Lagrangian and Hamiltonian Mechanics
(Symon Chapter Nine)

Physics A301*

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Contents

This chapter is concerned with perhaps the most significant and far-reaching development in classical mechanics: the formulation of the laws of motion in terms of a scalar quantity which obeys equations which can be easily derived in any coördinates describing the configuration of the system.

1 Lagrange's Equations

1.1 Newton's Laws Reformulated for a Single Particle in Cartesian Coördinates

We begin with an observation. Consider a single particle moving under the influence of a force field $\vec{F}(x, y, z, t)$ which can be derived from a potential energy $V(x, y, z, t)$. Newton's second law tells us

$$\dot{\vec{p}} = m\ddot{\vec{r}} = \vec{F} = -\vec{\nabla} V \tag{1.1}$$

This vector equation has components

$$\dot{p}_x = m\ddot{x} = F_x = -\frac{\partial V}{\partial x} \tag{1.2a}$$

$$\dot{p}_y = m\ddot{y} = F_y = -\frac{\partial V}{\partial y} \tag{1.2b}$$

$$\dot{p}_z = m\ddot{z} = F_z = -\frac{\partial V}{\partial z} \tag{1.2c}$$

So the force, a vector quantity, has components which are partial derivatives of a scalar quantity, the potential energy $V(x, y, z, t)$. On the other hand. the left-hand side of the vector equation concerns the time derivative of the momentum, a vector quantity. The most closely related scalar quantity is the kinetic energy

$$T(\dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 \tag{1.3}$$

its partial derivatives with respect to its arguments are

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x} = p_x \tag{1.4a}$$

$$\frac{\partial T}{\partial \dot{y}} = m\dot{y} = p_y \tag{1.4b}$$

$$\frac{\partial T}{\partial \dot{z}} = m\dot{z} = p_z \tag{1.4c}$$

which means Newton's second law can be written as the following three non-vector equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = -\frac{\partial V}{\partial x} \tag{1.5a}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = -\frac{\partial V}{\partial y} \tag{1.5b}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) = -\frac{\partial V}{\partial z} \tag{1.5c}$$

which are now statements about the partial derivatives of the scalar functions $V(x,y,z,t)$ and $T(\dot{x},\dot{y},\dot{z})$.

This would be just an intellectual curiosity, but when dealing with systems best described in terms of something other than the Cartesian coördinates of a bunch of particles, it turns out to be easier to formulate a general prescription for the equations of motion similar to (1.5) than to resolve the vector equation (1.1) in general coördinates.

1.2 Examples of non-Cartesian Coördinates

To get a feel for this, we'll keep an eye on three alternate coördinate systems we've analyzed before, where we always had to start with Newton's laws in Cartesian coördinates and then make special allowances for the non-standard nature of the coördinates, rather than immediately writing down the components of an equation as in (1.2).

- First is the case of curvilinear coördinates, such as **polar coördinates** for a particle moving in a plane. These are related to ordinary Cartesian coördinates by

$$x = r\cos\phi \tag{1.6a}$$
$$y = r\sin\phi \tag{1.6b}$$

  Because $\vec{r} = x\hat{x} + y\hat{y} = r\hat{r}$ has no ϕ component, and because of the ϕ dependence of the unit vectors $\hat{r}$ and $\hat{\phi}$, the equations analogous to (1.2) do not hold:

$$m\ddot{r} \neq \hat{r}\cdot\vec{F} \tag{1.7a}$$
$$m\ddot{\phi} \neq \hat{\phi}\cdot\vec{F} \tag{1.7b}$$

  Instead, we had to consider the implicit time dependence of the unit vectors and read of the components of

$$\vec{F} = m\ddot{\vec{r}} = m(\ddot{r} - r\dot{\phi}^2)\hat{r} + m(r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi} \tag{1.8}$$

  to find

$$m\ddot{r} = \hat{r}\cdot\vec{F} + r\dot{\phi}^2 \tag{1.9a}$$
$$mr\ddot{\phi} = \hat{\phi}\cdot\vec{F} - 2\dot{r}\dot{\phi} \tag{1.9b}$$

- Another case which we just studied is **rotating coördinates**, e.g.,

$$x^* = x\cos\omega t + y\sin\omega t \tag{1.10a}$$
$$y^* = -x\sin\omega t + y\cos\omega t \tag{1.10b}$$

  Again, because of the time dependence of the starred basis vectors the equations of motion are not the same as in the non-rotating coördinates:

$$m\ddot{x}^* \neq \hat{x}^*\cdot\vec{F} \tag{1.11a}$$
$$m\ddot{y}^* \neq \hat{y}^*\cdot\vec{F} \tag{1.11b}$$

Instead, the modified equations of motion contained correction terms in the form of "fictitious forces" to adjust for the non-standard coördinates:

$$m\,[\ddot{x}\hat{x} + \ddot{y}\hat{y}] = \vec{F} - m[\omega\hat{z}^*] \times \left([\omega\hat{z}^*] \times [x^*\hat{x}^* + y^*\hat{y}^*]\right) - 2m[\omega\hat{z}^*] \times [\dot{x}^*\hat{x}^* + \dot{y}^*\hat{y}^*] \quad (1.12)$$

- Finally, it will sometimes also be useful to consider coördinates which can't be isolated to describe just one particle or another. For example, in the **two-body problem** we replaced the position vectors $\vec{r}_1$ and $\vec{r}_2$ of the two particles with the combinations

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \quad (1.13a)$$

$$\vec{R} = \frac{m_1\vec{r}_1 - m_2\vec{r}_2}{m_1 + m_2} \quad (1.13b)$$

  This is more complicated than just a different choice of basis vectors; how would one even define the "component" of a force corresponding to the coördinate $x = \hat{x} \cdot r$ or $Z = \hat{z} \cdot \vec{R}$.

In each of these cases, we started with fixed Cartesian coördinates, worked out the equations of motion, and translated them into the non-standard coördinates. Scalars like T and V can more easily be written down directly in non-standard coördinates, so if we can find an equivalent of (1.5) which works in arbitrary coördinates, we should be able to skip the step of working out the equations in Cartesian coördinates.

1.3 Derivation of Lagrange's Equations

1.3.1 Cartesian and Generalized Coördinates

In general, if we consider a system of N particles moving in three dimensions, there are $3N$ Cartesian coördinates needed to define the locations of all the particles. It will be convenient to define the notation $\{X_\ell | \ell = 1 \ldots 3N\}$, or simply $\{X_\ell\}$ for short, to refer to the set of $3N$ Cartesian coördinates. For concreteness, the correspondence is

$$X_1 = x_1, \qquad X_2 = y_1, \qquad X_3 = z_1, \qquad X_4 = x_2, \qquad \cdots, \qquad X_{3N} = z_N \quad (1.14)$$

Now, instead of these $3N$ Cartesian coördinates, we could use a different set of $3N$ quantities $\{q_k | k = 1 \ldots 3N\}$, called *generalized coördinates*. If we're dealing with two good sets of coördinates, we should be able to work out the generalized coördinates of a point given its Cartesian coördinates, i.e., for each k from 1 to $3N$, there should be a function

$$q_k = q_k(\{X_\ell\}, t) \quad (1.15)$$

- For example, in two-dimensional **polar coördinates**, we have "$3N''$ = 2, $X_1 = x$, $X_2 = y$, $q_1 = r$, $q_2 = \phi$ and the transformations

$$q_1(\{X_\ell\}, t) = r(x, y, t) = \sqrt{x^2 + y^2} \quad (1.16a)$$

$$q_2(\{X_\ell\}, t) = \phi(x, y, t) = \tan^{-1}\frac{y}{x} \quad (1.16b)$$

Similarly, the transformation (1.15) should be invertible to give a function

$$X_\ell = X_\ell(\{q_k\}, t) \tag{1.17}$$

for each ℓ from 1 to $3N$.

- For example, in two-dimensional **polar coördinates**,

$$X_1(\{q_k\}, t) = x(r, \phi, t) = r\cos\phi \tag{1.18a}$$
$$X_2(\{q_k\}, t) = y(r, \phi, t) = r\sin\phi \tag{1.18b}$$

- An example of a time-dependent mapping is from **rotating coördinates**, where

$$x^*(x, y, t) = x\cos\omega t + y\sin\omega t \tag{1.19a}$$
$$y^*(x, y, t) = -x\sin\omega t + y\cos\omega t \tag{1.19b}$$

1.3.2 Equations of Motion in Cartesian Coördinates

To write the multi-particle equivalent of (1.5), we first need to write the kinetic and potential energies in terms of the $3N$ coördinates (and their time derivatives).

The total kinetic energy of all the particles is

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i = \sum_{i=1}^{N} \left(\frac{1}{2} m_i \dot{x}_i^2 + \frac{1}{2} m_i \dot{y}_i^2 + \frac{1}{2} m_i \dot{x}_i^2 \right) \tag{1.20}$$

We can write this in terms of our notation for the full set of $3N$ Cartesian coördinates if we define $\{M_\ell | \ell = 1 \ldots 3N\}$ by

$$M_1 = m_1, \qquad M_2 = m_1, \qquad M_3 = m_1, \qquad M_4 = m_2, \qquad \cdots, \qquad M_{3N} = m_N \tag{1.21}$$

so that

$$T(\{\dot{X}_\ell\}) = \sum_{\ell=1}^{3N} \frac{1}{2} M_\ell \dot{X}_\ell^2 \tag{1.22}$$

Meanwhile, one can typically define an overall potential energy $V(\{X_\ell\}, t)$ such that the force $\vec{F}_i$ on the ith particle has components

$$F_{ix} = -\frac{\partial V}{\partial x_i} \tag{1.23a}$$
$$F_{iy} = -\frac{\partial V}{\partial y_i} \tag{1.23b}$$
$$F_{iz} = -\frac{\partial V}{\partial z_i} \tag{1.23c}$$

which makes Newton's laws

$$M_\ell \ddot{X}_\ell = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{X}_\ell} \right) = -\frac{\partial V}{\partial X_\ell} \tag{1.24}$$

1.3.3 Kinetic and Potential Energy in Generalized Coördinates

The kinetic energy T and potential energy V can be written in terms of either the Cartesian or generalized coördinates. Mathematically, this is a composition of functions. Formally, the potential energy is

$$V(\{q_k\}, t) = V\big(\{X_\ell(\{q_k\}, t)\}, t\big) \tag{1.25}$$

and the kinetic energy is

$$T(\{q_k\}, \{\dot{q}_k\}, t) = T\big(\{\dot{X}_\ell(\{q_k\}, \{\dot{q}_k\}, t)\}, t\big) \tag{1.26}$$

Note that while the kinetic energy written in Cartesian coördinates only depends on the "velocities" $\{\dot{X}_\ell\}$, when written in terms of the generalized coördinates, it depends in general on both the "velocities" $\{\dot{q}_k\}$ and coördinates $\{q_k\}$, because the Cartesian velocities $\{\dot{X}_\ell\}$ depend on both the generalized velocities $\{\dot{q}_k\}$ and the generalized coördinates $\{q_k\}$.

