Moving Coördinate Systems
(Symon Chapter Seven)

Physics A301*

Spring 2004

Contents

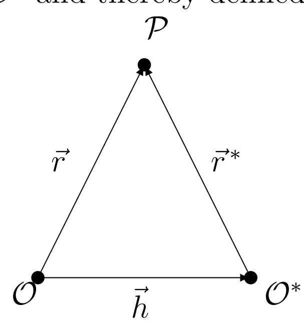
Up until now, we've formulated kinematics in a fixed coördinate system. In particular, we defined the position vector of a point $\mathcal{P}$ relative to some origin $\mathcal{O}$ which was not moving. We could resolve $\vec{r}$ into components which were the Cartesian coördinates of the point $\mathcal{P}$:

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \tag{0.1}$$

The basis vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ were also implicitly assumed not to change in time. (We did sometimes work in terms of spherical or cylindrical basis vectors which were different at different points in *space* and thus had an indirect time dependence as a result of the location of the particle changing.)

1 Translation of the Origin

The choice of the origin $\mathcal{O}$ was somewhat arbitrary. (Although sometimes the geometry of the problem made one origin more convenient than another.) We could also have chosen a different origin, called $\mathcal{O}^*$ and thereby defined a different position vector $\vec{r}^*$ for the point $\mathcal{P}$:



If we define $\vec{h}$ as the displacement vector from $\mathcal{O}$ to $\mathcal{O}^*$, the two position vectors are related by

$$\vec{r} = \vec{r}^* + \vec{h} \tag{1.1}$$

or, equivalently,

$$\vec{r}^* = \vec{r} - \vec{h} \tag{1.2}$$

Note that this defines the relationship between two different position vectors for the same point. If we resolve both position vectors along the *same* set of basis vectors (which means the two sets of coördinate axes are parallel to each other):

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \tag{1.3a}$$

$$\vec{r}^* = x^*\hat{x} + y^*\hat{y} + z^*\hat{z} \tag{1.3b}$$

then the coördinates of the point in the two coördinate system are related by

$$x = x^* + h_x \tag{1.4a}$$

$$y = y^* + h_y \tag{1.4b}$$

$$z = z^* + h_z \tag{1.4c}$$

1.1 Moving Origin

The formalism of a translated origin has interesting consequences for physics if we allow the second origin $\mathcal{O}^*$ to move relative to the first, so that the displacement vector $\vec{h}(t)$ is time-dependent. We can then refer to the velocity

$$\vec{v}_h = \frac{d\vec{h}}{dt} \tag{1.5}$$

of the second origin $\mathcal{O}^*$ relative to the first (non-moving) origin $\mathcal{O}$.

By the sum rule, we can find the relationship between the velocity relative to the unstarred and starred origins:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}^*}{dt} + \frac{d\vec{h}}{dt} = \vec{v}^* + \vec{v}_h \tag{1.6}$$

and likewise for the acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{v}^*}{dt} + \frac{d\vec{v}_h}{dt} = \vec{a}^* + \vec{a}_h \tag{1.7}$$

Now, our description of the laws of physics has assumed an absolute meaning of "at rest", so for all we know, there is a preferred "reference frame", a notion of absolute space, for which the laws of Physics work. So let $\mathcal{O}$ be an origin which is fixed in that system. If $\vec{F}$ is the total force on a particle of mass m, Newton's second law tells us that

$$\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2} \tag{1.8}$$

If we measure the position, velocity and acceleration with respect to a moving origin, the corresponding equation is

$$m\frac{d^2\vec{r}^*}{dt^2} = m\vec{a}^* = m\vec{a} - m\vec{a}_h = \vec{F} - m\vec{a}_h \tag{1.9}$$

So even though Newton's laws were only meant to be applied relative to the fixed origin $\mathcal{O}$, we can apply them relative to the moving origin, if we add to the "real" force $\vec{F}$ an additional *fictitious force* $-m\vec{a}_h$.

Notice three things about this fictitious force:

1. Physically, this is a familiar effect; in an accelerating car, you "feel" a fictitious force pushing you in the opposite direction to the car's acceleration.

2. If $\vec{v}_h$ is a constant, then $\vec{a}_h = \vec{0}$ and $m\vec{a}^* = \vec{F}$. so you can do Physics just as well in a uniformly moving reference frame as in a stationary one. No experiment can detect a constant absolute velocity. This is the principle of Galilean relativity. (A reference frame moving at a constant velocity is called an inertial reference frame, and any inertial reference frame works equally well for the application of Newtonian mechanics.)

3. The fictitious force $-m\vec{a}_h$ looks an awful lot like the effects of a uniform gravitational field $\vec{g} = -\vec{a}_h$. This is a manifestation of the equivalence principle.

2 Rotation of the Coördinate Axes

Translation of the origin is one way that different Cartesian coördinate systems can be related to one another. Another is by rotating the axes, which leads to the position vector being resolved along a different set of basis vectors.

For simplicity, we'll put aside the question of translation for the moment, and let the two origins coïncide ($\mathcal{O} = \mathcal{O}^*$). This means that there is only one position vector, and the different coördinates are resolutions of that vector along different basis vectors:

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = x^*\hat{x}^* + y^*\hat{y}^* + z^*\hat{z}^* \tag{2.1}$$

In fact, *any* vector has different components when resolved in the starred and unstarred bases:

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} = A_{x^*}\hat{x}^* + A_{y^*}\hat{y}^* + A_{z^*}\hat{z}^* \tag{2.2}$$

Note that Symon refers to the component along $\hat{x}^*$ as A_x^* rather than A_{x^*}. We prefer to put the asterisk on the subscript to emphasize that A_x and A_{x^*} are components of the *same* vector $\vec{A}$ along *different* basis vectors.

We can write the starred basis vectors in terms of the unstarred ones by resolving the starred basis vectors, like any other vector, into components along the unstarred basis vectors:

$$\hat{x}^* = (\hat{x}^* \cdot \hat{x})\hat{x} + (\hat{x}^* \cdot \hat{y})\hat{y} + (\hat{x}^* \cdot \hat{z})\hat{z} \tag{2.3}$$

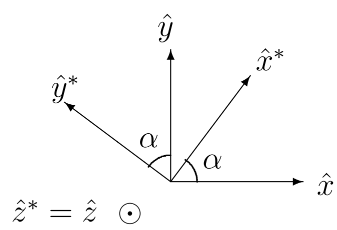
and likewise for $\hat{y}^*$ and $\hat{y}^*$. The shorthand of matrix algebra makes a nice way to write this relationship:

$$\begin{pmatrix} \hat{x}^* \\ \hat{y}^* \\ \hat{z}^* \end{pmatrix} = \begin{pmatrix} \hat{x}^* \cdot \hat{x} & \hat{x}^* \cdot \hat{y} & \hat{x}^* \cdot \hat{z} \\ \hat{y}^* \cdot \hat{x} & \hat{y}^* \cdot \hat{y} & \hat{y}^* \cdot \hat{z} \\ \hat{z}^* \cdot \hat{x} & \hat{z}^* \cdot \hat{y} & \hat{z}^* \cdot \hat{z} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \tag{2.4}$$

The 3×3 matrix of dot products is a *rotation matrix* which transforms the unstarred basis into the starred one.

2.1 Example: Starred Axes Rotated by α About $\hat{z}$

Consider a case where the starred and unstarred z axes agree, and the starred x and y axes make an angle of α with the unstarred ones:



In this case, the interesting dot products are

$$\hat{x}^* \cdot \hat{x} = \cos\alpha \tag{2.5a}$$

$$\hat{x}^* \cdot \hat{y} = \cos(90^\circ - \alpha) = \sin\alpha \tag{2.5b}$$

$$\hat{y}^* \cdot \hat{x} = \cos(90^\circ + \alpha) = -\sin\alpha \tag{2.5c}$$

$$\hat{y}^* \cdot \hat{y} = \cos\alpha \tag{2.5d}$$

which makes the transformation

$$\begin{pmatrix} \hat{x}^* \\ \hat{y}^* \\ \hat{z}^* \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \tag{2.6}$$

2.2 Relationship Between Components

We can find the starred components of a vector $\vec{A}$ in terms of the unstarred ones by dotting the starred basis vectors into that vector, again using linear algebra as a bookkeeping aid:

$$\begin{pmatrix} A_{x^*} \\ A_{y^*} \\ A_{z^*} \end{pmatrix} = \begin{pmatrix} \hat{x}^* \\ \hat{y}^* \\ \hat{z}^* \end{pmatrix} \cdot \vec{A} = \begin{pmatrix} \hat{x}^* \\ \hat{y}^* \\ \hat{z}^* \end{pmatrix} \cdot \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \hat{x}^* \cdot \hat{x} & \hat{x}^* \cdot \hat{y} & \hat{x}^* \cdot \hat{z} \\ \hat{y}^* \cdot \hat{x} & \hat{y}^* \cdot \hat{y} & \hat{y}^* \cdot \hat{z} \\ \hat{z}^* \cdot \hat{x} & \hat{z}^* \cdot \hat{y} & \hat{z}^* \cdot \hat{z} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \tag{2.7}$$

We note that the transformation is accomplished with the same rotation matrix which appears in (2.4).

