

# Gravitation

## (Symon Chapter Six)

Physics A301\*

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# 1 The Gravitational Force

## 1.1 Force Between Two Point Masses

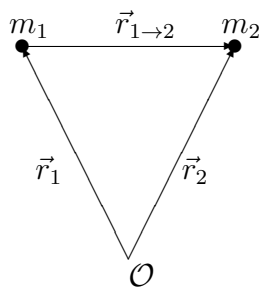
We've already introduced the gravitational force between two objects small enough to be idealized as point masses, which is:

1. Proportional to the product of the masses
2. Inversely proportional to the square of the distance between them
3. Attractive and directed on a line from one mass to the other

To represent this mathematically in a vector equation, we consider the gravitational force on a mass  $m_2$  with position vector  $r_2$  due to a mass  $m_1$  with position vector  $r_1$ . The key geometrical quantity is the vector

$$\vec{r}_{1 \rightarrow 2} = \vec{r}_2 - \vec{r}_1 \quad (1.1)$$

which points from one to the other:



The distance between the two masses is

$$r_{1 \rightarrow 2} = |\vec{r}_{1 \rightarrow 2}| \quad (1.2)$$

and the unit vector pointing from mass 1 to mass 2 is

$$\hat{r}_{1 \rightarrow 2} = \frac{\vec{r}_{1 \rightarrow 2}}{r_{1 \rightarrow 2}} \quad (1.3)$$

We then write the force on mass 2 due to mass 1 as

$$\vec{F}_{1 \rightarrow 2} = -G \frac{m_1 m_2}{r_{1 \rightarrow 2}^2} \hat{r}_{1 \rightarrow 2} \quad (1.4)$$

Some important features of this force:

1. It satisfies Newton's third law

$$\vec{F}_{2 \rightarrow 1} = -G \frac{m_2 m_1}{r_{2 \rightarrow 1}^2} \hat{r}_{2 \rightarrow 1} = G \frac{m_1 m_2}{r_{1 \rightarrow 2}^2} \hat{r}_{1 \rightarrow 2} = -\vec{F}_{1 \rightarrow 2} \quad (1.5)$$

2. Because the gravitational “charge” is just the same as the inertial mass, the acceleration experienced by particle 2 is

$$\vec{a}_2 = \frac{\vec{F}_{2 \rightarrow 1}}{m_2} = -G \frac{m_1}{r_{1 \rightarrow 2}^2} \hat{r}_{1 \rightarrow 2} \quad (1.6)$$

which depends on particle 2’s location but not any properties of the particle itself. This is called the *Equivalence Principle* and was instrumental in the development of Einstein’s General Theory of Relativity, which describes gravity in terms of the geometry of spacetime.

3. The coupling constant has been numerically determined as

$$G = 6.673 \times 10^{-11} \frac{\text{m}^3}{\text{kg s}^2} \quad (1.7)$$

We can check the units on this by considering two one-kilogram masses located one meter away from each other:

$$\left| \vec{F} \right| = 6.673 \times 10^{-11} \frac{\text{m}^3}{\text{kg s}^2} \frac{1 \text{ kg} \times 1 \text{ kg}}{(1 \text{ m})^2} = 6.673 \times 10^{-11} \frac{\text{kg m}}{\text{s}^2} = 6.673 \times 10^{-11} \text{ N} \quad (1.8)$$

We see not only that Newton’s constant  $G$  has the right units, but that the gravitational force between everyday objects is very small. This is why it is still difficult to determine Newton’s constant to the same accuracy as other constants of nature. Most things which are big enough to exert an appreciable gravitational force (like moons, planets, and stars) are too big to have their masses directly compared to everyday objects whose masses we know in kilograms. So we know the combination  $GM$  pretty well for things like the Earth and the Sun, from e.g., Kepler’s third law, but to get an independent measure of  $G$  (and hence of the masses of planet-sized things) required sophisticated experiments to measure the gravitational forces exerted by laboratory-sized objects.