- This is easiest to see in an example. In **polar coördinates**, taking the time derivatives of (1.6) gives

$$\dot{x}(r, \phi, \dot{r}, \dot{\phi}) = \dot{r}\cos\phi - r\sin\phi\,\dot{\phi} \tag{1.27a}$$

$$\dot{y}(r, \phi, \dot{r}, \dot{\phi}) = \dot{r}\sin\phi + r\cos\phi\,\dot{\phi} \tag{1.27b}$$

  and substituting those into the kinetic energy gives

$$T(r, \phi, \dot{r}, \dot{\phi}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 \tag{1.28}$$

1.3.4 Applications of the Chain Rule

Since we have equations involving the derivatives of T and V with respect to $\{X_\ell\}$ and $\{\dot{X}_\ell\}$, and we're trying to construct analogous equations for their derivatives with respect to $\{q_k\}$ and $\{\dot{q}_k\}$, we want to use the relationships between these two sets of derivatives, which are a manifestation of the chain rule for multi-variable calculus.

The simplest one of these is for the potential energy. We are going back and forth between the two sets of arguments $\{X_\ell\}, t$ and $\{q_k\}, t$. This makes the derivative with respect to one of the $\{q_k\}$

$$\frac{\partial V}{\partial q_k} = \sum_{\ell=1}^{3N} \frac{\partial V}{\partial X_\ell}\frac{\partial X_\ell}{\partial q_k} + \frac{\partial V}{\partial t}\overbrace{\frac{\partial t}{\partial q_k}}^{0} \tag{1.29}$$

When we take partial derivatives with respect to a particular q_k, we hold constant all of the arguments in that set, i.e., $\{q_{k'} | k' \neq k\}$ as well as t, which is why the last term vanishes.

We can also use the chain rule to relate the generalized and Cartesian velocities. The velocity $\dot{X}_\ell = \frac{dX_\ell}{dt}$ is what's known as a *total derivative*. This is the change of the particle's coördinates with time as it follows its actual trajectory. In contrast, one can also take the partial derivative of the coördinate transformation $X_\ell(\{q_k\}, t)$ with respect to one of its arguments, holding all the other arguments constant to obtain $\frac{\partial X_\ell}{\partial q_k}$ or $\frac{\partial X_\ell}{\partial t}$. The chain rule is used to find the total derivative of

$$X_\ell(t) = X_\ell\big(\{q_k(t)\}, t\big) \tag{1.30}$$

taking into consideration both the implicit time dependence via $q_k(t)$ and the explicit time dependence. Thus

$$\dot{X}_\ell = \frac{dX_\ell}{dt} = \sum_{k=1}^{3N} \frac{\partial X_\ell}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial X_\ell}{\partial t}\frac{dt}{dt} = \sum_{k=1}^{3N} \frac{\partial X_\ell}{\partial q_k}\dot{q}_k + \frac{\partial X_\ell}{\partial t} \tag{1.31}$$

- Explicitly, in the example of **rotating coördinates**, the partial derivatives of

$$x = x^* \cos\omega t - y^* \sin\omega t \tag{1.32a}$$
$$y = x^* \sin\omega t + y^* \cos\omega t \tag{1.32b}$$

are

$$\frac{\partial x}{\partial x^*} = \cos\omega t \tag{1.33a}$$
$$\frac{\partial x}{\partial y^*} = -\sin\omega t \tag{1.33b}$$
$$\frac{\partial x}{\partial t} = -\omega x^* \sin\omega t - \omega y^* \cos\omega t \tag{1.33c}$$
$$\frac{\partial y}{\partial x^*} = \sin\omega t \tag{1.33d}$$
$$\frac{\partial y}{\partial y^*} = \cos\omega t \tag{1.33e}$$
$$\frac{\partial y}{\partial t} = \omega x^* \cos\omega t - \omega y^* \sin\omega t \tag{1.33f}$$
$$\tag{1.33g}$$

and the total derivatives are

$$\dot{x} = \overbrace{\dot{x}^* \cos\omega t}^{\frac{\partial x}{\partial x^*}\frac{dx^*}{dt}} \overbrace{-\dot{y}^* \sin\omega t}^{\frac{\partial x}{\partial y^*}\frac{dy^*}{dt}} \overbrace{-\omega x^* \sin\omega t - \omega y^* \cos\omega t}^{\frac{\partial x}{\partial t}} \tag{1.34a}$$
$$\dot{y} = \underbrace{\dot{x}^* \sin\omega t}_{\frac{\partial y}{\partial x^*}\frac{dx^*}{dt}} + \underbrace{\dot{y}^* \cos\omega t}_{\frac{\partial y}{\partial y^*}\frac{dy^*}{dt}} + \underbrace{\omega x^* \cos\omega t - \omega y^* \sin\omega t}_{\frac{\partial y}{\partial t}} \tag{1.34b}$$

1.3.5 Partial Derivatives of the Kinetic and Potential Energy

In Cartesian coördinates, the kinetic energy T is a function only of the velocities $\{\dot{X}_\ell\}$:

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_1\dot{z}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \ldots \frac{1}{2}m_N\dot{z}_N^2 = \sum_{\ell=1}^{3N} \frac{1}{2}M_\ell \dot{X}_\ell^2 \tag{1.35}$$

(where we have defined the notation $M_1 = M_2 = M_3 = m_1$, $M_4 = M_5 = M_6 = m_2$, $\ldots M_{3N-2} = M_{3N-1} = M_{3N} = m_N$). But since the velocities $\{\dot{X}_\ell\}$ are functions in general of the generalized coördinates $\{q_k\}$ and time t as well as the generalized velocities $\{\dot{q}_k\}$, the kinetic energy depends on all of those things when written in generalized coördinates:

$$T(\{\dot{X}_\ell\}) = T(\{q_k\}, \{\dot{q}_k\}, t) \tag{1.36}$$

So the derivative we're interested in is $\frac{\partial T}{\partial \dot{q}_k}$, a partial derivative with respect to one $\dot{q}_k$ with all the other $\dot{q}_{k'}$s, plus *all* of the $q_{k'}$s and t treated as constants. The chain rule tells us that

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_{\ell=1}^{3N} \frac{\partial T}{\partial \dot{X}_\ell} \frac{\partial \dot{X}_\ell}{\partial \dot{q}_k} \tag{1.37}$$

So the thing we need to relate one set of partial derivatives to the other is $\frac{\partial \dot{X}_\ell}{\partial \dot{q}_k}$ for each possible choice of ℓ and k. We can actually look explicitly at the $\dot{q}_k$ dependence by writing out (1.31):

$$\dot{X}_\ell = \frac{\partial X_\ell}{\partial q_1}\dot{q}_1 + \frac{\partial X_\ell}{\partial q_2}\dot{q}_2 + \ldots + \frac{\partial X_\ell}{\partial q_{3N}}\dot{q}_{3N} + \frac{\partial X_\ell}{\partial t} \tag{1.38}$$

As we've noted before, all the coëfficients like $\frac{\partial X_\ell}{\partial q_1}$ and $\frac{\partial X_\ell}{\partial t}$ will depend only on $\{q_{k'}\}$ and t, and not on $\dot{q}_k$. In fact, for each possible value of k, only one term contains $\dot{q}_k$; for example, if $k = 5$ the only term which gives a non-zero contribution to the partial derivative $\frac{\partial \dot{X}_\ell}{\partial \dot{q}_5}$ is

$$\frac{\partial \dot{X}_\ell}{\partial \dot{q}_5} = \frac{\partial}{\partial \dot{q}_5}\left(\frac{\partial X_\ell}{\partial q_5}\dot{q}_5\right) = \frac{\partial X_\ell}{\partial q_5} \tag{1.39}$$

So for a general k, we find that

$$\frac{\partial \dot{X}_\ell}{\partial \dot{q}_k} = \frac{\partial X_\ell}{\partial q_k} \tag{1.40}$$

which means that

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_{\ell=1}^{3N} \frac{\partial T}{\partial \dot{X}_\ell} \frac{\partial \dot{X}_\ell}{\partial \dot{q}_k} = \sum_{\ell=1}^{3N} \frac{\partial T}{\partial \dot{X}_\ell} \frac{\partial X_\ell}{\partial q_k} \tag{1.41}$$

So now we have $\frac{\partial T}{\partial \dot{q}_k}$ from (1.41) and $\frac{\partial V}{\partial q_k}$ from (1.29). To write something analogous to (1.26), we need to take the (total) time derivative of this. So we differentiate (1.41), using the sum and product rules:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) &= \sum_{\ell=1}^{3N}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}_\ell}\right)\frac{\partial X_\ell}{\partial q_k} + \frac{\partial T}{\partial \dot{X}_\ell}\frac{d}{dt}\left(\frac{\partial X_\ell}{\partial q_k}\right)\right] \\ &= \sum_{\ell=1}^{3N}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}_\ell}\right)\frac{\partial X_\ell}{\partial q_k} + \sum_{\ell=1}^{3N}\frac{\partial T}{\partial \dot{X}_\ell}\frac{d}{dt}\left(\frac{\partial X_\ell}{\partial q_k}\right) \end{aligned} \tag{1.42}$$

The first sum can be simplified by using the equation of motion (1.26) and the chain rule:

$$\sum_{\ell=1}^{3N}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}_\ell}\right)\frac{\partial X_\ell}{\partial q_k} = \sum_{\ell=1}^{3N}\left(-\frac{\partial V}{\partial X_\ell}\right)\frac{\partial X_\ell}{\partial q_k} = -\frac{\partial V}{\partial q_k} \tag{1.43}$$

The second sum, which is the correction arising from the use of generalized coördinates, can be evaluated by noting that $\frac{\partial X_\ell}{\partial q_k}$, like X_ℓ, is a function of all the $q_{k'}$s as well as time, so we can write its total time derivative as

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial X_\ell}{\partial q_k}\right) &= \sum_{k'=1}^{3N} \frac{\partial}{\partial q_{k'}}\left(\frac{\partial X_\ell}{\partial q_k}\right)\frac{dq_{k'}}{dt} + \frac{\partial}{\partial t}\left(\frac{\partial X_\ell}{\partial q_k}\right) = \sum_{k'=1}^{3N}\left(\frac{\partial^2 X_\ell}{\partial q_{k'}\partial q_k}\right)\dot{q}_{k'} + \left(\frac{\partial X_\ell}{\partial t \partial q_k}\right) \\ &= \frac{\partial}{\partial q_k}\underbrace{\left[\sum_{k'=1}^{3N}\left(\frac{\partial X_\ell}{\partial q_{k'}}\right)\dot{q}_{k'} + \left(\frac{\partial^2 X_\ell}{\partial t}\right)\right]}_{\frac{dX_\ell}{dt}} = \frac{\partial \dot{X}_\ell}{\partial q_k}\end{aligned} \tag{1.44}$$

Plugging this back into (1.42) gives us

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = -\frac{\partial V}{\partial q_k} + \sum_{\ell=1}^{3N}\frac{\partial T}{\partial \dot{X}_\ell}\frac{\partial \dot{X}_\ell}{\partial q_k} = -\frac{\partial V}{\partial q_k} + \frac{\partial T}{\partial q_k} \tag{1.45}$$

where in the last step we have used the chain rule applied to

$$T(\{q_k\},\{\dot{q}_k\},t) = T\left(X_\ell(\{\{q_k\},\{\dot{q}_k\},t)\right) \tag{1.46}$$

So this means the equations of motion (Newton's second law) are equivalent to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)\frac{\partial}{\partial q_k}(T-V) \tag{1.47}$$

Now, since V is just a potential energy and depends only on the coördinates and time,

$$\frac{\partial V}{\partial \dot{q}_k} = 0 \tag{1.48}$$

so we can make the equation look more symmetric by adding zero to the left-hand side and writing

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_k}(T-V)\right) = \frac{\partial}{\partial q_k}(T-V) \tag{1.49}$$

The quantity appearing in parentheses on both sides of the equation is called the Lagrangian

$$L = T - V \tag{1.50}$$

Note that this is kinetic *minus* potential energy, as opposed to the total energy, which is kinetic *plus* potential. We've now shown that for any set of generalized coördinates[1], Newton's second law is equivalent to the full set of *Lagrange Equations*:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad k = 1, 2, \ldots, 3N \tag{1.51}$$

This is the generic form we were looking for. All we need to do is write the kinetic and potential energies in terms of the generalized coördinates, their time derivatives, and time, and we can find the equations of motion by taking partial derivatives without ever needing to use inertial Cartesian coördinates.