2.3 Rotating Axes

We now consider what happens when the starred coördinate axes are not just rotated with respect to the fixed, unstarred ones, but also rotating with time, so that the derivatives $\frac{d\hat{x}^*}{dt}$, $\frac{d\hat{y}^*}{dt}$, and $\frac{d\hat{z}^*}{dt}$ are non-zero. The components of the vector $\frac{d\vec{A}}{dt}$ with respect to the rotating basis will not just be the time derivatives of the components of $\vec{A}$:

$$\begin{aligned} \frac{d\vec{A}}{dt} &= \frac{dA_x}{dt}\hat{x} + \frac{dA_y}{dt}\hat{y} + \frac{dA_z}{dt}\hat{z} = \frac{d}{dt}(A_{x^*}\hat{x}^*) + \frac{d}{dt}(A_{y^*}\hat{y}^*) + \frac{d}{dt}(A_{z^*}\hat{z}^*) \\ &= \frac{dA_{x^*}}{dt}\hat{x}^* + \frac{dA_{y^*}}{dt}\hat{y}^* + \frac{dA_{z^*}}{dt}\hat{z}^* + A_{x^*}\frac{d\hat{x}^*}{dt} + A_{y^*}\frac{d\hat{y}^*}{dt} + A_{z^*}\frac{d\hat{z}^*}{dt} \end{aligned} \tag{2.8}$$

The derivative of the vector was trivial when the unstarred (constant) basis vectors were involved, but since the starred (rotating) basis vectors are time-dependent, we had to use the product rule to get the final expression.

The first three terms are what we would have written as the starred components of $\frac{d\vec{A}}{dt}$ if we had forgotten that the basis vectors were time-dependent. Following Symon, we define a shorthand for this:

$$\frac{d^*\vec{A}}{dt} = \frac{dA_{x^*}}{dt}\hat{x}^* + \frac{dA_{y^*}}{dt}\hat{y}^* + \frac{dA_{z^*}}{dt}\hat{z}^* \tag{2.9}$$

which means that

$$\frac{d\vec{A}}{dt} = \frac{d^*\vec{A}}{dt} + A_{x^*}\frac{d\hat{x}^*}{dt} + A_{y^*}\frac{d\hat{y}^*}{dt} + A_{z^*}\frac{d\hat{z}^*}{dt} \tag{2.10}$$

We'll always maintain the perspective that $\frac{d\vec{A}}{dt}$ is the "real" time derivative of the vector $\vec{A}$, but it will often prove useful to consider the components of $\frac{d^*\vec{A}}{dt}$. For instance, an observer looking only at the starred coördinates of a particle (the starred components of its position vector r) will associate with that particle a "velocity vector"[1] $\frac{d^*\vec{r}}{dt}$.

[1]This is sometimes referred to as the velocity in the "starred reference frame".

2.3.1 Change in Basis Vectors Induced by Infinitesimal Rotation

The expression (2.10) tells us the relationship between the time evolution of the components of a vector $\vec{A}$ in a rotating basis and the time derivatives of the basis vectors. But in order to apply it, we need to say what those time derivatives are. To find those, we need to consider how a vector (e.g., any of the starred basis vectors) behaves under an infinitesimal rotation.

Any evolution of the basis vectors which keeps them orthonormal (i.e., keeps them perpendicular unit vectors so $\hat{x}^* \cdot \hat{x}^* = 1$, $\hat{x}^* \cdot \hat{y}^* = 0$, etc.) can be described at any instant by an instantaneous angular velocity $\vec{\omega}$. (This is proved in Symon.) This means that in an infinitesimal amount of time dt, the entire triad of basis vectors will have rotated by an angle

$$d\theta = |\vec{\omega}|\ dt \tag{2.11}$$

about an axis

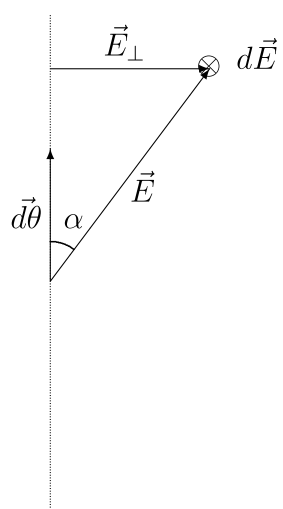
$$\hat{n} = \frac{\vec{\omega}}{|\vec{\omega}|} \tag{2.12}$$

We refer to this as an infinitesimal rotation

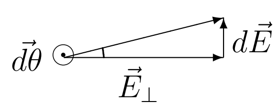
$$\vec{d\theta} = d\theta\,\hat{n} = \vec{\omega}\,dt \tag{2.13}$$

Note that the vector sign is over d and θ, not just θ, because finite rotations cannot be treated as vectors. (In particular, it matters in what order you add them; on the homework, you show that in the limit of infinitesimal rotations, that order is irrelevant.)

We want to rotate a vector we'll call $\vec{E}$ through the infinitesimal angle $\vec{d\theta}$. The vector after rotation will be called $\vec{E} + d\vec{E}$. Call the angle between $\vec{E}$ and the axis of rotation (and hence between $\vec{E}$ and $\vec{d\theta}$) α. Look at everything in the plane defined by $\vec{E}$ and the rotation axis:



"side view"



"top view"

If we define $\vec{E}_\perp$ to be the projection of $\vec{E}$ perpendicular to the axis of rotation, its magnitude is

$$\left|\vec{E}_\perp\right| = \left|\vec{E}\right| \sin\alpha \tag{2.14}$$

The infinitesimal change $d\vec{E}$ is perpendicular to both $\vec{E}$ and the axis of rotation because the tip of $\vec{E}$ would trace out a circle around the axis if we continued the rotation. As it is, for the infinitesimal rotation, it moves in a tiny piece of an arc of radius $\left|\vec{E}_\perp\right|$, subtending an angle $d\theta$. This means the magnitude of the change in $\vec{E}$ is

$$\left|d\vec{E}\right| = d\theta \left|\vec{E}_\perp\right| = \left|d\vec{\theta}\right| \left|\vec{E}\right| \sin\alpha \tag{2.15}$$

We note that this is the magnitude of the cross product

$$d\vec{\theta} \times \vec{E} \tag{2.16}$$

and we also note that according to the right-hand rule, $d\vec{\theta} \times \vec{E}$ points in the direction of $d\vec{E}$. Thus

$$d\vec{E} = d\vec{\theta} \times \vec{E} \tag{2.17}$$

If this occurs in an infinitesimal time dt, we can divide by dt to obtain the result in terms of the instantaneous angular velocity $\vec{\omega}$:

$$\frac{d\vec{E}}{dt} = \frac{d\vec{\theta}}{dt} \times \vec{E} = \vec{\omega} \times \vec{E} \tag{2.18}$$

2.3.2 Time Derivative of a Vector Resolved in a Rotating Basis

If the starred coördinate axes are rotating with an instantaneous angular velocity $\vec{\omega}$, we know from the result of the previous section that

$$\frac{d\hat{x}^*}{dt} = \vec{\omega} \times \hat{x}^* \tag{2.19a}$$

$$\frac{d\hat{y}^*}{dt} = \vec{\omega} \times \hat{y}^* \tag{2.19b}$$

$$\frac{d\hat{z}^*}{dt} = \vec{\omega} \times \hat{z}^* \tag{2.19c}$$

which we can now substitute into (2.10) to get

$$\begin{aligned} \frac{d\vec{A}}{dt} &= \frac{d^*\vec{A}}{dt} + A_{x^*}(\vec{\omega} \times \hat{x}^*) + A_{y^*}(\vec{\omega} \times \hat{y}^*) + A_{z^*}(\vec{\omega} \times \hat{z}^*) = \frac{d^*\vec{A}}{dt} + \vec{\omega} \times (A_{x^*}\hat{x}^* + A_{y^*}\hat{y}^* + A_{z^*}\hat{z}^*) \\ &= \frac{d^*\vec{A}}{dt} + \vec{\omega} \times \vec{A} \end{aligned} \tag{2.20}$$

This tells us how the time derivatives of the components of a vector in the starred (rotating) basis are related to the time derivative of the vector itself (which can be calculated by differentiating its components in the unstarred [fixed] basis).

