms of m_0, m_1, m_2, $\vec{r}_1$, and $\vec{r}_2$, and their magnitudes. (Here $M = m_1 + m_2$ is the total mass and $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/M$ is the center of mass vector of the two freely-moving particles.)

f) Using the formalism of the two-body problem, find an exact expression for $\mu\ddot{\vec{r}}$, where $\mu = m_1m_2/M$ is the reduced mass of the two freely-moving particles. Note that Symon's equation (4.96) does *not* hold, and you will need to retain a term

$$\vec{F}^t(\vec{r}_1, \vec{r}_2) = \mu\left(\frac{\vec{F}_1^e}{m_1} - \frac{\vec{F}_2^e}{m_2}\right) \tag{1.3}$$

Evaluate $\vec{F}^t$ explicitly in terms of $\vec{r}_1$ and $\vec{r}_2$.

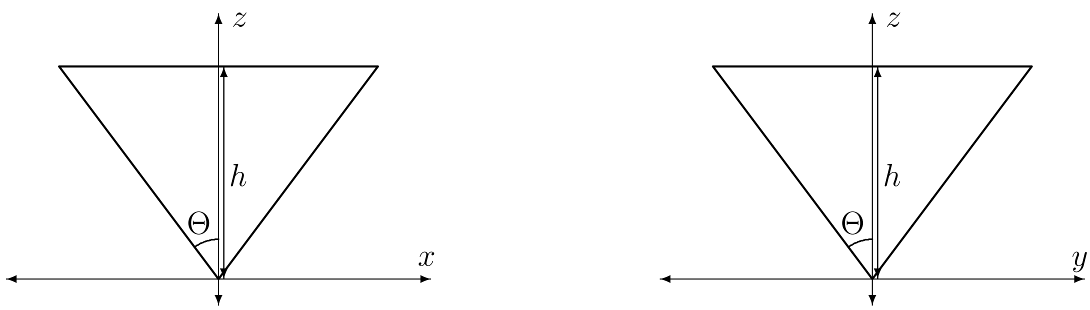
g) (bonus) In the limit that particles 1 and 2 are much closer to each other than they are to the origin, $R = \left|\vec{R}\right| \gg r$, things simplify somewhat. Using Symon's (4.92–4.93), expand your result for $M\ddot{\vec{R}}$ to zeroth order and your result for $\vec{F}^t$ to first order in the small parameter $\xi = r/R$. Explicitly, this means

   i) In your expression for $M\ddot{\vec{R}}$, just replace $\vec{r}_1$ and $\vec{r}_2$ with $\vec{R}$ (and similarly for their magnitudes) and you should get an answer which depends only on the properties of the 1-2 system as a whole (M and $\vec{R}$). What does this correspond to physically?

   ii) In your expression for $\vec{F}^t$, you'll need to substitute in Symon's (4.92–4.93) with $\vec{r} = \xi R\hat{r}$ and $r = \xi R$ into the results of part f) and then Taylor expand the result and keep the terms linear in ξ. Then you should be able to replace ξ with r/R and end up with an expression which depends on M, μ, $\vec{r}$, and $\vec{R}$. This describes the effects of the tidal field of the mass m_0 on the two-body system of masses 1 and 2.

2 Properties of a Cone

Consider a cone with opening angle $\Theta < \pi/2$ and height h with its apex at the origin and its axis of symmetry along the positive z axis, the $y = 0$ and $x = 0$ cross-sections of which are shown below:



2.1 Integrals in Various Coördinate Systems

a) Construct but *do not evaluate* triple integrals for the volume of this cone in Cartesian coördinates as follows:

   i) For what range of z values is *some* of the cone present?

   ii) Now consider a given z within that range. The constant-z cross-section of the cone is a circle. What is the radius of that circle as a function of z?

   iii) What is the condition on y such that *some* of the circle lies at that y value for a given z?

   iv) Given values of z and y, what is the range of x such that (x, y, z) lies within the circle?

   v) Using the results of previous parts, write but *do not evaluate* a triple integral for the volume of the cone in Cartesian coördinates, with explicit limits on all three of the integrals.

b) Construct but *do not evaluate* triple integrals for the volume of this cone in cylindrical coördinates as follows:

   i) In part a) you found the range of z values for which *some* of the cone lies at that z, and the radius of the corresponding constant-z circular cross-section. What is the condition on $q = \sqrt{x^2 + y^2}$ such that the point (q, ϕ, z) lies inside the circle for a given z? Is any restriction on ϕ involved?

ii) Using the results of previous parts, write but *do not evaluate* a triple integral for the volume of the cone in cylindrical coördinates, with explicit limits on all three of the integrals. Be sure to use the appropriate cylindrical coördinate volume element.

c) Construct but *do not evaluate* triple integrals for the volume of this cone in spherical coördinates as follows:

i) What is the range of ϕ values for which part of the cone is present? Does being at a particular value of ϕ restrict the range of r and θ values included in the cone?

ii) What is the range of θ values for which pare of the cone is present?

iii) Using the range of z values included in the cone and writing $z = r\cos\theta$, give the range of r values for which the point (r, θ, ϕ) lies inside the cone for a given θ and ϕ.

iv) Using the results of previous parts, write but *do not evaluate* a triple integral for the volume of the cone in spherical coördinates, with explicit limits on all three of the integrals. Be sure to use the appropriate spherical coördinate volume element.

2.2 Properties of the Mass Distribution

Assume the cone has a constant density ρ, and perform integrals in cylindrical coördinates.

a) Find the total mass M of the cone.

b) If we convert the components of the position vector $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ into cylindrical coördinates, but keep the constant Cartesian basis vectors, we can write

$$\vec{r} = q\cos\phi\,\hat{x} + q\sin\phi\,\hat{y} + z\,\hat{z} \tag{2.1}$$

Use this form to find the center of mass position vector $\vec{R}$ by performing a triple integral in cylindrical coördinates. Note that your answer should be expressed in terms of the Cartesian basis vectors (or equivalently consist of three expressions for the X, Y, and Z appearing in $\vec{R} = X\hat{x} + Y\hat{y} + Z\hat{z}$) and should not contain any q, ϕ, or z (which you will already have integrated over).

c) Assume the cone is rotating about the z axis with uniform angular velocity ω. Write the velocity vector $\vec{v}(q, \phi, z)$ of a piece of the cone located at cylindrical coördinates (q, ϕ, z). Again, the vector should be resolved into x, y, and z components, but the components should be functions of q, ϕ, and z, i.e., it should be in the form

$$\vec{v}(q, \phi, z) = v_x(q, \phi, z)\,\hat{x} + v_y(q, \phi, z)\,\hat{y} + v_z(q, \phi, z)\,\hat{z} \tag{2.2}$$

d) Using the form (2.1) of $\vec{r}$ and the form of $\vec{v}$ found in part c), calculate the total angular momentum

$$\vec{L} = \iiint \rho\,(\vec{r} \times \vec{v})\,d^3V \tag{2.3}$$

of the rotating cone, performing the integrals in cylindrical coördinates

e) Calculate the moment of inertia

$$I = \iiint \rho\,q^2\,d^3V \tag{2.4}$$

about the z axis (again by performing the triple integral in cylindrical coördinates), and verify that

$$\vec{L} = I\omega\hat{z} \tag{2.5}$$