[1] which can in principle be related by a coördinate transformation to the Cartesian coördinates for a problem with forces arising solely from a potential energy

1.4 Examples

Let's return to our three examples of generalized coördinates and see how we obtain the correct equations of motion in each case:

1.4.1 Polar Coördinates

Recall that here the total number of coördinates "$3N$" is 2, and the Cartesian and generalized coördinates are

$$\begin{aligned} X_1 &= x \qquad & q_1 &= r \\ X_2 &= y \qquad & q_2 &= \phi \end{aligned}$$

The kinetic energy is

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 \tag{1.53}$$

which we can obtain either directly by considering the infinitesimal distance

$$ds^2 = dr^2 + r^2\, d\phi^2 \tag{1.54}$$

associated with changes in r and ϕ, or by starting with

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \tag{1.55}$$

and making the substitutions

$$\dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi \tag{1.56a}$$

$$\dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi \tag{1.56b}$$

Meanwhile the potential energy $V(r, \phi)$ is a scalar field which can be described equally well as a function of the polar coördinates. So the Lagrangian is

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - V(r, \phi) \tag{1.57}$$

Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \tag{1.58a}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \tag{1.58b}$$

so to apply them we need to take the relevant partial derivatives:

$$\frac{\partial L}{\partial r} = mr\dot{\phi}^2 - \frac{\partial V}{\partial r} \tag{1.59a}$$

$$\frac{\partial L}{\partial \phi} = -\frac{\partial V}{\partial \phi} \tag{1.59b}$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \text{radial compt of momentum} \tag{1.59c}$$

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} \qquad \text{angular momentum} \tag{1.59d}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r} \tag{1.59e}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = mr^2\ddot{\phi} + 2mr\dot{r}\dot{\phi} \tag{1.59f}$$

So the equations of motion are

$$m\ddot{r} = mr\dot{\phi}^2 - \frac{\partial V}{\partial r} \tag{1.60a}$$

$$mr^2\ddot{\phi} + 2mr\dot{r}\dot{\phi} = -\frac{\partial V}{\partial \phi} \tag{1.60b}$$

We can verify that these are the same equations of motion given by

$$m\ddot{\vec{r}} = -\vec{\nabla}V \tag{1.61}$$

in the vector approach, in which

$$-\vec{\nabla}V = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{\phi} \tag{1.62}$$

and, thanks to the position and therefore time-dependent basis vectors

$$\frac{d\hat{r}}{dt} = \dot{\phi}\hat{\phi} \tag{1.63a}$$

$$\frac{d\hat{\phi}}{dt} = -\dot{\phi}\hat{r} \tag{1.63b}$$

the acceleration is calculated via

$$\vec{r} = r\hat{r} \qquad \Rightarrow \qquad \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} \qquad \Rightarrow \qquad \ddot{\vec{r}} = \ddot{r}\hat{r} + 2\dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} - r\dot{\phi}^2\hat{r} \tag{1.64}$$

to give the equations of motion

$$m\ddot{\vec{r}} = \left(m\ddot{r} - mr\dot{\phi}^2\right)\hat{r} + \left(mr\ddot{\phi} + 2m\dot{r}\dot{\phi}\right) \tag{1.65}$$

Equating the r and ϕ components of the vector expressions (1.65) and (1.62) gives the same equations of motion as (1.60).

1.4.2 Rotating Coördinates

Here again the total number of coördinates "$3N$" is 2, and the Cartesian and generalized coördinates are

$$\begin{aligned} X_1 &= x & q_1 &= x^* \\ X_2 &= y & q_2 &= y^* \end{aligned}$$

To get the kinetic energy, we really need to start with the inertial form and transform it using

$$\dot{x} = (\dot{x}^* - \omega y^*)\cos\omega t - (\dot{y}^* + \omega x^*)\sin\omega t \tag{1.67a}$$

$$\dot{y} = (\dot{x}^* - \omega y^*)\sin\omega t - (\dot{y}^* + \omega x^*)\cos\omega t \tag{1.67b}$$

which makes the kinetic energy

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2) = \frac{1}{2}m(\dot{x}^* - \omega y^*)^2 + \frac{1}{2}m(\dot{y}^* + \omega x^*)^2 \\ &= \frac{m}{2}\left[(\dot{x}^*)^2 + (\dot{y}^*)^2 - 2\omega\dot{x}^* y^* + 2\omega\dot{y}^* x^* + \omega^2(x^*)^2 + \omega^2(y^*)^2\right] \end{aligned} \tag{1.68}$$

and the Lagrangian

$$L = T - V(x^*, y^*, t) \tag{1.69}$$

so that the partial derivatives are

$$\frac{\partial L}{\partial x^*} = m\omega\dot{y}^* + m\omega^2 x^* - \frac{\partial V}{\partial x^*} \tag{1.70a}$$

$$\frac{\partial L}{\partial y^*} = -m\omega\dot{x}^* + m\omega^2 y^* - \frac{\partial V}{\partial y^*} \tag{1.70b}$$

$$\frac{\partial L}{\partial \dot{x}^*} = m\dot{x}^* - m\omega y^* \tag{1.70c}$$

$$\frac{\partial L}{\partial \dot{y}^*} = m\dot{y}^* + m\omega x^* \tag{1.70d}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^*}\right) = m\ddot{x}^* - m\omega\dot{y}^* \tag{1.70e}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^*}\right) = m\ddot{y}^* + m\omega\dot{y}^* \tag{1.70f}$$

So that the equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^*} - \frac{\partial L}{\partial x^*} = m\ddot{x}^* - \left(2m\omega\dot{y}^* + m\omega^2 x^* - \frac{\partial V}{\partial x^*}\right) = 0 \tag{1.71a}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}^*} - \frac{\partial L}{\partial y^*} = m\ddot{y}^* - \left(-2m\omega\dot{x}^* + m\omega^2 y^* - \frac{\partial V}{\partial y^*}\right) = 0 \tag{1.71b}$$

And one can easily check (exercise!) that these equations of motion are the same as those given by the non-inertial coördinate system method including fictitious forces.

1.4.3 Two-Body Problem

In this case, $3N = 6$ and the Cartesian coördinates are the six components of the two position vectors $\vec{r}_1$ and $\vec{r}_2$, while the generalized coördinates are the six components of the vectors $\vec{R}$ and $\vec{r}$. The most interesting part about the construction of the Lagrangian is the kinetic energy, which can be obtained by differentiating the inverse coördinate transformations:

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1 + m_2}\vec{r} \tag{1.72a}$$

$$\vec{r}_2 = \vec{R} - \frac{m_1}{m_1 + m_2}\vec{r} \tag{1.72b}$$

to get

$$\dot{\vec{r}}_1 = \dot{\vec{R}} + \frac{m_2}{m_1 + m_2}\dot{\vec{r}} \tag{1.73a}$$

$$\dot{\vec{r}}_2 = \dot{\vec{R}} - \frac{m_1}{m_1 + m_2}\dot{\vec{r}} \tag{1.73b}$$

The kinetic energy is thus

$$\begin{aligned} T = \frac{1}{2}m_1\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_1 + \frac{1}{2}m_2\dot{\vec{r}}_2 \cdot \dot{\vec{r}}_2 &= \frac{1}{2}m_1\dot{\vec{R}} \cdot \dot{\vec{R}} + \cancel{\frac{m_1 m_2}{m_1 + m_2}\dot{\vec{r}} \cdot \dot{\vec{R}}} + \frac{1}{2}\frac{m_1 m_2^2}{(m_1 + m_2)^2}\dot{\vec{r}} \cdot \dot{\vec{r}} \\ &+ \frac{1}{2}m_2\dot{\vec{R}} \cdot \dot{\vec{R}} - \cancel{\frac{m_1 m_2}{m_1 + m_2}\dot{\vec{r}} \cdot \dot{\vec{R}}} + \frac{1}{2}\frac{m_1^2 m_2}{(m_1 + m_2)^2}\dot{\vec{r}} \cdot \dot{\vec{r}} \\ = \frac{1}{2}(m_1 + m_2)\dot{\vec{R}} \cdot \dot{\vec{R}} + \frac{1}{2}\frac{\cancel{(m_1 + m_2)}m_1 m_2}{(m_1 + m_2)^{\cancel{2}}}\dot{\vec{r}} \cdot \dot{\vec{r}} &= \frac{1}{2}M\dot{\vec{R}} \cdot \dot{\vec{R}} + \frac{1}{2}\mu\dot{\vec{r}} \cdot \dot{\vec{r}} \end{aligned} \tag{1.74}$$

And this is actually a more direct way of showing the equivalence to the one-body problem. Furthermore, if the potential energy depends only on the distance $|\vec{r}_1 - \vec{r}_2| = |\vec{r}| = r$ between the two bodies (central force motion with no external forces), the Lagrangian becomes

$$L = \frac{1}{2}M\dot{\vec{R}} \cdot \dot{\vec{R}} + \frac{1}{2}\mu\dot{\vec{r}} \cdot \dot{\vec{r}} - V(r) \tag{1.75}$$

2 Systems Subject to Constraints

2.0 Recap

Recall that a system of N particles moving in 3 dimensions has a set of $3N$ Cartesian coördinates $\{X_\ell\}$:

$$X_1 = x_1, \qquad X_2 = y_1, \qquad X_3 = z_1, \qquad \ldots \qquad X_{3N} = z_N \tag{2.1}$$

Newton's second law in in the presence of a potential V is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}_\ell}\right) = M_\ell \ddot{X}_\ell = -\frac{\partial V}{\partial X_\ell} \qquad \ell = 1, 2, \ldots 3N \tag{2.2}$$

where

$$T(\{\dot{X}_\ell\}) = \sum_{\ell=1}^{N} \frac{1}{2} M_\ell (\dot{X}_\ell)^2 \tag{2.3}$$

is the kinetic energy, and

$$M_1 = M_2 = M_3 = m_1, \qquad \ldots \qquad M_{3N-2} = M_{3N-1} = M_{3N} = m_N \tag{2.4}$$