We can get an expression for the second derivative by applying (2.20) with $\frac{d\vec{A}}{dt}$ in place of $\vec{A}$:

$$\begin{aligned}\frac{d^2\vec{A}}{dt^2} &= \frac{d}{dt}\left(\frac{d\vec{A}}{dt}\right) = \frac{d}{dt}\left(\frac{d^*\vec{A}}{dt}\right) + \frac{d\vec{\omega}}{dt}\times\vec{A} + \vec{\omega}\times\frac{d\vec{A}}{dt} \\ &= \frac{d^{*2}\vec{A}}{dt} + \vec{\omega}\times\frac{d^*\vec{A}}{dt} + \left(\frac{d^*\vec{\omega}}{dt} + \vec{\omega}\times\vec{\omega}\right)\times\vec{A} + \vec{\omega}\times\left(\frac{d^*\vec{A}}{dt} + \vec{\omega}\times\vec{A}\right) \\ &= \frac{d^{*2}\vec{A}}{dt} + \vec{\omega}\times(\vec{\omega}\times\vec{A}) + 2\vec{\omega}\times\frac{d^*\vec{A}}{dt} + \frac{d^*\vec{\omega}}{dt}\times\vec{A}\end{aligned} \tag{2.21}$$

To see how Newtonian physics looks to a rotating observer, we use the position vector $\vec{r}$ as our $\vec{A}$, and apply first (2.20):

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d^*\vec{r}}{dt} + \vec{\omega}\times\vec{r} \tag{2.22}$$

and then (2.21):

$$\vec{a} = \frac{d^2\vec{r}}{dt} = \frac{d^{*2}\vec{r}}{dt} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + 2\vec{\omega}\times\frac{d^*\vec{r}}{dt} + \frac{d^*\vec{\omega}}{dt}\times\vec{r} \tag{2.23}$$

Note that for the time derivative of the angular velocity vector $\vec{\omega}$, it doesn't matter if we write $\frac{d^*\vec{\omega}}{dt}$ or $\frac{d\vec{\omega}}{dt}$ because the difference between the two is $\vec{\omega}\times\vec{\omega} = \vec{0}$.

If an observer rotating along with the starred basis vectors looks at the derivatives of the starred coördinates, he or she will construct an "acceleration"

$$\frac{d^2x^*}{dt^2}\hat{x}^* + \frac{d^2y^*}{dt^2}\hat{y}^* + \frac{d^2z^*}{dt^2}\hat{z}^* = \frac{d^{*2}\vec{r}}{dt} = \vec{a} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2\vec{\omega}\times\frac{d^*\vec{r}}{dt} - \frac{d\vec{\omega}}{dt}\times\vec{r} \tag{2.24}$$

In order to try and apply Newton's laws in the rotating coördinate system:

$$m\frac{d^{*2}\vec{r}}{dt} = \vec{F} - 2m\vec{\omega}\times\frac{d^*\vec{r}}{dt} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - m\frac{d\vec{\omega}}{dt}\times\vec{r} \tag{2.25}$$

the observer would need to add three fictitious forces. The last one, due to variable rotation of the reference frame, doesn't have a name. The term

$$\vec{F}_{\text{cent}} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r}) \tag{2.26}$$

is called the "centrifugal force", and is the fictitious force away from the rotational axis familiar to anyone who's ridden on a merry-go-round. Its magnitude is

$$\left|\vec{F}_{\text{cent}}\right| = mr_\perp\omega^2 \tag{2.27}$$

Where $r_\perp$ is the perpendicular distance from the axis of rotation. The fictitious force

$$\left|\vec{F}_{\text{cori}}\right| = -2m\vec{\omega}\times\frac{d^*\vec{r}}{dt} \tag{2.28}$$

is called the *Coriolis force* and can best be understood in terms of conservation of angular momentum. For example, moving away from the axis of rotation means you have to start rotating more slowly (as seen by an inertial observer) to make up for the longer lever arm. This manifests itself in the rotating reference frame as an apparent force *clockwise* (backwards) around the rotation axis.

3 Simultaneous Translation and Rotation

Now that we've considered translation and rotation individually, let's return to the most general relationship between fixed Cartesian coördinates (x, y, z) and moving Cartesian coördinates (x^*, y^*, z^*): not only can the origin $\mathcal{O}^*$ move relative to $\mathcal{O}$ (so that there are two different position vectors $\vec{r}$ and $\vec{r}^* = \vec{r} - \vec{h}$ to consider, relative to the different origins), but the starred axes can rotate relative to the unstarred ones, so there are two different sets of basis vectors with respect to which we take components. The relationships between the position vectors and the coördinates are thus

$$\vec{r}^* = x^*\hat{x}^* + y^*\hat{y}^* + z^*\hat{z}^* = \vec{r} - \vec{h} \tag{3.1a}$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = \vec{r}^* + \vec{h} \tag{3.1b}$$

Meanwhile, the same vector $\vec{A}$ can be resolved along either set of axes:

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} = A_{x^*}\hat{x}^* + A_{y^*}\hat{y}^* + A_{z^*}\hat{z}^* \tag{3.2}$$

Again we define the "starred derivative" as a shorthand for a vector made up out of the time derivatives of the components in a rotating basis:

$$\frac{d^*\vec{A}}{dt} = \frac{dA_{x^*}}{dt}\hat{x}^* + \frac{dA_{y^*}}{dt}\hat{y}^* + \frac{dA_{z^*}}{dt}\hat{z}^* \tag{3.3}$$

If we follow the "starred" coördinates of a particle, the "velocity" we construct is

$$\frac{dx^*}{dt}\hat{x}^* + \frac{dy^*}{dt}\hat{y}^* + \frac{dz^*}{dt}\hat{z}^* = \frac{d^*\vec{r}^*}{dt} \tag{3.4}$$

Note that because the starred origin is moving and its axes are rotating, we have the starred derivative of the vector $\vec{r}^*$. The relationship between this and the velocity measured by a non-moving observer is

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}^*}{dt} + \frac{d\vec{h}}{dt} = \frac{d^*\vec{r}^*}{dt} + \vec{\omega} \times \vec{r}^* + \frac{d\vec{h}}{dt} \tag{3.5}$$

and similarly, the acceleration measured by a non-moving observer is

$$\vec{a} = \frac{d^2\vec{r}}{dt} = \frac{d^2\vec{r}^*}{dt} + \frac{d^2\vec{h}}{dt} = \frac{d^{*2}\vec{r}^*}{dt} + 2\vec{\omega} \times \frac{d^*\vec{r}^*}{dt} + \vec{\omega} \times (\vec{\omega} \times \vec{r}^*) + \frac{d^*\vec{\omega}}{dt} \times \vec{r}^* + \frac{d^2\vec{h}}{dt} \tag{3.6}$$

This means that the "equation of motion" for the second time derivatives of the starred components of a particle of mass m is

$$m\frac{d^{*2}\vec{r}^*}{dt} = \vec{F} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}^*) - 2m\vec{\omega} \times \frac{d^*\vec{r}^*}{dt} - m\frac{d\vec{\omega}}{dt} \times \vec{r}^* - \frac{d^2\vec{h}}{dt} \tag{3.7}$$

where $\vec{F} = m\vec{a}$ is the actual force measured by an inertial observer, and the additional terms are fictitious forces.

4 Physics on the Rotating Earth

As a special application, consider the Earth, assumed to be rigidly rotating with constant angular velocity $\vec{\omega}$, where

$$|\vec{\omega}| = \frac{2\pi}{1\,\text{day}} \tag{4.1}$$

Let the unstarred coördinates be fixed in space and the starred coördinates be rotating with the Earth.