While the form (1.4) emphasizes that the magnitude of the gravitational force is inversely proportional to the *square* of the distance between the masses, for practical calculations, it’s more useful to replace  $\hat{r}_{1 \rightarrow 2}$  using (1.3) and say

$$\vec{F}_{1 \rightarrow 2} = -Gm_1m_2 \frac{\vec{r}_{1 \rightarrow 2}}{r_{1 \rightarrow 2}^3} = -Gm_1m_2 \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} \quad (1.9)$$

which is the form of Symon’s equation (6.3).<sup>1</sup> The extra factor of  $r_{1 \rightarrow 2}$  in the denominator serves to cancel out the factor of distance in the vector  $\vec{r}_{1 \rightarrow 2}$ .

For the sake of completeness, we spell out the form of the gravitational interaction in terms of the  $x$ ,  $y$ , and  $z$  coordinates of the masses involved. The position vectors are

$$\vec{r}_1 = x_1 \hat{x} + y_1 \hat{y} + z_1 \hat{z} \quad (1.10a)$$

$$\vec{r}_2 = x_2 \hat{x} + y_2 \hat{y} + z_2 \hat{z} \quad (1.10b)$$

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<sup>1</sup>Symon puts the minus sign in a different place by talking about  $\vec{r}_1 - \vec{r}_2$ , but the two expressions are equivalent.

which makes the displacement vector from object 1 to object 2

$$\vec{r}_{1 \rightarrow 2} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\hat{x} + (y_2 - y_1)\hat{y} + (z_2 - z_1)\hat{z} \quad (1.11)$$

The distance  $r_{1 \rightarrow 2}$  between the objects is the length of this vector

$$r_{1 \rightarrow 2} = |\vec{r}_{1 \rightarrow 2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad (1.12)$$

So we can write the gravitational force in gory detail as

$$\vec{F}_{1 \rightarrow 2} = -Gm_1m_2 \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} = -Gm_1m_2 \frac{(x_2 - x_1)\hat{x} + (y_2 - y_1)\hat{y} + (z_2 - z_1)\hat{z}}{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{3/2}} \quad (1.13)$$

## 1.2 Force Due to a Distribution of Masses

Because Newtonian gravity obeys the principle of superposition, we can build up the gravitational force on a point mass due to a collection of other point masses, or due to a continuous mass distribution, by adding up the effects due to all of the individual source masses.

We're interested in the gravitational force *on* a point mass, located at a location often referred to as the *field point*. When the source of the gravitational force was also a point mass, we called the mass at the field point  $m_2$  and its position vector  $\vec{r}_2$ . Now we refer to it simply as  $m$ , and its position vector as  $\vec{r}$ . Instead of a single source mass  $m_1$  at a location  $\vec{r}_1$ , we now have a set of  $N$  sources of masses  $\{m_i | i = 1 \dots N\}$  located at positions  $\{\vec{r}_i | i = 1 \dots N\}$ , respectively. The force on our mass  $m$  due to the  $i$ th source mass is, by analogy with (1.9),

$$\vec{F}_{\text{due to } i} = -Gmm_i \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3} \quad (1.14)$$

which makes the total force

$$\vec{F} = \sum_{i=1}^N \vec{F}_{\text{due to } i} = - \sum_{i=1}^N Gmm_i \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3} \quad (1.15)$$

which is equivalent to Symon's equation (6.6). As we did when calculating centers of mass in Chapter 5, we can replace the sum over point masses with an integral over a mass distribution of density  $\rho(\vec{r}')$ , with the infinitesimal mass  $\rho(\vec{r}')d^3V'$ :

$$\vec{F} = - \iiint Gm\rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} d^3V' \quad (1.16)$$

The integral is over the volume occupied by the mass distribution, although as Symon points out, it can be thought of as an integral over all space if we define  $\rho(\vec{r}') = 0$  for points  $\vec{r}'$  outside the mass distribution.