This is a special case of the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_\ell}\right) - \frac{\partial L}{\partial X_\ell} = 0 \qquad \ell = 1, 2, \ldots 3N \tag{2.5}$$

where $L = T - V$ is the Lagrangian. Similar equations hold in any set of $3N$ "generalized" coördinates $\{q_k\}$, i.e.,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad k = 1, 2, \ldots 3N \tag{2.6}$$

2.1 The Springy Pendulum

As an example, consider a pointlike pendulum bob of mass m attached to the end of a spring of spring constant k and unstretched length b, moving in a plane under the influence of a constant gravitational field of magnitude g. Choose as generalized coördinates the (actual, instantaneous) length ℓ of the spring and the angle α between the spring and the vertical. These are something like polar coördinates, so the kinetic energy of the bob is

$$T = \frac{1}{2} m \dot{\ell}^2 + \frac{1}{2} m \ell^2 \dot{\alpha}^2 \tag{2.7}$$

There are two sources of potential energy: gravity and the spring. The gravitational potential energy can be worked out with a little trigonometry as $-mg\ell\cos\alpha$, while the potential energy in the spring is $\frac{1}{2}k(\ell - b)^2$. So the Lagrangian is

$$L = \frac{1}{2} m \dot{\ell}^2 + \frac{1}{2} m \ell^2 \dot{\alpha}^2 + mg\ell\cos\alpha - \frac{1}{2}k(\ell - b)^2 \tag{2.8}$$

The relevant derivatives are

$$\frac{\partial L}{\partial \ell} = m\ell\dot{\alpha}^2 + mg\cos\alpha - k(\ell - b) \tag{2.9a}$$

$$\frac{\partial L}{\partial \alpha} = -mg\sin\alpha \tag{2.9b}$$

$$\frac{\partial L}{\partial \dot{\ell}} = m\dot{\ell} \tag{2.9c}$$

$$\frac{\partial L}{\partial \dot{\alpha}} = m\ell^2\dot{\alpha} \tag{2.9d}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\ell}}\right) = m\ddot{\ell} \tag{2.9e}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}}\right) = m\ell^2\ddot{\alpha} + 2m\ell\dot{\ell}\dot{\alpha} \tag{2.9f}$$

which makes the equations of motion

$$m\ddot{\ell} = m\ell\dot{\alpha}^2 + mg\cos\alpha - k(\ell - b) \tag{2.10a}$$

$$m\ell^2\ddot{\alpha} + 2m\ell\dot{\ell}\dot{\alpha} = -mg\ell\sin\alpha \tag{2.10b}$$

Note that, if we define a unit vector $\hat{\ell}$ in the direction of increasing ℓ (i.e., radially outward), the force due to the spring is

$$\vec{F}_{\text{spring}} = -k(\ell - b)\hat{\ell} \tag{2.11}$$

and this is reflected by a $-k(\ell - b)$ which appears in the $\ddot{\ell}$ equation.

What if there were a rod rather than a spring, so that the ℓ coördinate were fixed to be b? Physically, the rod would provide an additional radial force (pushing or tension) which was just what was needed to keep ℓ constant.

The key thing about a constraining force (e.g., tension, normal force, static friction) is that we don't know it *a priori* at each instant or at each point in space. It has to be just enough to keep the particle's motion consistent with the constraint.

If we write the tension in the rod as λ (which could be negative if the rod is pushing rather than pulling on the pendulum bob), the equations of motion are

$$m\ddot{\ell} = m\ell\dot{\alpha}^2 + mg\cos\alpha - \lambda \tag{2.12a}$$

$$m\ell^2\ddot{\alpha} + 2m\ell\dot{\ell}\dot{\alpha} = -mg\ell\sin\alpha \tag{2.12b}$$

And additionally there is a constraint

$$\ell = b \tag{2.12c}$$

The tension λ at any time is only worked out after the fact from the actual trajectory.

There are two ways to obtain the equations of motion (2.12) from a Lagrangian formalism:

1) Since one of the coördinates is a constant, we can make the substitutions $\ell = b$, $\dot{\ell} = 0$, and $\ddot{\ell} = 0$ into (2.12b) and get

$$mb^2\ddot{\alpha} = -mgb\sin\alpha \tag{2.13}$$

which are Lagrange's equations if we start with the reduced Lagrangian

$$L^{\text{red}}(\alpha, \dot{\alpha}, t) = \frac{1}{2}mb^2\dot{\alpha}^2 + mgb\cos\alpha \tag{2.14}$$

In the reduced Lagrangian, the constraint is satisfied and only the unconstrained "degree of freedom" α is treated as a generalized coördinate.

2) We can instead treat ℓ, α, *and* λ as the variables and look for a Lagrangian $\widetilde{L}(\ell, \alpha, \lambda, \dot{\ell}, \dot{\alpha}, \dot{\lambda}, t)$ which gives the equations of motion (2.12).

If we note that

$$-\lambda = \frac{\partial}{\partial\ell}(-\ell\lambda) \tag{2.15}$$

we see that

$$m\ddot{\ell} - m\ell\dot{\alpha}^2 = mg\cos\alpha - \lambda = \frac{\partial}{\partial\ell}\left(mg\ell\cos\alpha - \lambda\ell\right) \tag{2.16}$$

so we would get the right equation for $\ddot{\ell}$ if we add $-\lambda\ell$ to the Lagrangian:

$$\widetilde{L} \stackrel{?}{=} \frac{1}{2}m\dot{\ell}^2 + \frac{1}{2}m\ell^2\dot{\alpha}^2 + mg\ell\cos\alpha - \lambda\ell \tag{2.17}$$

That's not quite right, though, since if we take the derivatives associated with the tension, we get

$$\frac{\partial\widetilde{L}}{\partial\lambda} = -\ell \tag{2.18a}$$

$$\frac{\partial\widetilde{L}}{\partial\dot{\lambda}} = 0 \tag{2.18b}$$

$$\frac{d}{dt}\left(\frac{\partial\widetilde{L}}{\partial\dot{\lambda}}\right) = 0 \tag{2.18c}$$

since the constraint is $\ell = b$, not $\ell = 0$. But if we add $\lambda(b-\ell)$ instead of $-\lambda\ell$, we don't change $\frac{\partial\widetilde{L}}{\partial\lambda}$ and we do get

$$\frac{\partial\widetilde{L}}{\partial\lambda} = b - \ell = 0 = \frac{d}{dt}\left(\frac{\partial\widetilde{L}}{\partial\dot{\ell}}\right) \tag{2.19}$$

so the constraint is just the Lagrange equation associated with the unknown tension λ.

So we can describe the pendulum with the Lagrangian

$$\widetilde{L} = \underbrace{\frac{1}{2}m\dot{\ell}^2 + \frac{1}{2}m\ell^2\dot{\alpha}^2}_{T} + \underbrace{mg\ell\cos\alpha}_{-V} + \underbrace{\lambda}_{\substack{\text{Lagrange}\\\text{multiplier}}} \underbrace{(b-\ell)}_{\text{constraint}} \tag{2.20}$$

In Cartesian coördinates (centered at the support point of the pendulum) the constraint is

$$b - \sqrt{x^2+y^2} = 0 \tag{2.21}$$

so we can't just eliminate one coördinate by setting it to a constant. In this case, the constraining force has components in both directions. Since

$$\cos\alpha = -\frac{y}{\sqrt{x^2+y^2}} \tag{2.22a}$$

$$\sin\alpha = \frac{x}{\sqrt{x^2+y^2}} \tag{2.22b}$$

if we call the tension $\vec{T}$ and let λ be $\pm\left|\vec{T}\right|$ (with the sign depending on whether $\vec{T}$ is a pull or a push), then

$$T_x = -\lambda\sin\alpha = -\frac{x}{x^2+y^2} = \lambda\frac{\partial}{\partial x}\left(b - \sqrt{x^2+y^2}\right) \tag{2.23a}$$

$$T_y = \lambda\cos\alpha = -\frac{y}{x^2+y^2} = \lambda\frac{\partial}{\partial y}\left(b - \sqrt{x^2+y^2}\right) \tag{2.23b}$$

so, again we get the right equations by adding λ times the constraint to the Lagrangian:

$$\widetilde{L}(x,y,\lambda,\dot{x},\dot{y},t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + mgy + \lambda\left(b - \sqrt{x^2+y^2}\right) \tag{2.24}$$

The circle defined by $\ell = b$ is called a *surface of constraint*. The tension in the rod is directed radially, i.e., perpendicular to that circle. In polar coördinates, we can write the constraint as $h(r,\phi) = b - r = 0$ and note that $\vec{\nabla}h = \hat{r}$ is perpendicular to the surface of the constraint, so the constraining force $-\lambda\hat{r}$ is parallel to $\vec{\nabla}h$.

2.2 Constrained Systems in General

Think about the general case with $3N$ Cartesian cördinates for N particles and require these to satisfy c constraints

$$h_1(\{X_\ell\}) = 0 \tag{2.25a}$$

$$h_2(\{X_\ell\}) = 0 \tag{2.25b}$$

$$\vdots$$

$$h_c(\{X_\ell\}) = 0 \tag{2.25c}$$

Each constraint has an associated constraining force which acts on each particle. For one particle, the force corresponding to the jth constraint, which might be called $\vec{F}_j$, will be perpendicular to the surface $h_j(\vec{r}) = 0$, and thus be parallel to $\vec{\nabla}h_j$. so we could write it as $\vec{F}_j^{\text{const}} = \lambda_j\vec{\nabla}h_j$. For N particles, the force associated with the jth constraint, acting on the ith particle, will be

$$\vec{F}_{ji}^{\text{const}} = \lambda_j\vec{\nabla}_i h_j \tag{2.26}$$

In our current notation for Cartesian coördinates, this makes the equations of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{X}_\ell}\right) = M_\ell\ddot{X}_\ell = -\frac{\partial V}{\partial X_\ell} + \sum_{j=1}^{c}\lambda_j\frac{\partial h_j}{\partial X_\ell} \tag{2.27}$$

Now, although we know the directions of the constraining forces, their strengths $\{\lambda_j|j = 1\ldots c\}$ are unknown, which means there are $3N + c$ unknowns

$$X_1, X_2, \ldots, X_{3N}, \lambda_1, \lambda_2, \ldots, \lambda_c \tag{2.28}$$

We have a total of $3N + c$ equations of motion: the $3N$ dynamical equations (2.27) plus the c constraints (2.25). Now, we can turn the unconstrained differential equations into the constrained ones by replacing V with $V - \sum_{j=1}^{c}\lambda_j h_j$ so if we define a *modified Lagrangian*

$$\widetilde{L} = L + \sum_{j=1}^{c}\lambda_j h_j = T - V + \sum_{j=1}^{c}\lambda_j h_j \tag{2.29}$$

we will get the right equations of motion. First, if we hold all the $\{\lambda_j\}$ constant in the derivatives,

$$\frac{d}{dt}\left(\frac{\partial\widetilde{L}}{\partial\dot{X}_\ell}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{X}_\ell}\right) = M\ddot{X}_\ell = -\frac{\partial V}{\partial X_\ell} + \sum_{j=1}^{c}\lambda_j\frac{\partial h_j}{\partial X_\ell} = \frac{\partial\widetilde{L}}{\partial X_\ell} \tag{2.30}$$