An object of mass m experiences a force $m\vec{g}$ due to the gravitational pull of the Earth (we allow $\vec{g}$ to depend on the location in space, and in particular don't assume the surface of the Earth is a perfect sphere) and $\vec{F}_{\text{oth}}$ due to other forces, so (taking the origins of both coördinate systems to be at the center of the Earth so that $\vec{h} = \vec{0}$ and noting that $\frac{d\vec{\omega}}{dt} = \vec{0}$)

$$m\frac{d^{*2}\vec{r}}{dt} = \vec{F}_{\text{oth}} + m\vec{g} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \frac{d^*\vec{r}}{dt} \tag{4.2}$$

Notice that both the centrifugal force and the gravitational force are proportional to the mass of the particle and depend only on its location in space. We can therefore combine the two into the effects of a single "effective gravitational field"

$$\vec{g}_{\text{eff}} = \vec{g} - \vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{4.3}$$

so that the equation of motion is

$$m\frac{d^{*2}\vec{r}}{dt} = \vec{F}_{\text{oth}} + m\vec{g}_{\text{eff}} - 2m\vec{\omega} \times \frac{d^*\vec{r}}{dt} \tag{4.4}$$

Laboratory experiments with spring-scales, surveyors' plumb-bobs, etc., will all measure $\vec{g}_{\text{eff}}$ rather than $\vec{g}$.

We can also understand the oblate (flattened) shape of the Earth in terms of $\vec{g}_{\text{eff}}$. The surface of the Earth is (on average) perpendicular to $\vec{g}_{\text{eff}}$. (If it were not, objects on the ground would experience a net force [the combined effect of the normal force and the "effective gravitational force"] in one direction or the other, and tend to slide.)

4.1 Another Perspective

For laboratory problems, however, it's often not very useful to put the origin of the rotating coördinates at the center of the Earth. Instead, we want to have the origin $\mathcal{O}^*$ be some point co-rotating with the Earth (usually but not always on the Earth's surface). Then the position vectors $\vec{r}$ and $\vec{r}^*$ are related by a vector $\vec{h} = \vec{r} - \vec{r}^*$ which is co-rotating with the Earth and therefore satisfies

$$\frac{d\vec{h}}{dt} = \vec{\omega} \times \vec{h} \tag{4.5a}$$

$$\frac{d^*\vec{h}}{dt} = \vec{0} \tag{4.5b}$$

and (using the fact that $\frac{d\vec{\omega}}{dt} = \vec{0}$)

$$\frac{d^2\vec{h}}{dt^2} = \vec{\omega} \times \frac{d\vec{h}}{dt} = \vec{\omega} \times (\vec{\omega} \times \vec{h}) \tag{4.6}$$

This then makes the "equation of motion" in the starred coördinates (which now have an origin moving in a circle and axes rotating at the same rate)

$$\begin{aligned} m\frac{d^{*2}\vec{r}^*}{dt} &= \vec{F}_{\text{oth}} + m\vec{g} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}^*) - 2m\vec{\omega} \times \frac{d^*\vec{r}^*}{dt} - m\underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{h})}_{\frac{d^2\vec{h}}{dt^2}} \\ &= \vec{F}_{\text{oth}} + m\left(\vec{g} - \vec{\omega} \times [\vec{\omega} \times (\vec{r}^* + \vec{h})]\right) - 2m\vec{\omega} \times \frac{d^*\vec{r}^*}{dt} \end{aligned} \tag{4.7}$$

Again, we have a centrifugal correction

$$-\vec{\omega} \times [\vec{\omega} \times (\vec{r}^* + \vec{h})] = -\vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{4.8}$$

although now some of it is counted as an effect of the moving origin and some as an effect of the rotating axes. And it doesn't matter which position vector we put into the Coriolis term, because

$$\frac{d^*\vec{r}}{dt} = \frac{d^*\vec{r}^*}{dt} + \underbrace{\frac{d^*\vec{h}}{dt}}_{\vec{0}} \tag{4.9}$$

4.2 Example: The Foucault Pendulum

A classic example of the non-inertial effects on the surface of the rotating Earth is the Foucault pendulum. This is a pendulum suspended from a point rotating with the Earth, undergoing small-amplitude oscillations. Without rotational effects, we know from freshman physics that this pendulum would swing back and forth in simple harmonic motion (assuming the amplitude is small enough that the small-angle formula applies). The effect of the centrifugal force is to change the magnitude and direction of the gravitational acceleration, meaning we should work with $\vec{g}_{\text{eff}}$ rather than $\vec{g}$. The effect of the Coriolis force is the interesting physics which we will now analyze. Note that the Coriolis force is proportional to (among other things) the speed of the pendulum, and that the maximum speed is in turn proportional to the square root of the pendulum's amplitude, so we treat it as a small effect.

In defining the geometry of the problem, two vectors are important: $\vec{g}_{\text{eff}}$, which defines a local notion of "down" and $\vec{\omega}$, the Earth's rotational angular velocity. The ang

In this problem, there are three "forces" (including fictitious ones) as viewed in the rotating reference frame of the Earth:

1. A force $\vec{T}$ due to the tension in the rope by which the pendulum is suspended;
2. A combined gravitational and centrifugal force $m\vec{g}_{\text{eff}}$;
3. A Coriolis force $-2m\vec{\omega} \times \vec{v}^*$

where $\vec{v}^*$ is the vector which a co-rotating observer would construct as the "velocity":

$$\vec{v}^* = \frac{d^*\vec{r}^*}{dt} = \frac{dx^*}{dt}\hat{x}^* + \frac{dy^*}{dt}\hat{y}^* + \frac{dz^*}{dt}\hat{z}^* \tag{4.10}$$

To do the problem, we set up a Cartesian coördinate system at the location of the pendulum. The z^* axis points straight up, i.e., anti-parallel to $\vec{g}_{\text{eff}}$. The x^* and y^* axes are tangent to the local (average) surface of the Earth, pointing in the East and North directions. (You can check that this is indeed a right-handed coördinate system.) The local (effective) gravitational field is this $\vec{g}_{\text{eff}} = -g_{\text{eff}}\hat{z}^*$. The starred components of $\vec{\omega}$ depend on the latitude λ at which the pendulum is located; the latitude is the angle between the equatorial plane and the local vertical direction $\hat{z}^*$, so

$$\vec{\omega} = \omega \cos\lambda\, \hat{y}^* + \omega \sin\lambda\, \hat{z}^* \tag{4.11}$$

One of the reasons why we used the latitude λ and not the traditional spherical coördinate θ (which is what Symon uses) is that λ is actually a spheroidal rather than a spherical coördinate. The spherical coördinate angle θ is the angle between $\hat{r}$ and $\hat{z}$, which means it's related to the angle between $\vec{g}$ and $\vec{\omega}$ (assuming the gravitational field, not including centrifugal effects, really does point towards the center of the Earth). On the other hand, the spheroidal coördinate λ is defined with respect to the perpendicular not to a sphere but to an ellipsoid whose shape is flattened (oblate) due to the Earth's rotation; this perpendicular is just $\hat{z}^*$. And it is this λ that cartographers use as latitude, whenever they're making precise enough measurements that the rotational oblateness of the Earth matters. To the extent that we can neglect the oblateness of the Earth, $\lambda \approx \frac{\pi}{2} - \theta$.

Now, to resolve the tension in the starred basis, we need to apply the small-angle formula. Let α be the small angle that the pendulum makes with the vertical, and define the origin of coördinates to be the equilibrium position of the pendulum bob, when it hangs straight down, so that the position vector of the suspension point $\mathcal{S}$ is $\vec{r}^*_{\mathcal{S}} = \ell\hat{z}^*$, where ℓ is the length of the pendulum. If we neglect terms which are second order and higher in α, we see that the motion is confined almost entirely to the x^*-y^* plane. This is because

$$q^* := \sqrt{(x^*)^2 + (y^*)^2} = \ell \sin\alpha \approx \ell\alpha \tag{4.12a}$$

$$z^* := \ell(\cos\alpha - 1) = \mathcal{O}(\alpha^2) \approx 0 \tag{4.12b}$$

Now, the tension points along the pendulum rope, so a little trigonometry shows us

$$\vec{T} = T\cos\alpha\,\hat{z}^* - T\sin\alpha\,\hat{q}^* \approx T\,\hat{z}^* - T\frac{q^*}{\ell}\,\hat{q}^* = T\,\hat{z}^* - T\frac{x^*}{\ell}\,\hat{x}^* - T\frac{y^*}{\ell}\,\hat{y}^* \tag{4.13}$$

where we are again neglecting terms of order α^2 and above.