Note that there are two position vectors in this expression: the source point

$$\vec{r}' = \hat{x}x' + \hat{y}y' + \hat{z}z' \quad (1.17)$$

which is integrated over (in Cartesian coordinates,  $d^3V' = dx'dy'dz'$ ), and the field point

$$\vec{r} = \hat{x}x + \hat{y}y + \hat{z}z \quad (1.18)$$

which is not. When evaluating this expression for particular mass distributions, one should note that the answer cannot depend on the “primed” coordinates  $x'$ ,  $y'$ , and  $z'$ , but only on the unprimed ones  $x$ ,  $y$ , and  $z$ . For a given mass distribution, we can often find the force which would be experienced by a point particle at an arbitrary position  $\vec{r}$  and consider this to be a function of  $\vec{r}$  and the mass  $m$  of the particle:

$$\vec{F}(\vec{r}) = \begin{cases} -\sum_{i=1}^N Gmm_i \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3} & \text{collection of point masses} \\ -\iiint Gm\rho(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} d^3V' & \text{continuous mass distribution} \end{cases} \quad (1.19)$$

This is a force field, just like we considered last semester.

## 2 Gravitational Field and Potential

### 2.1 Gravitational Field

As we noted in the case of a single point source, the acceleration experienced by a point particle due to gravitational forces is actually independent of the particle’s mass.<sup>2</sup> It is useful to think of acceleration, dependent on the position but not the mass of the particle in question, as a vector field, which we call the *gravitational field*

$$\vec{g}(\vec{r}) = \frac{\vec{F}(\vec{r})}{m} = \begin{cases} -\sum_{i=1}^N Gm_i \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3} & \text{collection of point masses} \\ -\iiint G\rho(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} d^3V' & \text{continuous mass distribution} \end{cases} \quad (2.1)$$

### 2.2 Gravitational Potential

The force field  $\vec{F}(\vec{r})$  defined by (1.19) is conservative, as one could demonstrate by explicitly calculating the curl  $\vec{\nabla} \times \vec{F}$  in Cartesian coordinates. However, it’s easier just to write down a potential energy  $V(\vec{r})$  which satisfies

$$\vec{F} = -\vec{\nabla}V \quad (2.2)$$

In fact, we’ve already worked out the gravitational potential energy associated with a point source (see our study of central force motion, in particular Symon’s equation (3.229), or problem 1 on Problem Set 10 from last semester). In the language of Section 1.1 of these notes, it’s

$$V(\vec{r}_2) = -\frac{Gm_1m_2}{r_{1\rightarrow 2}} \quad (2.3)$$

which generalizes to the superposition case as

$$V(\vec{r}) = \begin{cases} -\sum_{i=1}^N \frac{Gmm_i}{|\vec{r}-\vec{r}_i|} & \text{collection of point masses} \\ -\iiint \frac{Gm\rho(\vec{r}')d^3V'}{|\vec{r}-\vec{r}'|} & \text{continuous mass distribution} \end{cases} \quad (2.4)$$

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<sup>2</sup>The one caveat is that the gravitational force of the “test particle” back on the source masses might cause them to move. But we can typically assume they are held in place by some other forces and consider their locations to be fixed.

Just as it's useful to divide out the mass  $m$  and produce a vector field  $\vec{g}(\vec{r})$  which depends only on the source masses, one can similarly define a *gravitational potential*

$$\varphi(\vec{r}) = \frac{V(\vec{r})}{m} = \begin{cases} -\sum_{i=1}^N \frac{Gm_i}{|\vec{r}-\vec{r}_i|} & \text{collection of point masses} \\ -\iiint \frac{G\rho(\vec{r}')d^3V'}{|\vec{r}-\vec{r}'|} & \text{continuous mass distribution} \end{cases} \quad (2.5)$$

In practice, it is often easier to calculate the potential  $\varphi$  with (2.5) and then differentiate it to get the gravitational field

$$\vec{g} = -\vec{\nabla}\varphi \quad (2.6)$$

than it is to calculate the field  $\vec{g}(\vec{r})$  directly using (2.1).