On the other hand if we take a partial derivative with respect to each λ_j, holding the $\{X_\ell\}$ and $\{\dot{X}_\ell\}$ constant,

$$\frac{\partial \widetilde{L}}{\partial X_\ell} = \sum_{j'=1}^{c} \underbrace{\frac{\partial \lambda_{j'}}{\partial \lambda_j}}_{\substack{0 \text{ if } j'\neq j \\ 1 \text{ if } j'=j}} h_{j'}(\{X_\ell\}) = h_j(\{X_\ell\}) = 0 = \frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{\lambda}_j}\right) \tag{2.31}$$

So the $3N + c$ equations of motion are

$$\frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{X}_\ell}\right) - \frac{\partial \widetilde{L}}{\partial X_\ell} = 0 \qquad \ell = 1, 2, \dots 3N \tag{2.32a}$$

$$\frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{\lambda}_k}\right) - \frac{\partial \widetilde{L}}{\partial \lambda_k} = 0 \qquad k = 1, 2, \dots c \tag{2.32b}$$

This is called the method of *Lagrange undetermined multipliers*. The modified Lagrangian is a function of $3N+c$ "coördinates" $\{X_1, X_2, \dots, X_{3N}, \lambda_1, \lambda_2, \dots, \lambda_c\}$ and their time derivatives (it happens to be independent of $\{\dot{\lambda}_j\}$)

$$\widetilde{L}(\{X_\ell\}, \{\lambda_j\}, \{\dot{X}_\ell\}, t) = L(\{X_\ell\}, \{\lambda_j\}, \{\dot{X}_\ell\}, t) + \sum_{j=1}^{c} \lambda_j h_j(\{X_\ell\}) \tag{2.33}$$

Now, we could replace these $3N+c$ coördinates with $3N+c$ different ones, and the equations of motion would still come from applying Lagrange's equations to $\widetilde{L}$.

Suppose we manage to choose those in such a way that c of them are just the constraints, c of them are still the Lagrange multipliers, and the remaining

$$3N - c =: f \tag{2.34}$$

are something else. Then f is the number of "degrees of freedom" which is just the number of unconstrained "directions" the system can move in. the old and new coördinates would be

Old:	X_1	X_2	$\dots$	X_f	X_{f+1}	$\dots$	X_{f+c}	λ_1	$\dots$	λ_c
New:	q_1	q_2	$\dots$	q_f	a_1	$\dots$	a_c	λ_1	$\dots$	λ_c

Substituting the inverse transformations $X_\ell = X_\ell(\{q_k | k = 1 \dots f\}, \{a_j | j = 1 \dots c\})$ into the modified Lagrangian, we'd get the form

$$\widetilde{L}(\{q_k\}, \{a_j\}, \{\lambda_j\}, \{\dot{q}_k\}, \{\dot{a}_j\}, t) = L(\{q_k\}, \{a_j\}, \{\dot{q}_k\}, \{\dot{a}_j\}, t) + \sum_{j=1}^{c} \lambda_j a_j \tag{2.35}$$

The Lagrange equations are then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} = \frac{\partial \widetilde{L}}{\partial q_k} \tag{2.36a}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}_j}\right) = \frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{a}_j}\right) = \frac{\partial L}{\partial a_j} = \frac{\partial \widetilde{L}}{\partial a_j} + \lambda_j \tag{2.36b}$$

$$0 = \frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{\lambda}_j}\right) = \frac{\partial L}{\partial \lambda_j} = a_j \tag{2.36c}$$

- Now, the last set of Lagrange equations tell us we can impose the constraints and set $a_j = 0$ for $j = 1 \ldots c$ in the other two.
- The second set is only really needed if we care about the constraining forces, i.e., the Lagrange multipliers λ_j.
- The first set tells us

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k}\right]_{\substack{a_1=0 \\ \vdots \\ a_c=0}} = 0 \qquad k = 1 \ldots f \tag{2.37}$$

If we define the *reduced Lagrangian*

$$L^{\text{red}}(\{q_k\}, \{\dot{q}_k\}, t) = L(\{q_k\}, \{a_j = 0\}, \{\dot{q}_k\}, \{\dot{a}_j = 0\}, t) \tag{2.38}$$

then the Lagrange equations are equivalent to

$$\frac{d}{dt}\left(\frac{\partial L^{\text{red}}}{\partial \dot{q}_k}\right) - \frac{\partial L^{\text{red}}}{\partial q_k} = 0 \qquad k = 1 \ldots f \tag{2.39}$$

This means that if a system is constrained to have only f degrees of freedom, we can formulate the Lagrangian as a function of the f generalized coördinates and apply the usual Lagrange equations. as long as we don't care about the constraining forces.

2.3 Example: The Atwood Machine

As an example of how to apply the method of Lagrange multipliers to a constrained system, consider the Atwood machine. This consists of two blocks, of masses m_1 and m_2, at opposite ends of a massless rope of (fixed) length ℓ, hung over a massless, frictionless pulley, subject to a uniform gravitational field of strength g. This is illustrated in Symon's Fig. 9.5. Symon considers this system in the reduced Lagrangian approach in Section 9.5.

We are looking for the acceleration of each block, and possibly also the tension in the rope.

2.3.1 Lagrange Multiplier Approach

Here we will use the modified Lagrangian approach, with a Lagrange multiplier corresponding to the unknown constraining force, here provided by the tension in the rope.

We define generalized coördinates x_1 and x_2, which are both supposed to be positive, and refer to the distance of each block *below* the pulley. The constraint here is that the fixed length of the rope is $\ell = x_1 + x_2$. To construct the modified Lagrangian, we need the kinetic energy

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2 \tag{2.40}$$

and the potential energy

$$V = -m_1 g x_1 - m_2 g x_2 \tag{2.41}$$

(since the height of each block *above* the pulley is $-x_1$ or $-x_2$). The modified Lagrangian is then

$$\widetilde{L}(x_1, x_2, \lambda, \dot{x}_1, \dot{y}_1, t) = T - V + \lambda(x_1 + x_2 - \ell) = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + m_1 g x_1 + m_2 g x_2 + \lambda(x_1 + x_2 - \ell) \tag{2.42}$$

and the equations of motion are

$$m_1\ddot{x}_1 = \frac{d}{dt}\left(\frac{\partial\widetilde{L}}{\partial\dot{x}_1}\right) = \frac{\partial L}{\partial x_1} = m_1 g + \lambda \tag{2.43a}$$

$$m_1\ddot{x}_2 = \frac{d}{dt}\left(\frac{\partial\widetilde{L}}{\partial\dot{x}_2}\right) = \frac{\partial L}{\partial x_2} = m_2 g + \lambda \tag{2.43b}$$

$$0 = \frac{d}{dt}\left(\frac{\partial\widetilde{L}}{\partial\dot{\lambda}}\right) = \frac{\partial L}{\partial \lambda} = x_1 + x_2 - \ell \tag{2.43c}$$

Since a positive tension will produce a force which tends to try to decrease x_1 and x_2, the tension in the rope is $(-\lambda) > 0$.

To get separate equations for $\ddot{x}_1$ and $\ddot{x}_2$, we take time derivatives of the constraint (2.43c) to say

$$\dot{x}_1 + \dot{x}_2 = 0 \tag{2.44a}$$

$$\ddot{x}_1 + \ddot{x}_2 = 0 \tag{2.44b}$$

and so we can substitute $\ddot{x}_2 = -\ddot{x}_1$. We might also need $x_2 = \ell - x_1$ in a more general problem, but here the equations of motion turn out to be independent of x_2.

To eliminate λ from (2.43a) we solve (2.43b) for

$$\lambda = m_2(\ddot{x}_2 - g) = m_2(-\ddot{x}_1 - g) \tag{2.45}$$

and then substitute in to get

$$m_1\ddot{x}_1 = m_1 g - m_2\ddot{x}_1 - m_2 g \tag{2.46}$$

which can be solved for

$$\ddot{x}_1 = \frac{m_1 - m_2}{m_1 + m_2}\, g \tag{2.47}$$

and then

$$\ddot{x}_2 = -\ddot{x}_1 = \frac{m_2 - m_1}{m_2 + m_1}\, g \tag{2.48}$$

and finally

$$\lambda = m_1(\ddot{x}_1 + g) = m_1 g \left(\frac{m_1 - m_2 - m_1 - m_2}{m_1 + m_2} \right) = -\frac{2m_1 m_2}{m_1 + m_2} g \tag{2.49}$$

which is minus the tension in the rope.

2.3.2 Reduced Lagrangian Approach

The other way to handle this problem is to make the replacements $x_2 = \ell - x_1$ and $\dot{x}_2 = -\dot{x}_1$ at the level of the Lagrangian, and obtain the reduced Lagrangian with one degree of freedom:

$$L(x_1, \dot{x}_1, t) = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (-\dot{x}_1)^2 + m_1 g x_1 + m_2 g (\ell - x_1) = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + (m_1 - m_2) g x_1 + m_2 g \ell \tag{2.50}$$

The equation of motion is then

$$(m_1 + m_2)\ddot{x}_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{\partial L}{\partial x_1} = (m_1 - m_2) g \tag{2.51}$$

or

$$\ddot{x}_1 = \frac{m_1 - m_2}{m_1 + m_2}\, g \tag{2.52}$$

It's a lot simpler to get the equation of motion from this approach, but we don't get the value of the tension.

3 Velocity-Dependent Potentials

So far we've started from the Newtonian picture and seen how to derive a Lagrangian for each class of problem. Once we'd justified the Lagrangian approach, we were able to construct the Lagrangian in generalized coördinates directly. Now, let's go a step further and start with a Lagrangian, and see what sort of systems it describes. The Lagrangian is, in Cartesian coördinates for a single particle,

$$L(\vec{r}, \dot{\vec{r}}, t) = \frac{1}{2} m \left|\dot{\vec{r}}\right|^2 - Q\varphi(\vec{r}, t) + Q\dot{\vec{r}} \cdot \vec{A}(\vec{r}, t) \tag{3.1}$$

where Q is a constant, $\varphi(\vec{r}, t)$ is a scalar field, and $\vec{A}(\vec{r}, t)$ is a vector field. Note that

$$Q\varphi(\vec{r}, t) - Q\dot{\vec{r}} \cdot \vec{A}(\vec{r}, t) \tag{3.2}$$

plays the role of the potential energy V, but now it depends on $\dot{\vec{r}}$ as well as $\vec{r}$. This sort of Lagrangian is associated with a "velocity-dependent potential".

Writing the Lagrangian out explicitly,

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 - Q\varphi + Q\dot{x}A_x + Q\dot{y}A_y + Q\dot{z}A_z \tag{3.3}$$

where A_x, A_y, A_z, and φ are all functions of x, y, z, and t.