Having resolved the (effective) gravitational and tension forces into starred components, we turn at last to the Coriolis force, which is given by

$$\begin{aligned} \vec{F}_{\text{cori}} = -2m\vec{\omega} \times \vec{v}^* &= -2m \begin{vmatrix} \hat{x}^* & \hat{y}^* & \hat{z}^* \\ 0 & \omega\cos\lambda & \omega\sin\lambda \\ \frac{dx^*}{dt} & \frac{dy^*}{dt} & 0 \end{vmatrix} \\ &= 2m\omega\sin\lambda\,\dot{y}^*\,\hat{x}^* - 2m\omega\sin\lambda\,\dot{x}^*\,\hat{y}^* + 2m\omega\cos\lambda\,\dot{x}^*\,\hat{z}^* \end{aligned} \tag{4.14}$$

where we have neglected $\dot{z}^*$ since it, like z^*, is of order α^2.

The equations of motion have the form

$$m\frac{d^{*2}\vec{r}^*}{dt} = \vec{T} + m\vec{g}_{\text{eff}} + \vec{F}_{cori} \tag{4.15}$$

which has components

$$m\ddot{x}^* = -T\frac{x^*}{\ell} + 2m\omega\sin\lambda\,\dot{y}^* \tag{4.16a}$$

$$m\ddot{y}^* = -T\frac{y^*}{\ell} - 2m\omega\sin\lambda\,\dot{x}^* \tag{4.16b}$$

$$0 = T - mg_{\text{eff}} + 2m\omega\cos\lambda\,\dot{x}^* \tag{4.16c}$$

where again, we have neglected $\ddot{z}^* = \mathcal{O}(\alpha^2)$. Now the third equation can be solved to give us

$$T = mg_{\text{eff}} - 2m\omega\cos\lambda\,\dot{x}^* = mg_{\text{eff}} + \mathcal{O}(\alpha) \tag{4.17}$$

we can then insert this in turn into the x^* and y^* equations of motion. But note that the terms containing T already have factors of order α (i.e., x^*/ℓ and y^*/ℓ) and thus the $\mathcal{O}(\alpha)$ correction to T would only contribute a term of order α^2, which we can neglect. So the equations of motion are

$$\ddot{x}^* = -\frac{g_{\text{eff}}}{\ell}x^* + 2\omega\sin\lambda\,\dot{y}^* \tag{4.18a}$$

$$\ddot{y}^* = -\frac{g_{\text{eff}}}{\ell}y^* - 2\omega\sin\lambda\,\dot{x}^* \tag{4.18b}$$

Now, without the Coriolis term, these would just be two uncoupled equations for simple harmonic oscillators with natural frequency $\omega_0 = \sqrt{g_{\text{eff}}/\ell}$. If we define the frequency $\Omega = -\omega\sin\lambda$ (which is negative in the Northern Hemisphere and positive in the Southern Hemisphere), the additional acceleration due to the Coriolis force is

$$2\Omega\hat{z}^* \times (\dot{x}^*\hat{x}^* + \dot{y}^*\hat{y}^*) \tag{4.19}$$

Note that this is at right angles to the instantaneous motion of the pendulum, so will not cause it to gain or lose energy. If we assume that the natural oscillation period is a lot less than a day ($\omega_0 \gg \omega \geq |\Omega|$), the Coriolis influence will be slow. If we consider the angular momentum of the pendulum about its suspension point, the torque due to the Coriolis force is always perpendicular to this, and will thus cause the plane of the pendulum's rotation to precess.

We can confirm this by guessing a precessing solution and verifying that it satisfies the differential equations when $\omega_0 \gg |\Omega|$. We write the coupled differential equations in matrix form as

$$\underbrace{\begin{pmatrix}\ddot{x}^* \\ \ddot{y}^*\end{pmatrix}}_{\ddot{\boldsymbol{\xi}}^*} + 2\Omega\underbrace{\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}}_{\boldsymbol{\sigma}}\underbrace{\begin{pmatrix}\dot{x}^* \\ \dot{y}^*\end{pmatrix}}_{\dot{\boldsymbol{\xi}}^*} + \omega_0^2\underbrace{\begin{pmatrix}x^* \\ y^*\end{pmatrix}}_{\boldsymbol{\xi}^*} = \begin{pmatrix}0 \\ 0\end{pmatrix} = \mathbf{0} \tag{4.20}$$

We will, as usual, solve this set of differential equations by guessing the right answer. Use, for initial conditions, the pendulum being released with from "rest" at the point $\vec{r}_0^* = x_0^*\hat{x}^* + y_0^*\hat{y}^*$. If there were no Coriolis term, the solution would be

$$\boldsymbol{\xi}^*_{NR}(t) = \begin{pmatrix} x^*(t) \\ y^*(t) \end{pmatrix} = \begin{pmatrix} x_0^* \cos\omega_0 t \\ y_0^* \cos\omega_0 t \end{pmatrix} = \boldsymbol{\xi}^*_0 \cos\omega_0 t \tag{4.21}$$

Now, we propose to look for a solution to our problem which is like this, only slowly rotating:

$$\boldsymbol{\xi}^*(t) = \begin{pmatrix} \cos\beta(t) & -\sin\beta(t) \\ \sin\beta(t) & \cos\beta(t) \end{pmatrix} \boldsymbol{\xi}^*_0 \cos\omega_0 t = \mathbf{R}(t)\boldsymbol{\xi}^*_{NR} \tag{4.22}$$

where $\boldsymbol{\xi}^*_{NR}$ is the non-rotating solution given in (4.21).

The time derivatives of (4.22) are

$$\dot{\boldsymbol{\xi}}^* = \dot{\mathbf{R}}\,\boldsymbol{\xi}^*_{NR} + \mathbf{R}\,\dot{\boldsymbol{\xi}}^*_{NR} \tag{4.23a}$$

$$\ddot{\boldsymbol{\xi}}^* = \ddot{\mathbf{R}}\,\boldsymbol{\xi}^*_{NR} + 2\dot{\mathbf{R}}\,\dot{\boldsymbol{\xi}}^*_{NR} + \mathbf{R}\,\ddot{\boldsymbol{\xi}}^*_{NR} \tag{4.23b}$$

which we can substitute into (4.20) to get

$$\begin{aligned} \mathbf{0} &= \ddot{\mathbf{R}}\,\boldsymbol{\xi}^*_{NR} + 2\dot{\mathbf{R}}\,\dot{\boldsymbol{\xi}}^*_{NR} + \mathbf{R}\,\ddot{\boldsymbol{\xi}}^*_{NR} + 2\Omega\boldsymbol{\sigma}\left(\dot{\mathbf{R}}\,\boldsymbol{\xi}^*_{NR} + \mathbf{R}\,\dot{\boldsymbol{\xi}}^*_{NR}\right) + \omega_0^2\mathbf{R}\,\boldsymbol{\xi}^*_{NR} \\ &= \mathbf{R}\,\ddot{\boldsymbol{\xi}}^*_{NR} + 2\left(\dot{\mathbf{R}} + \Omega\boldsymbol{\sigma}\mathbf{R}\right)\dot{\boldsymbol{\xi}}^*_{NR} + \left(\ddot{\mathbf{R}} + 2\Omega\boldsymbol{\sigma}\dot{\mathbf{R}} + \omega_0^2\mathbf{R}\right)\boldsymbol{\xi}^*_{NR} \end{aligned} \tag{4.24}$$

Now, the thing that's complicating the differential equations is the $\dot{\boldsymbol{\xi}}^*_{NR}$ term. We can make this go away by choosing the as-yet-unspecified rotational angle $\beta(t)$ so that

$$\begin{aligned} \mathbf{0} = \dot{\mathbf{R}} + \Omega\boldsymbol{\sigma}\mathbf{R} &= \begin{pmatrix} -\dot{\beta}\sin\beta & -\dot{\beta}\cos\beta \\ \dot{\beta}\cos\beta & -\dot{\beta}\sin\beta \end{pmatrix} + \Omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \\ &= \begin{pmatrix} -\dot{\beta}\sin\beta + \Omega\sin\beta & -\dot{\beta}\cos\beta + \Omega\cos\beta \\ \dot{\beta}\cos\beta - \Omega\cos\beta & -\dot{\beta}\sin\beta + \Omega\sin\beta \end{pmatrix} \end{aligned} \tag{4.25}$$

Which is satisfied for $\dot{\beta} = \Omega$; a solution of this is $\beta = \Omega t$, and we will take that form for β.