### 2.2.1 Warnings about Symon

1. There is a typo in equation (6.11). The left-hand side should read  $V_{mm_i}$  rather than  $Vmm_i$ .
2. Symon defines a “gravitational field”  $\mathcal{G}(\vec{r}) = -V(\vec{r})/m$  with the opposite sign to  $\varphi(\vec{r})$  and claims this sign convention is standard. Perhaps it was in 1971, but I've never seen it before. We'll work with  $\varphi(\vec{r})$  instead, since the sign convention preserves the symmetrical relationship among  $\vec{F}$ ,  $\vec{g}$ ,  $V$ , and  $\varphi$ .

## 2.3 Example: Gravitational Field of a Spherical Shell

Let  $\mathcal{S}$  be a spherical shell of constant density and total mass  $M$  with inner radius  $a$  and outer radius  $b$ , centered on the origin. Find the gravitational field  $\vec{g}(\vec{r})$  resulting from this shell, at an arbitrary position  $\vec{r}$ .

This is slightly different than the example given in Symon, since we allow the shell to have finite thickness, and we solve the problem by a slightly different method, integrating for the gravitational field directly rather than finding the potential first.

For economy of notation, we'll call the density  $\rho$ . We can relate  $\rho$  to  $M$  by performing a volume integral:

$$M = \iiint_{\mathcal{S}} \rho d^3V = 4\pi \frac{b^3 - a^3}{3} \rho \quad (2.7)$$

In the calculations, we'll just work with a constant  $\rho$  and then relate it to  $M$  at the end.

Now, the gravitational field is given by the integral

$$\vec{g}(\vec{r}) = -\iiint G\rho(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} d^3V' \quad (2.8)$$

In general, we'd need to perform a triple integral over  $\vec{r}'$  for each of the three components  $g_x$ ,  $g_y$ , and  $g_z$ , but in this case we can take advantage of the spherical symmetry of the problem to restrict the form of  $\vec{g}(\vec{r})$ . Because no direction is preferred over any other direction, the magnitude of the field must depend only on the radial coordinate  $r = |\vec{r}|$  and not on the direction of the position vector  $\vec{r}$ . Likewise, the field must point in the radial direction,

because there's nothing to pick out one direction over another. (When we do the integral, there will be non-radial components of the gravitational field due to little pieces of the shell, but they will integrate to zero.) So the field must have the form

$$\vec{g}(\vec{r}) = g_r(r)\hat{r} \quad (2.9)$$

and we just need to find the radial component

$$g_r(r) = \hat{r} \cdot \left( - \iiint G\rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} d^3V' \right) \quad (2.10)$$

If we write  $\vec{r} = r\hat{r}$ , this becomes

$$g_r(r) = - \iiint G\rho(\vec{r}') \frac{r - \hat{r} \cdot \vec{r}'}{|\hat{r} - \vec{r}'|^3} d^3V' \quad (2.11)$$

where the magnitude in the denominator is

$$|r\hat{r} - \vec{r}'| = \sqrt{r^2 + r'^2 - r\hat{r} \cdot \vec{r}'} \quad (2.12)$$

So the crucial piece of geometry is the dot product  $\hat{r} \cdot \vec{r}'$ . Due to the geometry, we want to do the integral  $d^3V'$  in spherical coordinates, but the choice of axis with which to define the  $\theta'$  and  $\phi'$  coordinates is basically arbitrary. So we choose the axis to lie along  $\hat{r}$  (which is fine because it's  $\vec{r}'$  and not  $\vec{r}$  which is varying in the integral) which means  $\hat{r} \cdot \vec{r}' = r' \cos \theta'$ .

