# Taylor Series Demystified

Physics A300*

Fall 2003

Generation after generation of Physics majors, looking to put the mystery of Taylor series behind them, gets the bad news that 80% of Physics is actually based on Taylor expansions. I maintain that the thing that makes us all initially uncomfortable with Taylor series is the typical presentation, which produces the series by fiat and then proceeds to prove a bunch of properties about it. A lot of the mystery goes away if you start by stating the desired properties of a series approximation to a function and then derive the form it has to have.

The eventual goal will be to find an arbitrary long polynomial in $x$ which is an arbitrarily good approximation to some function $f(x)$:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

what you have at your disposal is the behavior of $f(x)$ at $x = 0$, in the form of its derivatives: $f(0)$ is its value at that point, $f'(0)$ is its slope, $f''(0)$ is its curvature, etc.

If we're going to use those derivatives to construct an approximation to our function, we'd better make sure the derivatives of the approximation match the derivatives of the function. So, to pick $a_0$ we require

$$f(0) = a_0 + a_1 0 + a_2 0^2 + a_3 0^3 + \ldots = a_0 \tag{2}$$

i.e.,

$$a_0 = f(0) \tag{3}$$

To match the first derivatives, we note that the derivative of the expansion tells us

$$f'(x) = a_1 + a_2(2x) + a_3(3x^2) + a_4(4x^3) + \ldots \tag{4}$$

Requiring this to hold for $x = 0$ tells us that

$$f'(0) = a_1 + a_2(2 \cdot 0) + a_3(3 \cdot 0^2) + a_4(4 \cdot 0^3) + \ldots = a_1 \tag{5}$$

i.e.,

$$a_1 = f'(0) \tag{6}$$

To match the second derivatives, we take the derivative of (4) and find

$$f''(x) = a_2(2) + a_3(3 \cdot 2x) + a_4(4 \cdot 3x^2) + a_5(5 \cdot 4x^3) + \dots \tag{7}$$

Requiring this to hold for $x = 0$ tells us that

$$f''(0) = a_2(2) + a_3(3 \cdot 2 \cdot 0) + a_4(4 \cdot 3 \cdot 0^2) + a_5(5 \cdot 4 \cdot 30^3) + \dots = 2a_2 \tag{8}$$

i.e.,

$$a_2 = \frac{f''(0)}{2} \tag{9}$$

One more derivative should be enough to make the pattern clear:

$$f'''(x) = a_3(3 \cdot 2) + a_4(4 \cdot 3 \cdot 2x) + a_5(5 \cdot 4 \cdot 3x^2) + a_6(6 \cdot 5 \cdot 4x^3) + \dots \tag{10}$$

Requiring this to hold for $x = 0$ tells us that

$$f'''(0) = a_3(3 \cdot 2) + a_4(4 \cdot 3 \cdot 2 \cdot 0) + a_5(5 \cdot 4 \cdot 3 \cdot 0^2) + a_6(6 \cdot 5 \cdot 4 \cdot 0^3) + \dots = (3 \cdot 2)a_3 \tag{11}$$

i.e.,

$$a_3 = \frac{f'''(0)}{3 \cdot 2} \tag{12}$$

So the $n$th derivative brings down another factor of $n$ and in general

$$f^n(0) = a_n \left. \frac{d^n}{dx^n} x^n \right|_{x=0} = a_n \cdot n \cdot (n-1) \dots \cdot 2 \tag{13}$$

or

$$a_n = \frac{f^{(n)}(0)}{n(n-1) \dots 2 \cdot 1} = \frac{f^{(n)}(0)}{n!} \tag{14}$$

where

$$n! = 1 \cdot 2 \cdot \dots \cdot (n-1) \cdot n \tag{15}$$

is the factorial of $n$.

This means that the infinite series expansion of a function $f(x)$, if it exists (which is the hard part) is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \tag{16}$$