Substituting $\dot{\mathbf{R}} = -\Omega\boldsymbol{\sigma}\mathbf{R}$ into (4.24) gives us

$$\mathbf{0} = \mathbf{R}\,\ddot{\boldsymbol{\xi}}^*_{NR} + \left(\Omega^2\boldsymbol{\sigma}^2\mathbf{R} - 2\Omega^2\boldsymbol{\sigma}^2\mathbf{R} + \omega_0^2\mathbf{R}\right)\boldsymbol{\xi}^*_{NR} \tag{4.26}$$

Now, we're assuming that $\omega_0 \gg \omega > |\Omega|$ (the pendulum is oscillating a lot faster than it's precessing). so we can neglect the Ω^2 terms relative to the ω_0^2 terms, leaving us with the differential equation

$$\mathbf{0} = \mathbf{R}\left(\ddot{\boldsymbol{\xi}}^*_{NR} + \omega_0^2\boldsymbol{\xi}^*_{NR}\right) \tag{4.27}$$

but the quantity in parentheses is just the left-hand side of differential equation in the non-rotating case, so it does indeed vanish.

So we've found that for a pendulum of length ℓ at a latitude λ on the Earth, where the effective gravitational field has magnitude g_{eff}, so that the natural frequency of oscillation is $\omega_0 = \sqrt{g_{\text{eff}}/\ell} \gg \omega$, if we release it from rest at a small $\ll \ell$ displacement from its equilibrium, the resulting motion will be oscillation at frequency ω_0 together with slow precession of the oscillation plane at a frequency $\Omega = -\omega \sin\lambda$:

$$\begin{pmatrix} x^*(t) \\ y^*(t) \end{pmatrix} \approx \begin{pmatrix} \cos\Omega t & -\sin\Omega t \\ \sin\Omega t & \cos\Omega t \end{pmatrix} \begin{pmatrix} x_0^* \cos\omega_0 t \\ y_0^* \cos\omega_0 t \end{pmatrix} \tag{4.28}$$

Note that we can check that the precession has the right sign by considering a pendulum at the North Pole $\lambda = \pi/2$. In that case, the suspension point is not moving in an inertial coördinate system, and we have a freely swinging pendulum with the Earth rotating beneath it. Seen from the perspective of the rotating Earth, this looks like a *clockwise* precession at frequency $\Omega = -\omega$. At lower latitudes, the suspension point of the pendulum is rotating along with the Earth, and analysis of the problem in a non-rotating reference frame would be rather more difficult than the solution we've followed here.

5 Application: Tidal Forces

Our final application of non-inertial reference frames is to the problem of tidal effects. This is in some ways similar to the restricted three-body problem considered in Section 7.6 of Symon, in that we have two massive bodies whose motion under their mutual gravitational influence can be determined exactly (gravitational two-body problem). We then add to the picture some additional matter which is not massive enough to disrupt the two-body motion appreciably, and consider its motion in under the influence of the two orbiting bodies. In the case of tides, we're interested in deformations to the surface of one of the bodies, so we choose the origin of the non-inertial coördinate system to be the center of one of the orbiting bodies rather than the center of mass of the system.

Tidal effects occur because of the variation of the gravitational field (e.g., of the Sun or the Moon) in space (e.g., from one part of the Earth to another). They are usually analyzed by subtracting off the average gravitational field to get a leftover "tidal field". By using a non-inertial reference frame centered on the Earth, we end up with a fictitious force which is the negative of this "average" field, making the tidal field manifest itself in a straightforward way, with less hand-waving needed.

The problem we're interested in is that of a body of mass M_1 (e.g., the Earth) and a body of mass M_2 (e.g., the Sun or the Moon; to make the problem general enough to cover both cases, we will not make any assumptions about the relative values of M_1 and M_2). These are to be orbiting one another at a constant distance a. The bodies are assumed to be approximately spherical, of radii much less than a. We'll want to obtain the equation of motion of a "test" body of mass $m \ll M_1, M_2$ in a coördinate system co-rotating with body #1.

First, we need the trajectories of both bodies in an inertial reference frame, which we take to have its origin at the center of mass. The two bodies have position vectors $\vec{r}_1$ and $\vec{r}_2$ related by

$$M_1\vec{r}_1 + M_2\vec{r}_2 = \vec{0} \tag{5.1}$$

We know from our analysis of the Kepler problem that a circular orbit is a possible solution, in which case the distance between them

$$|\vec{r}_1 - \vec{r}_2| = a \tag{5.2}$$

is a constant. We could start from the effective one-body problem and construct $\vec{r}_1$ and $\vec{r}_2$ from $\vec{r}_{2\to1}$ and $\vec{R}=\vec{0}$, but it's easier to get what we need by noting that the acceleration of body #1 as it moves in its circular orbit is derived from the gravitational force on it due to body #2:

$$\ddot{\vec{r}}_1 = \frac{\vec{F}_{2\to1}}{M_1} = \frac{1}{M_1}\left(-GM_1M_2\frac{\vec{r}_1-\vec{r}_2}{|\vec{r}_1-\vec{r}_2|^3}\right) = \frac{GM_2}{a^3}(\vec{r}_2-\vec{r}_1) \tag{5.3}$$

Also note that both planets are rotating with orbital angular momentum $\vec{\omega}_{\text{orb}}$ which is directed perpendicular to the orbital plane and has magnitude (from Kepler's third law)

$$|\vec{\omega}_{\text{orb}}| = \frac{2\pi}{T} = \sqrt{\frac{G(M_1+M_2)}{a^3}} \tag{5.4}$$

and in particular, both position vectors are rotating with constant angular velocity $\vec{\omega}_{\text{orb}}$:

$$\frac{d\vec{r}_1}{dt} = \vec{\omega}_{\text{orb}}\times\vec{r}_1 \tag{5.5a}$$

$$\frac{d\vec{r}_2}{dt} = \vec{\omega}_{\text{orb}}\times\vec{r}_2 \tag{5.5b}$$

Now we change to a coördinate system (i.e., the starred coördinate system) attached to body #1. Its origin is at the location of the first body:

$$\vec{h} = \vec{r}_1 \tag{5.6}$$

which means that the position vectors of the bodies are

$$\vec{r}_1^{\,*} = \vec{r}_1 - \vec{h} = \vec{0} \tag{5.7a}$$

$$\vec{r}_2^{\,*} = \vec{r}_2 - \vec{h} = \vec{r}_2 - \vec{r}_1 \tag{5.7b}$$

and therefore

$$\ddot{\vec{h}} = \ddot{\vec{r}}_1 = \frac{GM_2\vec{r}_2^{\,*}}{a^3} \tag{5.8}$$

In addition to a new origin $\mathcal{O}^*$, the "starred" coördinates also have axes with rotate along with body #1. This means the reference frame is rotating at some angular velocity $\vec{\omega}$. In general this will be *different* from the orbital angular momentum $\vec{\omega}_{\text{orb}}$. For example, if body #1 is the Earth, $|\vec{\omega}| = 2\pi/(1\,\text{siderial day})$,[2] and if body #2 is the Sun, $|\vec{\omega}_{\text{orb}}| = 2\pi/(1\,\text{yr})$.