This makes the integral

$$\begin{aligned} g_r(0, 0, r) &= -G\rho \int_0^{2\pi} \int_0^\pi \int_a^b \frac{(r - r' \cos \theta')}{[r^2 + r'^2 - rr' \cos \theta']^{3/2}} r'^2 dr' \sin \theta' d\theta' d\phi' \\ &= -2\pi G\rho \int_a^b \int_0^\pi \frac{(r - r' \cos \theta')}{[r^2 + r'^2 - rr' \cos \theta']^{3/2}} \sin \theta' d\theta' r'^2 dr' \end{aligned} \quad (2.13)$$

The quantity inside the square brackets is

$$r'^2 + r^2 - 2r'r \cos \theta' \quad (2.14)$$

Note that this is always a positive number, which is equal to  $(r' - r)^2$  at  $\theta' = 0$ , increasing with  $\theta'$  all the way to  $(r' + r)^2$  at  $\theta' = \pi$ . So we can change variables, replacing  $\theta'$  with

$$u = \sqrt{r'^2 + r^2 - 2r'r \cos \theta'} \quad (2.15)$$

to get the differential, we note that

$$2u du = d(u^2) = d(r'^2 + r^2 - 2r'r \cos \theta') = 2r'r \sin \theta' d\theta' \quad (2.16)$$

so

$$\sin \theta' d\theta' = \frac{u du}{r'r} \quad (2.17)$$

We also note that

$$r' \cos \theta' = \frac{r'^2 + r^2 - u^2}{2r} \quad (2.18)$$

which tells us

$$\begin{aligned}
g_r(0, 0, r) &= -2\pi G\rho \int_a^b \int_{|r-r'|}^{r+r'} \left( r - \frac{r'^2 + r^2 - u^2}{2r} \right) u^{-3} \frac{u}{r'r} r'^2 dr' \\
&= -\frac{2\pi G\rho}{r^2} \int_a^b \underbrace{\left( \int_{|r-r'|}^{r+r'} \frac{r^2 - r'^2 + u^2}{2u} du \right)}_{I(r')} r' dr'
\end{aligned} \tag{2.19}$$

Looking at the integral

$$I(r') = \frac{1}{2} \int_{|r-r'|}^{r+r'} \left( \frac{r^2 - r'^2}{u^2} + 1 \right) du = \frac{1}{2} \left[ \frac{r'^2 - r^2}{u} + u \right]_{|r-r'|}^{r+r'} \tag{2.20}$$

we see the value depends on whether  $r'$  is greater or less than  $r$ . Considering each case separately, we find

$$I(r' > r) = \frac{1}{2} \left( \frac{(r' + r)(r' - r)}{r' + r} + (r' + r) - \frac{(r' + r)(r' - r)}{r' - r} - (r' - r) \right) = 0 \tag{2.21}$$

and

$$\begin{aligned}
I(r' < r) &= \frac{1}{2} \left( -\frac{(r + r')(r - r')}{r + r'} + (r + r') + \frac{(r + r')(r - r')}{r - r'} - (r - r') \right) \\
&= \frac{-r + r' + r + r' + r + r' - r + r'}{2} = 2r'
\end{aligned} \tag{2.22}$$

Armed with the result that

$$I(r') = \begin{cases} 2r' & r' < r \\ 0 & r' > r \end{cases} \tag{2.23}$$

we need to think about how the possible values of  $r'$  compare to  $r$  in the integral

$$g_r(0, 0, r) = -\frac{2\pi G\rho}{r^2} \int_a^b I(r') r' dr' \tag{2.24}$$

There are three cases, depending on the value of  $r$

**0 < r < a** In this case, all of the values  $a \leq r' \leq b$  in the integral are larger than  $r$ , and therefore

$$g_r(0, 0, r) = -\frac{2\pi G\rho}{r^2} \int_a^b (0) r' dr' = 0 \quad \text{when } 0 < r < a \tag{2.25}$$

**r > b** Here, all the possible values of  $r'$  are smaller than  $r$  and thus

$$g_r(0, 0, r) = -\frac{2\pi G\rho}{r^2} \int_a^b (2r') r' dr' = -\frac{4\pi G\rho}{r^2} \frac{b^3 - a^3}{3} = -\frac{GM}{r^2} \quad \text{when } r > b \tag{2.26}$$



$\mathbf{a} < \mathbf{r} < \mathbf{b}$  In this case,  $r'$  can be larger or smaller than  $r$ , but the only non-zero contributions are from  $r' < r$  so the integral becomes

$$g_r(0, 0, r) = -\frac{2\pi G\rho}{r^2} \int_a^r (2r')r' dr' = -\frac{4\pi G\rho}{r^2} \frac{r^3 - a^3}{3} = -\frac{GM}{r^2} \frac{r^3 - a^3}{b^3 - a^3} \quad \text{when } r > b \quad (2.27)$$