Let's focus on one Lagrange equation, the one corresponding to the x coördinate. The relevant derivatives are

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + QA_x \tag{3.4}$$

and then, using the chain rule to evaluate the total time derivative,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} + Q\frac{dA_x}{dt} = m\ddot{x} + Q\frac{\partial A_x}{\partial x}\dot{x} + Q\frac{\partial A_x}{\partial y}\dot{y} + Q\frac{\partial A_x}{\partial z}\dot{z} + Q\frac{\partial A_x}{\partial t} \tag{3.5}$$

and finally, because the Lagrangian depends on x through the fields $\vec{A}$ and φ,

$$\frac{\partial L}{\partial x} = -Q\frac{\partial \varphi}{\partial x} + Q\dot{x}\frac{A_x}{\partial x} + Q\dot{y}\frac{\partial A_y}{\partial x} + Q\dot{z}\frac{\partial A_z}{\partial x} \tag{3.6}$$

Now, the Lagrange equation is

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \\ &= m\ddot{x} + \cancel{Q\dot{x}\frac{\partial A_x}{\partial x}} + Q\dot{y}\frac{\partial A_x}{\partial y} + Q\dot{z}\frac{\partial A_x}{\partial z} + Q\frac{\partial A_x}{\partial t} + Q\frac{\partial \varphi}{\partial x} - \cancel{Q\dot{x}\frac{A_x}{\partial x}} - Q\dot{y}\frac{\partial A_y}{\partial x} - Q\dot{z}\frac{\partial A_z}{\partial x} \\ &= m\ddot{x} + Q\left(\frac{\partial A_x}{\partial t} + \frac{\partial \varphi}{\partial x}\right) + Q\dot{y}\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right) + Q\dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \end{aligned} \tag{3.7}$$

which we can solve for

$$m\ddot{x} = Q\left(\left[-\frac{\partial \varphi}{\partial x} - \frac{\partial A_x}{\partial t}\right] + \dot{y}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] - \dot{z}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\right) \tag{3.8}$$

The first expression in brackets is the x component of $-\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$, the second expression in brackets is the z component of $\vec{\nabla} \times \vec{A}$, and the third expression in brackets is the y component of $\vec{\nabla} \times \vec{A}$. But if we interpret φ as the scalar potential and $\vec{A}$ as the vector potential of electrodynamics, these are just components of the electric and magnetic fields

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} \tag{3.9a}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{3.9b}$$

so

$$m\ddot{x} = Q\big(E_x + \underbrace{\dot{y}B_z - \dot{z}B_y}_{x \text{ compt of } \dot{\vec{r}} \times \vec{B}}\big) = \hat{x} \cdot \left[Q\left(\vec{E} + \dot{\vec{r}} \times \vec{B}\right)\right] \tag{3.10}$$

The calculations for the y and z components of the acceleration are similar and the Lagrange equations arising from

$$L = \frac{1}{2}m\left|\dot{\vec{r}}\right|^2 - Q\varphi + Q\dot{\vec{r}}\cdot\vec{A} \tag{3.11}$$

are just the components of the Lorentz force law

$$m\ddot{\vec{r}} = Q\left(\vec{E} + \dot{\vec{r}}\times\vec{B}\right) \tag{3.12}$$

More on this example in the future.

4 Conservation Laws

4.0 Brans Review

Covered: q_k=Ignorable coord $\Leftrightarrow$ Symmetry along q_k $\Leftrightarrow$ $\frac{\partial L}{\partial \dot{q}_k}$=const=conserved.

Big deal: Symmetry $\Leftrightarrow$ Conservation Law

Defined "generalized momentum" = "momemntum canonically conjugate to q_k"

Commented on notational + terminology problems: $p_\phi = \frac{\partial L}{\partial \dot{\phi}} \neq \vec{p}\cdot\hat{\phi}$

In fact, they picked up on $p_\phi = L_z$ right away: $p_\phi = \hat{z}\cdot(\vec{r}\times\vec{p})$.

Discussed general importance of using $\frac{\partial L}{\partial \dot{q}_k}$ not just $\frac{\partial T}{\partial \dot{q}_k}$.

Then wondered about "symmetry" $\frac{\partial L}{\partial t} = 0$. Defined "$X$"$= \sum \dot{q}_k\frac{\partial L}{\partial \dot{q}_k} - L$ + went through $\frac{dX}{dt} = 0$ in gory detail.

We eased into identification of "X" with energy, after Cartesian example. Didn't call it "H"

Went over $\frac{\partial L}{\partial t} = 0$ vs $\frac{dL}{dt} = 0$.

We set up the spherical pendulum via L after discussing probable difficulties of doing it in Cartesian starting from $\vec{F} = m\vec{a}$.

Used p_ϕ=const, E=const. to get one 1st order ODE in $\dot{\theta}$ in terms of constants E, p_ϕ, saying this was a nice tool.

4.1 Conservation of Momentum and Ignorable Coördinates

Recall conservation of momentum in Newtonian physics:

$$\frac{d\vec{p}}{dt} = m\ddot{\vec{r}} = \vec{F} = -\vec{\nabla}V \tag{4.1}$$

$$p_x = m\dot{x} = \text{constant} \qquad \text{if } \frac{\partial V}{\partial x} = 0 \tag{4.2a}$$

$$p_y = m\dot{y} = \text{constant} \qquad \text{if } \frac{\partial V}{\partial y} = 0 \tag{4.2b}$$

$$p_z = m\dot{z} = \text{constant} \qquad \text{if } \frac{\partial V}{\partial z} = 0 \tag{4.2c}$$

If V is independent of one of the Cartesian coördinates, the corresponding component of the momentum is a constant of the motion.

In Lagrangian mechanics, with generalized coördinates,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k} \tag{4.3}$$

so if L is independent if one of the coördinates q_k, the corresponding quantity $\frac{\partial L}{\partial \dot{q}_k}$ is a constant. We call this the *generalized momentum*

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \tag{4.4}$$

For Cartesian coördinates, this is just one of the components of the momentum of one of the particles.

$$p_{xi} = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i \tag{4.5a}$$

$$p_{yi} = \frac{\partial L}{\partial \dot{y}_i} = m\dot{y}_i \tag{4.5b}$$

$$p_{zi} = \frac{\partial L}{\partial \dot{z}_i} = m\dot{z}_i \tag{4.5c}$$

(i is particle label)

Examples of generalized momenta for non-Cartesian coördinates, e.g., show in polar coördinates (r, ϕ), $p_\phi = L_z$. Stress that $p_\phi \neq \hat{\phi} \cdot \vec{p}$. (This entails a slight change in notation; if we need it, we can use the name $p_{\hat{\phi}} = \hat{\phi} \cdot \vec{p}$.)

If L is independent of q_k, p_k is a constant and q_k is called an *ignorable coördinate*.

Emphasize that L needs to be independent of q_k, not just V. (Example where coördinate only shows up in T, perhaps spherical coörds?)

4.1.1 Examples

4.2 Conservation of Energy and Definition of the Hamiltonian

Recall conservation of energy, $T + V = E =$ constant, i.e., $\frac{d}{dt}(T + V) = 0$. How did that come about in the simplest situation, one particle in one dimension with $T = \frac{1}{2}m\dot{x}^2$?

$$\frac{d}{dt}(T + V) = \frac{dT}{dt} + \frac{dV}{dt} \tag{4.6}$$

where

$$\frac{dT}{dt} = \frac{dT}{d\dot{x}}\frac{d\dot{x}}{dt} = m\dot{x}\ddot{x} = \dot{x}F(x) = \dot{x}\left(-\frac{dV}{dx}\right) \tag{4.7}$$

(using Newton's 2nd law) while

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = \frac{dV}{dx}\dot{x} \tag{4.8}$$

so

$$\frac{dT}{dt} = -\frac{dV}{dt} \tag{4.9}$$

Note this only works because V is independent of time. If we had $V(x,t)$ with explicit time dependence as well as that implicit in the x, we'd get

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial t}\frac{dt}{dt} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial t} \tag{4.10}$$

Now $L = T - V$; can we use Lagrange eqns to derive conservation of energy? Use chain rule to find time derivative of $L(t) = L(\{q_k(t)\}, \{\dot{q}_k(t)\}, t)$

$$\begin{aligned}\frac{dL}{dt} &= \sum_{k=1}^{f}\left(\frac{\partial L}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt}\right) + \frac{\partial L}{\partial t} = \sum_{k=1}^{f}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k\right) + \frac{\partial L}{\partial t} \\ &= \frac{d}{dt}\left(\sum_{k=1}^{f}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right) + \frac{\partial L}{\partial t}\end{aligned} \tag{4.11}$$

Or, putting all the total derivatives on one side,

$$\frac{d}{dt}\left(\sum_{k=1}^{f}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L\right) = -\frac{\partial L}{\partial t} \tag{4.12}$$

If the Lagrangian has no explicit time dependence ($\frac{\partial L}{\partial t} = 0$), then

$$H = \sum_{k=1}^{f}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L \tag{4.13}$$

is a constant of the motion. H is called the Hamiltonian, and is often but not always equal to the total energy $T + V$.

Recall that $T(\{q_k\}, \{\dot{q}_k\}, t)$ is in general made up of pieces which are quadratic, linear and independent of the generalized velocities $\{\dot{q}_k\}$:

$$T = \underbrace{\sum_{k'=1}^{f}\sum_{k''=1}^{f}\frac{1}{2}A_{k'k''}(\{q_k\},t)\dot{q}_{k'}\dot{q}_{k''}}_{T_2(\{q_k\},\{\dot{q}_k\},t)} + \underbrace{\sum_{k'=1}^{f}B_{k'k''}(\{q_k\},t)\dot{q}_{k'}}_{T_1(\{q_k\},\{\dot{q}_k\},t)} + T_0(\{q_k\},t) \tag{4.14}$$

Show that

$$\sum_{k=1}^{f}\frac{\partial T_2}{\partial \dot{q}_k}\dot{q}_k = 2T_2 \tag{4.15a}$$

$$\sum_{k=1}^{f}\frac{\partial T_1}{\partial \dot{q}_k}\dot{q}_k = T_1 \tag{4.15b}$$

$$\sum_{k=1}^{f}\frac{\partial T_0}{\partial \dot{q}_k}\dot{q}_k = 0 \tag{4.15c}$$

So in general

$$H = 2T_2 + T_1 - (T_2 + T_1 + T_0 - V) = T_2 - T_0 - V \tag{4.16}$$

So the Hamiltonian is the same as the total energy $T + V$ if the kinetic energy is purely quadratic, $T_1 = 0 = T_0$ so that $T = T_2$.

4.2.1 Examples

Examples where H is or is not the total energy.

5 Hamiltonian Mechanics

So far: system w/f degrees of freedom described by f generalized coördinates $\{q_k|k=1\dots f\}$

Construct a Lagrangian $L(\{q_k\},\{\dot{q}_k\},t)$ which is usually $T-V$.

The actual trajectory $\{q_k(t)\}$ satisfies

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)=\frac{\partial L}{\partial q_k} \qquad \text{for all } k=1\dots f \tag{5.1}$$

Note, these are second-order differential equations because $\frac{\partial L}{\partial \dot{q}_k}$ will usually contain time derivatives $\{\dot{q}_{k'}\}$.