Now consider a test particle of mass m with position vector $\vec{r}^{\,*}$ in these coördinates. It experiences gravitational forces due to bodies #1 and #2 as well as any other force $\vec{F}_{\text{oth}}$, making the total force

$$\vec{F} = \vec{F}_{\text{oth}} - GM_1m\frac{\vec{r}^{\,*}}{r^{*3}} - GM_2m\frac{\vec{r}^{\,*}-\vec{r}_2^{\,*}}{|\vec{r}^{\,*}-\vec{r}_2^{\,*}|^3} \tag{5.9}$$

[2] A sidereal day is the time it takes the Earth to rotate once with respect to the fixed stars, and is 23 hours and 56 minutes long.

adding to this the usual fictitious forces gives the equation of motion

$$
\begin{aligned}
m\frac{d^{*2}\vec{r}^*}{dt} &= \vec{F}_{\text{oth}} - GM_1 m\frac{\vec{r}^*}{r^{*3}} - GM_2 m\frac{\vec{r}^* - \vec{r}_2^*}{|\vec{r}^* - \vec{r}_2^*|^3} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}^*) - 2m\vec{\omega}\times\frac{d^*\vec{r}^*}{dt} - m\underbrace{\frac{GM_2\vec{r}_2^*}{a^3}}_{\frac{d^2\vec{h}}{dt^2}} \\
&= \vec{F}_{\text{oth}} - 2m\vec{\omega}\times\frac{d^*\vec{r}^*}{dt} + m\vec{g}_{\text{eff}}
\end{aligned} \tag{5.10}
$$

Again, everything except the Coriolis and "other" forces can be combined into an effective gravitational field. If we define unit vectors $\hat{r}^* = \vec{r}^*/r^*$ and $\hat{r}_2^* = \vec{r}_2^*/a$, this can be written as

$$
\vec{g}_{\text{eff}} = \underbrace{-\frac{GM_1}{r^{*2}}\hat{r}^*}_{\vec{g}_{\text{earth}}} \underbrace{-\vec{\omega}\times(\vec{\omega}\times\vec{r}^*)}_{\vec{g}_{\text{cent}}} + \underbrace{GM_2\left(\frac{a\hat{r}_2^* - r^*\hat{r}^*}{|a\hat{r}_2^* - r^*\hat{r}^*|^3} - \frac{\hat{r}_2^*}{a^2}\right)}_{\vec{g}_{\text{tidal}}} \tag{5.11}
$$

The effective gravitational field breaks up nicely into the gravitational field of the Earth, the centrifugal correction, and the tidal field.

Note that the unit vector $\hat{r}_2^*$ does not have constant components in either the inertial or rotating coördinate system. This is because

$$
\begin{aligned}
\frac{d^*\hat{r}_2^*}{dt} &= \frac{d\hat{r}_2^*}{dt} - \vec{\omega}\times\hat{r}_2^* = \frac{d}{dt}\frac{\vec{r}_2 - \vec{r}_1}{a} - \vec{\omega}\times\hat{r}_2^* = \frac{\vec{\omega}_{\text{orb}}\times\vec{r}_2 - \vec{\omega}_{\text{orb}}\times\vec{r}_1}{a} - \vec{\omega}\times\hat{r}_2^* \\
&= -(\vec{\omega} - \vec{\omega}_{\text{orb}})\times\hat{r}_2^*
\end{aligned} \tag{5.12}
$$

So, as seen by an observer co-rotating with body #1, the direction to body #2 rotates at an angular velocity $-(\vec{\omega} - \vec{\omega}_{\text{orb}})$. Note that if body #1 is the Earth and body #2 is the Sun,

$$
\frac{2\pi}{|\vec{\omega}|} = 1\,\text{sidereal day} = 23\,\text{h} + 56\,\text{m} \tag{5.13a}
$$

$$
\frac{2\pi}{|\vec{\omega} - \vec{\omega}_{\text{orb}}|} = 1\,\text{solar day} = 24\,\text{h} \tag{5.13b}
$$

If the test body is a lot closer to body #1 than to body #2, so that $r^* \ll a$, we can approximate the tidal field $\vec{g}_{\text{tidal}}$. First, we note that

$$
\begin{aligned}
|a\hat{r}_2^* - r^*\hat{r}^*|^{-3} &= \left[a^2 - 2ar^*(\hat{r}^*\cdot\hat{r}_2^*) + (r^*)^2\right]^{-3/2} = a^{-3}\left[1 - 2\frac{r^*}{a}(\hat{r}^*\cdot\hat{r}_2^*) + \left(\frac{r^*}{a}\right)^2\right]^{-3/2} \\
&= a^{-3}\left(1 + 3\frac{r^*}{a}(\hat{r}^*\cdot\hat{r}_2^*) + \mathcal{O}\left[\left(\frac{r^*}{a}\right)^2\right]\right)
\end{aligned} \tag{5.14}
$$

where $\mathcal{O}\left[\left(\frac{r^*}{a}\right)^2\right]$ represents any terms quadratic and higher in the "small" ratio r^*/a and we have used the Taylor expansion

$$
(1+\chi)^\alpha = 1 + \alpha\chi + \mathcal{O}(\chi^2)\ . \tag{5.15}
$$

If we multiply (5.14) by the denominator from (5.11) we get

$$\begin{aligned}\frac{a\hat{r}_2^* - r^*\hat{r}^*}{|a\hat{r}_2^* - r^*\hat{r}^*|^3} &= a^{-3}\left(1 + 3\frac{r^*}{a}(\hat{r}^*\cdot\hat{r}_2^*) + \mathcal{O}\left[\left(\frac{r^*}{a}\right)^2\right]\right)(a\hat{r}_2^* - r^*\hat{r}^*) \\ &= a^{-2}\left(1 + 3\frac{r^*}{a}(\hat{r}^*\cdot\hat{r}_2^*) + \mathcal{O}\left[\left(\frac{r^*}{a}\right)^2\right]\right)\left(\hat{r}_2^* - \frac{r^*}{a}\hat{r}^*\right) \\ &= a^{-2}\left(\hat{r}_2^* - \frac{r^*}{a}\hat{r}^* + 3\hat{r}_2^*\frac{r^*}{a}(\hat{r}^*\cdot\hat{r}_2^*) + \mathcal{O}\left[\left(\frac{r^*}{a}\right)^2\right]\right)\end{aligned} \tag{5.16}$$

where we have again absorbed terms with more than one power of r^*/a into $\mathcal{O}\left[\left(\frac{r^*}{a}\right)^2\right]$. This then allows us to write the tidal field as

$$\vec{g}_{\text{tidal}} = \frac{GM_2}{a^2}\left(\hat{r}_2^* - \frac{r^*}{a}\hat{r}^* + 3\hat{r}_2^*\frac{r^*}{a}(\hat{r}^*\cdot\hat{r}_2^*) + \mathcal{O}\left[\left(\frac{r^*}{a}\right)^2\right] - \hat{r}_2^*\right) \approx \frac{GM_2 r^*[-\hat{r}^* + 3\hat{r}_2^*(\hat{r}^*\cdot\hat{r}_2^*)]}{a^3} \tag{5.17}$$

So we see the strength of the tidal field is inversely proportional to the *cube* of the distance to the second body. This is why the Moon exerts a greater tidal force than the Sun, even though the Sun exerts a greater overall gravitational force:

$$\frac{M_☽}{a_☽^2} < \frac{M_\odot}{a_\odot^2} \qquad \text{but} \qquad \frac{M_☽}{a_☽^3} > \frac{M_\odot}{a_\odot^3} \tag{5.18}$$

Now, we can actually find the approximate size and shape of the Earth's tidal bulge by the same method used on the homework to find the rotational oblateness. As before, the surface of the Earth must be normal to the local direction of $\vec{g}_{\text{eff}}$.

We assume that the Earth (body #1) is a sphere of radius R plus a slight deformation, which is not significant enough to alter its gravitational field. The surface of the Earth is thus defined by

$$r^* = R + \delta R_{\text{rot}}(\theta^*, \phi^*) + \delta R_{\text{tide}}(\theta^*, \phi^*) \tag{5.19}$$

We recall from Calc III that if we can write the equation defining a three-dimensional surface as $F(\vec{r}) = \text{constant}$, the vector $\vec{\nabla}F$ will be normal to the surface. So if we write the equation for the surface of the Earth as

$$R = r^* - \delta R_{\text{rot}} - \delta R_{\text{tide}} \tag{5.20}$$

and any vector perpendicular to the surface must be proportional to

$$\vec{\nabla}^*(r^* - \delta R_{\text{rot}} - \delta R_{\text{tide}}) = \hat{r}^* - \vec{\nabla}^*(\delta R_{\text{rot}}) - \vec{\nabla}^*(\delta R_{\text{tide}}) \tag{5.21}$$

where

$$\vec{\nabla}^* = \hat{x}^*\frac{\partial}{\partial x^*} + \hat{y}^*\frac{\partial}{\partial y^*} + \hat{z}^*\frac{\partial}{\partial z^*} \tag{5.22}$$

is the gradient with respect to the starred coördinates.