Putting it all together, we get

$$\vec{g}(0, 0, r) = \begin{cases} 0 & 0 \leq r \leq a \\ -\frac{GM}{r^2} \frac{r^3 - a^3}{b^3 - a^3} \hat{r} & a \leq r \leq b \\ -\frac{GM}{r^2} \hat{r} & r \geq b \end{cases} \quad (2.28)$$

and thus

$$\vec{g}(\vec{r}) = \begin{cases} 0 & 0 \leq r \leq a \\ -\frac{GM}{r^2} \frac{r^3 - a^3}{b^3 - a^3} \hat{r} & a \leq r \leq b \\ -\frac{GM}{r^2} \hat{r} & r \geq b \end{cases} \quad (2.29)$$

where  $r = |\vec{r}|$  and  $\hat{r} = \vec{r}/r$  as usual.

Note that this means that outside a spherical shell of matter, the gravitational field is the same as if the whole mass were concentrated at the center, while inside a spherical shell, there is no gravitational field due to the shell. And since any spherically symmetric distribution can be described as a superposition of spherical shells, it means the gravitational field due to such a spherically symmetric distribution, a distance  $r$  from the center, is just the field due to the mass closer to the center:

$$\vec{g}(r) = -\frac{GM(r)}{r^2} \hat{r} \quad (2.30)$$

where

$$M(r) = 4\pi \int_0^r \rho(r') r'^2 dr' \quad (2.31)$$

is the mass inside a sphere of radius  $r$ .

### 3 Gauss's Law

Section 6.3 of Symon concerns “gravitational field theory” which is analogous to electrostatic field theory. Here we derive one of the key results, which concerns the flux of the gravitational field through a closed surface. The notation we'll use refers to the surface as  $\partial\mathcal{V}$ , and the volume it encloses as  $\mathcal{V}$ . The flux through  $\partial\mathcal{V}$  is thus

$$\oiint_{\partial\mathcal{V}} \vec{g} \cdot \vec{d^2A} \quad (3.1)$$

where the vector-valued area element for the surface is

$$\vec{d^2A} = \vec{n} d^2A \quad (3.2)$$

with  $\vec{n}$  being an outward-directed unit vector normal (perpendicular) to the surface  $\partial\mathcal{V}$ . Symon uses geometrical arguments to evaluate the integral, but we'll rely on a result from vector calculus, the divergence theorem, which says

$$\oiint_{\partial\mathcal{V}} \vec{g} \cdot \vec{d^2A} = \iiint_{\mathcal{V}} (\vec{\nabla} \cdot \vec{g}) d^3V \quad (3.3)$$

which we can apply whenever the divergence  $\vec{\nabla} \cdot \vec{g}$  is well-defined.

First, we consider the flux of the gravitational field due to a point mass  $M$ . If we define our coördinates so this point source is at the origin, the field is

$$\vec{g}(\vec{r}) = -\frac{GM}{r^2} \hat{r} \quad (3.4)$$

At any point other than the origin (at which  $r = 0$ ) we can calculate the divergence:

$$\vec{\nabla} \cdot \vec{g} = \left( \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \cdot (g_r(r) \hat{r}) \quad (3.5)$$

Now, we could look up the form of the divergence in spherical coördinates, but the relatively simple form of  $\vec{g}$  means we can also just work with the derivative operators, if we recall how the unit vector  $\hat{r}$  varies from point to point:

$$\frac{\partial \hat{r}}{\partial r} = \vec{0} \quad (3.6a)$$

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta} \quad (3.6b)$$

$$\frac{\partial \hat{r}}{\partial \phi} = \sin \theta \hat{\phi} \quad (3.6c)$$

Using this and the product rule, we have

$$\begin{aligned} \vec{\nabla} \cdot \vec{g} &= \frac{dg_r}{dr} (\hat{r} \cdot \hat{r}) + g_r \left( \frac{1}{r} (\hat{\theta} \cdot \hat{\theta}) + \frac{1}{r \sin \theta} (\hat{\phi} \cdot \sin \theta \hat{\phi}) \right) = \frac{dg_r}{dr} + \frac{2}{r} g_r = -GM \left( -2r^{-3} + \frac{2}{r} r^2 \right) \\ &= 0 \quad \text{if } r \neq 0 \end{aligned} \quad (3.7)$$