Last week you saw that $p_k=\frac{\partial L}{\partial \dot{q}_k}$ was a constant of the motion if $\left(\frac{\partial L}{\partial q_k}\right)_{\{q_{k'\neq k}\},\{\dot{q}_{k'}\},t}=0$, because $\frac{dp_k}{dt}=\frac{\partial L}{\partial q_k}$.

You also saw that "X"$=\sum_{k=1}^{f}\dot{q}_k\frac{\partial L}{\partial \dot{q}_k}-L$ was conserved if L had no explicit time dependence, and this "X" was often the total energy $T+V$. This quantity is called the Hamiltonian, and we use the symbol H to refer to it.

We thought of p_k as some function of the $\{q_{k'}\}$ & $\{\dot{q}_{k'}\}$ but since we're taking a total derivative, it could also be thought of as some function of time which is determined by the trajectory, so $\dot{p}_k=\frac{\partial L}{\partial q_k}$ as a consequence of the Lagrange equations.

This is an interesting situation:

$$p_k=\frac{\partial L}{\partial \dot{q}_k} \tag{5.2a}$$

$$\dot{p}_k=\frac{\partial L}{\partial q_k} \tag{5.2b}$$

A good way to summarize partial derivatives is to think about the infinitesimal change in a function associated with infinitesimal changes in its arguments. So for $f(x,y)$, we have

$$df=\frac{\partial f}{\partial x}\,dx+\frac{\partial f}{\partial y}\,dy \tag{5.3}$$

This is a distillation of the chain rule (with a hint of implicit differentiation thrown in).

Apply this to $L(\{q_k\},\{\dot{q}_k\},t)$:

$$\begin{aligned} dL&=\frac{\partial L}{\partial q_1}\,dq_1+\frac{\partial L}{\partial q_2}\,dq_2+\ldots+\frac{\partial L}{\partial q_f}\,dq_f+\frac{\partial L}{\partial \dot{q}_1}\,d\dot{q}_1+\frac{\partial L}{\partial \dot{q}_2}\,d\dot{q}_2+\ldots+\frac{\partial L}{\partial \dot{q}_f}\,d\dot{q}_f+\frac{\partial L}{\partial t}\,dt\\ &=\sum_{k=1}^{f}\left(\frac{\partial L}{\partial q_k}\,dq_k+\frac{\partial L}{\partial \dot{q}_k}\,d\dot{q}_k\right)+\frac{\partial L}{\partial t}\,dt=\sum_{k=1}^{f}\left(\dot{p}_k\,dq_k+p_k\,d\dot{q}_k\right)+\frac{\partial L}{\partial t}\,dt \end{aligned} \tag{5.4}$$

Now think about the definition of the Hamiltonian

$$H = \sum_{k=1}^{f} p_k \dot{q}_k - L \tag{5.5}$$

well,

$$\begin{aligned} dH &= \sum_{k=1}^{f} (\cancel{p_k\, d\dot{q}_k} + \dot{q}_k\, dp_k) - \sum_{k=1}^{f} (\dot{p}_k\, dq_k + \cancel{p_k\, d\dot{q}_k}) - \frac{\partial L}{\partial t}\, dt \\ &= \sum_{k=1}^{f} (\dot{q}_k\, dp_k - \dot{p}_k\, dq_k) - \frac{\partial L}{\partial t}\, dt \end{aligned} \tag{5.6}$$

The $d\dot{q}$ terms cancel. This means that it's much easier to think of H as a function of $\{q_k\}$, $\{p_k\}$ and t rather than $\{q_k\}$, $\{\dot{q}_k\}$ and t. Its partial derivatives are

$$\left(\frac{\partial H}{\partial p_k}\right)_{\{q_{k'\neq k}\},\{p_{k'}\},t} = \dot{q}_k \tag{5.7a}$$

$$\left(\frac{\partial H}{\partial q_k}\right)_{\{q_{k'}\},\{p_{k'\neq k}\},t} = -\dot{p}_k \tag{5.7b}$$

$$\left(\frac{\partial H}{\partial t}\right)_{\{q_k\},\{p_k\}} = -\left(\frac{\partial L}{\partial t}\right)_{\{q_k\},\{\dot{q}_k\}} \tag{5.7c}$$

The first two equations are called *Hamilton's equations* and contain the same information as the Lagrange equations.

What we have just performed is called a *Legendre transformation* from $L(\{q_k\},\{\dot{q}_k\},t)$ to $H(\{q_k\},\{p_k\},t)$:

- Construct $p_k(\{q_{k'}\},\{\dot{q}_{k'}\},t) = \frac{\partial L}{\partial p_k}$
- Invert to get $\dot{q}_k = \dot{q}_k(\{q_{k'}\},\{p_{k'}\},t)$
- $$H(\{q_k\},\{p_k\},t) = \sum_{k'=1}^{f} p_{k'}\dot{q}_k(\{q_{k'}\},\{p_{k'}\},t) - LL(\{q_k\},\{\dot{q}_k(\{q_{k'}\},\{p_{k'}\},t)\},t) \tag{5.8}$$

5.1 Examples

5.1.1 Cartesian Coördinates

For N particles in three dimensions, we have as usual

$$L = \sum_{\ell=1}^{3N} \frac{1}{2} M_\ell \dot{X}_\ell^2 - V(\{X_\ell\}) \tag{5.9}$$

The canonical momentum conjugate to a particular X_ℓ is

$$P_\ell = \frac{\partial L}{\partial \dot{X}_\ell} = M_\ell \dot{X}_\ell \tag{5.10}$$

which can be inverted to get

$$\dot{X}_\ell = \frac{P_\ell}{M_\ell} \tag{5.11}$$

and then we find the Hamiltonian

$$\begin{aligned} H &= \sum_{\ell=1}^{3N} P_\ell \dot{X}_\ell - \sum_{\ell=1}^{3N} \frac{1}{2} M_\ell \dot{X}_\ell^2 + V(\{X_\ell\}) \\ &= \sum_{\ell=1}^{3N} \frac{P_\ell^2}{M_\ell} - \sum_{\ell=1}^{3N} \frac{1}{2} M_\ell \left(\frac{P_\ell}{M_\ell}\right)^2 + V(\{X_\ell\}) \\ &= \sum_{\ell=1}^{3N} \frac{P_\ell^2}{2M_\ell} + V(\{X_\ell\}) \end{aligned} \tag{5.12}$$

which then has Hamilton equations of

$$\frac{\partial H}{\partial P_\ell} = \frac{P_\ell}{M_\ell} = \dot{X}_\ell \tag{5.13a}$$

$$\frac{\partial H}{\partial X_\ell} = \frac{\partial V}{\partial X_\ell} = -\dot{P}_\ell \tag{5.13b}$$

5.1.2 Polar Coördinates

In terms of polar coördinates r and ϕ, the Lagrangian is

$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 - V(r, \phi) \tag{5.14}$$

The conjugate momenta are

$$p_r = m\dot{r} \tag{5.15a}$$

$$p_\phi = m r^2 \dot{\phi} \tag{5.15b}$$

which are, physically, the radial component of momentum and the angular momentum, respectively.

Inverting this gives

$$\dot{r} = \frac{p_r}{m} \tag{5.16a}$$

$$\dot{\phi} = \frac{p_\phi}{m r^2} \tag{5.16b}$$

Note the r dependence of $\dot{\phi}$.

The Hamiltonian is then

$$
\begin{aligned}
H &= p_r\dot{r} + p_\phi\dot{\phi} - \frac{1}{2}m\dot{r}^2 - \frac{1}{2}mr^2\dot{\phi}^2 + V(r,\phi) \\
&= \frac{p_r^2}{m} + \frac{p_\phi}{mr^2} - \frac{m}{2}\left(\frac{p_r}{m}\right)^2 - \frac{mr^2}{2}\left(\frac{p_\phi}{2mr^2}\right)^2 + V(r,\phi) \\
&= \frac{p_r^2}{2m} + \frac{p_\phi}{2mr^2} + V(r,\phi)
\end{aligned}
\tag{5.17}
$$

and Hamilton's equations are

$$\frac{\partial H}{\partial p_r} = \frac{p_r}{m} = \dot{r} \tag{5.18a}$$

$$\frac{\partial H}{\partial r} = -\frac{p_\phi^2}{mr^3} + \frac{\partial V}{\partial r} = -\dot{p}_r \tag{5.18b}$$

$$\frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2} = \dot{\phi} \tag{5.18c}$$

$$\frac{\partial H}{\partial r} = \frac{\partial V}{\partial \phi} = -\dot{p}_\phi \tag{5.18d}$$

Note that if V is $V(r)$, as in the case of a central force, ϕ is an "ignorable coördinate" and p_ϕ is a constant.

5.2 More Features of the Hamiltonian

Last time, we constructed a Hamiltonian out of a Lagrangian and then showed Lagrange's equations implied Hamilton's equations. To stress the completeness of the Hamiltonian picture, let's state the formulation without reference to the Lagrangian picture:

Given a Hamiltonian $H(\{q_k\},\{p_k\},t)$, Hamilton's equations are

$$\frac{\partial H}{\partial q_k} = -\dot{p}_k \qquad k = 1,\dots,f \tag{5.19a}$$

$$\frac{\partial H}{\partial p_k} = \dot{q}_k \qquad k = 1,\dots,f \tag{5.19b}$$

What is this Hamiltonian? Starting from the Lagrangian $L(\{q_k\},\{\dot{q}_k\},t)$, construct $p_k = \frac{\partial L}{\partial p_k} = p_k(\{q_{k'}\},\{\dot{q}_{k'}\},t)$ and $H = \sum_{k=1}^{f} p_k\dot{q}_k - L$, using the inverse $\dot{q}_k = \dot{q}_k(\{q_{k'}\},\{p_{k'}\},t)$ to write H as a function of the ps and qs but not the $\dot{q}$s.

The Hamiltonian is "often" the total energy $E = T + V$. Basically, they're the same whenever V is a function only of the $\{q_k\}$ and T is purely quadratic in the $\{\dot{q}_k\}$. Look at an example for $f = 1$ to make this explicit without worrying about subscripts ...

From the kinetic energy

$$T = \frac{1}{2}a(q,t)\,\dot{q}^2 + b(q,t)\,\dot{q} + c(q,t) \tag{5.20}$$

and the potential energy $V = V(q,t)$ we can construct the Lagrangian

$$L = \frac{1}{2}a(q,t)\,\dot{q}^2 + b(q,t)\,\dot{q} + c(q,t) - V(q,t) \tag{5.21}$$

and the generalized momentum

$$p = \frac{\partial L}{\partial \dot{q}} = a(q,t)\,\dot{q} + b(q,t) \tag{5.22}$$

which makes the Hamiltonian

$$\begin{aligned} H = p\dot{q} - L &= a(q,t)\,\dot{q}^2 + \cancel{b(q,t)\,\dot{q}} - \frac{1}{2}a(q,t)\,\dot{q}^2 - \cancel{b(q,t)\,\dot{q}} - c(q,t) + V(q,t) \\ &= \frac{1}{2}a(q,t)\,\dot{q}^2 - c(q,t) + V(q,t) \end{aligned} \tag{5.23}$$

The first two terms are only equal to T if $b = 0$ and $c = 0$, i.e., if $T = \frac{1}{2}a(q,t)\,\dot{q}^2$.