The gravitational field $\vec{g}_{\text{earth}}$ is purely radial; the corrections δR_{rot} and δR_{tide} are needed to correct for the non-radial parts of $\vec{g}_{\text{cent}}$ and $\vec{g}_{\text{tidal}}$. When studying rotational oblateness, you constructed a small dimensionless parameter ε proportional to ω^2. For the tidal force, the small parameter will turn out to be $\gamma = \frac{M_2}{M_1}\frac{R^3}{a^3}$. The corrections due to the centrifugal force (e.g., δR_{rot}) will be of order ε, while the corrections due to tidal effects (e.g., δR_{tide}) will be of order γ). We assume that these are both small numbers, and that we can always neglect them unless the other terms in an expression cancel out. In particular, we can ignore centrifugal effects when considering tidal effects, and vice-versa. The errors we make in doing this will be corrections to the corrections, of order $\varepsilon\gamma$, and can be neglected compared with the corrections which are proportional to only one of the two small parameters.

So let's consider the correction δR_{tide} needed to make

$$\hat{r}^* - \vec{\nabla}^*(\delta R_{\text{tide}}) \tag{5.23}$$

parallel to

$$\vec{g}_{\text{earth}} + \vec{g}_{\text{tidal}} \approx -\frac{GM_1}{r^{*2}}\hat{r}^* + \frac{GM_2 r^*[-\hat{r}^* + 3\hat{r}_2^*(\hat{r}^* \cdot \hat{r}_2^*)]}{a^3} \tag{5.24}$$

Since we want to distinguish radial from non-radial coördinates, we'll want to so this in a spherical coördinate system. As usual, we choose this spherical coördinate system to have its axis in the preferred direction, which is in this case $\hat{r}_2^*$. We call these spherical coördinates r^*, Θ^*, and Φ^*. (The radial coördinate is just the distance from the origin $\mathcal{O}^*$, but we choose the names Θ^*, and Φ^* rather than θ^*, and ϕ^* to emphasize that these angles are not measured with respect to a more familiar fixed direction in space, like the Earth's rotation axis.) Now, $\hat{r}_2^*$ is basically the "z axis" for these spherical coördinates, which means the usual trigonometry tells us

$$\hat{r}_2^* = \hat{r}^* \cos\Theta^* - \hat{\Theta}^* \sin\Theta^* \tag{5.25}$$

and in particular

$$\hat{r}^* \cdot \hat{r}_2^* = \cos\Theta^* \tag{5.26}$$

which makes

$$\vec{g}_{\text{earth}} + \vec{g}_{\text{tidal}} \approx \left(-\frac{GM_1}{r^{*2}} - \frac{GM_2 r^*(-1 + 3\cos^2\Theta^*)}{a^3}\right)\hat{r}^* + \frac{3GM_2 r^* \cos\Theta^* \sin\Theta^*}{a^3}\hat{\Theta}^* \tag{5.27}$$

Now, we can apply the approximation that $R \ll a$ as follows: first, at the surface of the Earth r^* is equal to R plus a correction, which we are assuming to be small, so we can go ahead and replace r^* with R. Second, the first term in the r^* component is much larger than the second term, so we can drop that second term. This leaves us with:

$$\vec{g}_{\text{earth}} + \vec{g}_{\text{tidal}} \approx -\frac{GM_1}{R^2}\hat{r}^* + \frac{3GM_2 R \cos\Theta^* \sin\Theta^*}{a^3}\hat{\Theta}^* \tag{5.28}$$

Note that we have to keep the small term in the $\hat{\Theta}^*$ component, because there is no larger term in that component.

Now, the normal to the Earth's surface must be a scalar times (5.28). This means $n_{\Phi^*} = 0$ and

$$\frac{n_{\Theta^*}}{n_{r^*}} = \frac{3GM_2 R \cos\Theta^* \sin\Theta^*/a^3}{-GM_1/R^2} = -3\frac{M_2}{M_1}\frac{R^3}{a^3}\cos\Theta^* \sin\Theta^* \tag{5.29}$$

We can now relate this back to the correction δR_{tide}. First, the fact that $n_{\Phi^*} = 0$ tells us that (5.23) has no Φ^* component, which means that δR_{tide} is a function only of Θ^*. That makes the normal to the surface proportional to

$$\hat{r}^* - \vec{\nabla}^*[\delta R_{\text{tide}}(\Theta^*)] = \hat{r}^* - \frac{\hat{\Theta}^*}{r^*}\delta R'_{\text{tide}}(\Theta^*) \tag{5.30}$$

The ratio of components of the normal vector must thus be

$$\frac{n_{\Theta^*}}{n_{r^*}} = -\frac{\delta R'_{\text{tide}}(\Theta^*)}{r^*} \approx -\frac{\delta R'_{\text{tide}}(\Theta^*)}{R} \tag{5.31}$$

Setting this equal to (5.29) tells us

$$\frac{\delta R'_{\text{tide}}(\Theta^*)}{R} \approx -3\frac{M_2}{M_1}\frac{R^3}{a^3}\cos\Theta^*\sin\Theta^* \tag{5.32}$$

which gives us the differential equation

$$\delta R'_{\text{tide}}(\Theta^*) = -3\frac{M_2}{M_1}\frac{R^3}{a^3}R\cos\Theta^*\sin\Theta^* \tag{5.33}$$

which can be solved for

$$\delta R_{\text{tide}} = \frac{M_2}{M_1}\frac{R^3}{a^3}R\frac{3\cos^2\Theta^* - 1}{2} = R\frac{M_2}{M_1}\frac{R^3}{a^3}\frac{3(\hat{r}^*\cdot\hat{r}_2^*)^2 - 1}{2} \tag{5.34}$$

(There's an arbitrary constant in the integration, but the choice above makes the lowest order correction to the volume of the Earth vanish.)

Note that we do indeed have

$$\frac{\delta R_{\text{tide}}}{R} = \mathcal{O}\left(\frac{M_2}{M_1}\frac{R^3}{a^3}\right) \tag{5.35}$$

6 Summary of Gravity and Non-Inertial Coördinates

6.1 Gravity

$\vec{F} = m\vec{g}$

- $\vec{g}$ from colletion of point masses or mass distribution
- φ by direct calculations (point masses or distribution)

  Note our φ and Symon's $\mathcal{G}$ have opposite sign.
- $\vec{g} = -\vec{\nabla}\varphi$; *vecg* from φ or φ from *vecg* (solve differential equations)
- Gauss's Law (apply)

Also physical applications e.g., equations of motion and trajectory or conservation of energy using $V = m\varphi$.

6.2 Moving Coördinates

Will be provided

$$\frac{d^{*2}\vec{r}^*}{dt^2} = \frac{d^2\vec{r}}{dt^2} - \vec{\omega} \times (\vec{\omega} \times \vec{r}^*) - 2\vec{\omega} \times \frac{d^*\vec{r}^*}{dt} - \frac{d\vec{\omega}}{dt} \times \vec{r}^* - \frac{d^2\vec{h}}{dt} \tag{6.1}$$

- Be able to apply to physical problems
- Be conversant with derivation

$$\vec{r}^* = \vec{r} - \vec{h} \tag{6.2}$$

$$\vec{r}^* = x^*\hat{x}^* + y^*\hat{y}^* + z^*\hat{z}^* \tag{6.3a}$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \tag{6.3b}$$

$$\frac{d}{dt}\begin{pmatrix}\hat{x}^* \\ \hat{y}^* \\ \hat{z}^*\end{pmatrix} = \vec{\omega} \times \begin{pmatrix}\hat{x}^* \\ \hat{y}^* \\ \hat{z}^*\end{pmatrix} \tag{6.4}$$

- Construction of $\vec{g}_{\text{eff}}$ for rotating planet → surface perpendicular to $\vec{g}_{\text{eff}}$
- Tidal force and distortion as application of the method

A Appendix: Correspondence to Class Lectures

Date	Sections	Pages
2004 January 22	1–2.3	2–5
2004 January 27	2.3.1–3	6–9
2004 January 29	4–4.2	10–13
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2004 February 5	5	15–20
2004 February 17	6	20–21