So, if the volume  $\mathcal{V}$  enclosed by the surface  $\partial\mathcal{V}$  does *not* include the origin (where the point source is located), we can use the divergence theorem to show that the gravitational flux through  $\partial\mathcal{V}$  vanishes.

If  $\mathcal{V}$  *does* include the origin, there must be some minimum radius  $a$  such that a ball of radius  $a$  centered on the origin (which we call  $\mathcal{B}_a$ ) is entirely inside  $\partial\mathcal{V}$ . Then we can split up  $\mathcal{V}$  into  $\mathcal{B}_a$  and  $\mathcal{V}'$ , which is the volume  $\mathcal{V}$  with a ball of radius  $a$  removed from it. If we're careful about the geometry, we can show that

$$\oiint_{\partial\mathcal{V}} \vec{g} \cdot \vec{d^2A} = \oiint_{\partial\mathcal{B}_a} \vec{g} \cdot \vec{d^2A} + \oiint_{\partial\mathcal{V}'} \vec{g} \cdot \vec{d^2A} \quad (3.8)$$

since the contribution from the inner boundary of  $\mathcal{V}'$  cancels out that through  $\partial\mathcal{B}_a$ . At any rate, it's clear that

$$\mathcal{V} = \mathcal{B}_a \cup \mathcal{V}' \quad (3.9)$$

and so, appealing to the divergence theorem,

$$\oiint_{\partial\mathcal{V}} \vec{g} \cdot \overrightarrow{d^2A} = \iiint_{\mathcal{V}} (\vec{\nabla} \cdot \vec{g}) d^3V = \iiint_{\mathcal{B}_a} (\vec{\nabla} \cdot \vec{g}) d^3V + \iiint_{\mathcal{V}'} \underbrace{(\vec{\nabla} \cdot \vec{g})}_{=0 \text{ for } \vec{r} \in \mathcal{V}'} d^3V = \oiint_{\partial\mathcal{B}_a} \vec{g} \cdot \overrightarrow{d^2A} \quad (3.10)$$

The flux through a sphere of radius  $a$  we can calculate directly, since  $r = a$  and  $\hat{n} = \hat{r}$  everywhere on the sphere, and thus

$$\oiint_{\partial\mathcal{B}_a} \vec{g} \cdot \overrightarrow{d^2A} = \int_0^{2\pi} \int_0^\pi \left( -\frac{GM}{a^2} \hat{r} \right) \cdot (\hat{r} a^2 \sin\theta d\theta d\phi) = -4\pi GM \quad (3.11)$$

So in general the flux through a closed surface due to a point source is zero if the source is not enclosed in the surface and  $-4\pi G$  times the mass, if it is. But any mass distribution can be built up out of point sources, so the general result is

$$\oiint_{\partial\mathcal{V}} \vec{g} \cdot \overrightarrow{d^2A} = -4\pi GM_{\text{enc}} \quad (3.12)$$

where  $M_{\text{enc}}$  is the enclosed mass

$$M_{\text{enc}} = \iiint_{\mathcal{V}} \rho(\vec{r}') d^3V' \quad (3.13)$$

This can be written in differential form as

$$\vec{\nabla} \cdot \vec{g} = -4\pi G\rho \quad (3.14)$$

which is analogous to Gauss's law in electrostatics.

## A Appendix: Correspondence to Class Lectures

Date	Sections	Pages
2004 January 13	Overview & A300 Final Review	
2004 January 15	1–2.2	2–6
2004 January 20	2.3–3	6–11