In the general case, $H = T + V$ if and only if $T = \sum_{k=1}^{f} \sum_{k'=1}^{f} A_{kk'}(\{q_{k''}\}, t)\,\dot{q}_k\,\dot{q}_{k'}$.

5.3 Example: 1-D Harmonic Oscillator

So, for a one-dimensional harmonic oscillator

$$T = \frac{1}{2}m\dot{x}^2 = \frac{p^2}{2m} \tag{5.24a}$$

$$V = \frac{1}{2}kx^2 \tag{5.24b}$$

which meets the conditions (quadratic kinetic energy, velocity-independent potential energy) so the Hamiltonian is just the total energy:

$$H(x,p) = T + V = \frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{5.25}$$

Hamilton's equations are

$$\frac{\partial H}{\partial x} = kx = -\dot{p} \tag{5.26a}$$

$$\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x} \tag{5.26b}$$

Now look at an example where T is *not* purely quadratic ...

5.4 Rotating Coördinates

As we've seen before,

$$L = \frac{m\dot{x}^{*2}}{2} + \frac{m\dot{y}^{*2}}{2} - m\omega\dot{x}^* y^* + m\omega\dot{y}^* x^* \underbrace{-V(x^*,y^*) + \frac{m\omega^2 x^{*2}}{2} + \frac{m\omega^2 y^{*2}}{2}}_{-V_{\text{eff}}(x^*,y^*)} \tag{5.27}$$

So the generalized momenta are

$$p_{x^*} = m\dot{x}^* - m\omega y^* \tag{5.28a}$$

$$p_{y^*} = m\dot{y}^* + m\omega x^* \tag{5.28b}$$

which can be inverted to give

$$\dot{x}^* = \frac{p_{x^*}}{m} + \omega y^* \tag{5.29a}$$
$$\dot{y}^* = \frac{p_{y^*}}{m} - \omega x^* \tag{5.29b}$$

We want to construct

$$H = p_{x^*}\dot{x}^* + p_{y^*}\dot{y}^* - L \tag{5.30}$$

The straightforward approach says to substitute for $\dot{x}^*$ and $\dot{y}^*$. But the algebra is easier if we substitute for p_{x^*} and p_{x^*} and then back again:

$$\begin{aligned} H &= m\dot{x}^{*2} - \cancel{m\omega\dot{x}^* y^*} + m\dot{y}^{*2} - \cancel{m\omega\dot{y}^* x^*} - \frac{m\dot{x}^{*2}}{2} - \frac{m\dot{y}^{*2}}{2} + \cancel{m\omega\dot{x}^* y^*} - \cancel{m\omega\dot{y}^* x^*} + V_{\text{eff}} \\ &= \frac{1}{2}m\left(\frac{p_{x^*}}{m} + \omega y^*\right)^2 + \frac{1}{2}m\left(\frac{p_{y^*}}{m} - \omega x^*\right)^2 + V(x^*, y^*) - \frac{m\omega^2 x^{*2}}{2} - \frac{m\omega^2 y^{*2}}{2} \\ &= \frac{p_{x^*}^2}{2m} + \frac{p_{y^*}^2}{2m} + \omega y^* p_{x^*} - \omega x^* p_{y^*} + V(x^*, y^*) \end{aligned} \tag{5.31}$$

From this Hamiltonian we get Hamilton's equations:

$$\frac{\partial H}{\partial p_{x^*}} = \frac{p_{x^*}}{m} + \omega y^* = \dot{x}^* \tag{5.32a}$$
$$\frac{\partial H}{\partial p_{y^*}} = \frac{p_{y^*}}{m} - \omega x^* = \dot{y}^* \tag{5.32b}$$
$$\frac{\partial H}{\partial x^*} = -\omega p_{y^*} + \frac{\partial V}{\partial x^*} = -\dot{p}_{x^*} \tag{5.32c}$$
$$\frac{\partial H}{\partial y^*} = \omega p_{x^*} + \frac{\partial V}{\partial y^*} = -\dot{p}_{y^*} \tag{5.32d}$$
$$\tag{5.32e}$$

5.4.1 Time-Dependence of Hamiltonian

Note one consequence of Hamilton's equations:

$$\frac{dH}{dt} = \sum_{k=1}^{f}\left(\frac{\partial H}{\partial q_k}\dot{q}_k + \frac{\partial H}{\partial p_k}\dot{p}_k\right) + \frac{\partial H}{\partial t} = \sum_{k=1}^{f}(-\dot{p}_k\,\dot{q}_k + \dot{q}_k\,\dot{p}_k) + \frac{\partial H}{\partial t} \tag{5.33}$$

I.e.,

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{5.34}$$

Because of Hamilton's equations, the implicit time dependence of the Hamiltonian "cancels out".

This tells us something we already know:

If H is a function of the $\{p_k\}$ and $\{q_k\}$ with no explicit time dependence, H is a constant of the motion.

We knew this because we saw last time

$$\left(\frac{\partial H}{\partial t}\right)_{\{q_k\},\{p_k\}} = -\left(\frac{\partial L}{\partial t}\right)_{\{q_k\},\{\dot{q}_k\}} \tag{5.35}$$

and we first saw the Hamiltonian as the thing which is conserved when $\frac{\partial L}{\partial t} = 0$.

6 Lightning Recap

1) For f *degrees of freedom* the Lagrangian depends on $\{q_k|k = 1\ldots f\}$ and $\{\dot{q}_k|k = 1\ldots f\}$ and maybe t. Physically

$$L(\{q_k\},\{\dot{q}_k\},t) = \underbrace{T}_{\text{kinetic}} - \underbrace{V}_{\text{potential; usually fcn only of }\{q_k\}} \tag{6.1}$$

Mechanics given by Lagrange's equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k} \qquad k = 1\ldots f \tag{6.2}$$

2) **Constraints and Lagrange Multipliers**

There are two different was to handle constraints: Either you can choose coördinates such that the constraints are automatically satisfied *or* use Lagrange multipliers (which are useful for getting the constraining forces).

In the Lagrange multiplier method, the Lagrangian

$$L(\{q_k|k = 1\ldots f+c\},\{\dot{q}_k|k = 1\ldots f+c\},t) = T - V \tag{6.3}$$

needs to be modified to enforce the constraints

$$h_j(\{q_k\}) = 0 \qquad j = 1\ldots c \tag{6.4}$$

The modified Lagrangian is

$$\widetilde{L}(\{q_k\},\{\lambda_j\},\{\dot{q}_k\},t) = L(\{q_k\},\{\dot{q}_k\},t) + \sum_{j=1}^{c}\lambda_j h_j(\{q_k\}) \tag{6.5}$$

The equations of motion are then the Lagrange equations for $\widetilde{L}$:

$$\frac{d}{dt}\frac{\partial \widetilde{L}}{\partial \dot{q}_k} = \frac{\partial \widetilde{L}}{\partial q_k} \tag{6.6a}$$

$$0 = \frac{\partial \widetilde{L}}{\partial \lambda_j} \tag{6.6b}$$

This works physically because the (unknown) constraining forces are perpendicular to the surface of constraint.

3) **Conserved quantities**

$$\frac{\partial L}{\partial q_k} = 0 \text{ for some } k \qquad \Rightarrow \qquad p_k = \frac{\partial L}{\partial \dot{q}_k} \text{ conserved for that } k \tag{6.7a}$$

$$\frac{\partial L}{\partial t} = 0 \qquad \Rightarrow \qquad H = \sum_{k=1}^{f} p_k \dot{q}_k - L \text{ conserved} \tag{6.7b}$$

7 Review of Lagrangian Mechanics

1) Basic Formulation: f degrees of freedom, generalized coördinates $\{q_k | k = 1 \ldots f\}$, Lagrangian $L(\{q_k\}, \{\dot{q}_k\}, t)$

   Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k} \tag{7.1}$$

   are equivalent to Newton's laws when $L = T - V$.

   a) Derivation starting from Newton's laws in Cartesian coördintes $\{X_\ell | \ell = 1 \ldots 3N\}$ for N particles in 3 dimensions with

$$T(\{\dot{X}_\ell\}) = \frac{1}{2} M_\ell \dot{X}_\ell^2 \qquad \text{and} \qquad V(\{X_\ell\}, t) \tag{7.2}$$

   Newton's laws are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}_\ell}\right) = \frac{d}{dt}(M_\ell \dot{X}_\ell) = -\frac{\partial V}{\partial X_\ell} \tag{7.3}$$

   (The one-particle version of this is $\frac{d}{dt}(m\dot{\vec{r}}) = \vec{\nabla} V$.)
   Converting derivatives using $X_\ell(\{q_k\}, t)$ gives $T(\{q_k\}, \{\dot{q}_k\}, t)$ and $V(\{q_k\}, t)$.

   b) Justification for $f < 3N$ comes from reduction of constraint problem.

   c) Can also start from Lagrangian, e.g., the electromagnetic Lagrangian

$$L = \frac{1}{2} m(\dot{\vec{r}} \cdot \dot{\vec{r}}) - Q\varphi(\vec{r}, t) + Q\dot{\vec{r}} \cdot \vec{A}(\vec{r}, t) \tag{7.4}$$

2) Constraints: Given $f + c$ coördinates but only f degrees of freedom, there are c constraints[2]

$$h_j(\{q_k\}, t) = 0 \qquad j = 1 \ldots c \tag{7.5}$$

   Modify ordinary Lagrangian $L(\{q_k\}, \{\dot{q}_k\}, t)$ by adding terms involving new "coördinates" $\{\lambda_j | j = 1 \ldots c\}$

$$\widetilde{L}(\{q_k\}, \{\lambda_j\}, \{\dot{q}_k\}, \{\dot{\lambda}_j\}, t) = L(\{q_k\}, \{\dot{q}_k\}, t) + \sum_{j=1}^{c} \lambda_j h_j(\{q_k\}, t) \tag{7.6}$$

[2] Sometimes the constraints are written as $h_j(\{q_k\}, t) = a_j$ where a_j is some constant, but we can always write them in the form (7.5) by constructing $h^{\text{new}}(\{q_k\}, t) = h^{\text{old}}(\{q_k\}, t) - a_j = 0$.

The equations of motion are then the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{q}_k}\right) = \frac{\partial \widetilde{L}}{\partial q_k} = \frac{\partial L}{\partial q_k} + \sum_{j=1}^{c} \lambda_j \frac{\partial h_j}{\partial q_k} \tag{7.7a}$$

$$0 = \frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{\lambda}_j}\right) = \frac{\partial L}{\partial \lambda_j} = h_j \tag{7.7b}$$

(7.7a) are the equations of motion including the constraining forces, and (7.7b) are the constraints.

a) Method works because constrainting forces are perpendiculat for surface of constraint $\vec{F}_{\text{constraint}} \propto \vec{\nabla} h$; the λ_j are proportional to the constraining forces.

b) **WARNING!** Do not impose constraints when constructing Lagrangian in Lagrange multiplier method. E.g., include $\frac{1}{2}m\dot{y}^2$ in Lagrangian even if $y = 0$ is a constraint.

c) Constraints can also be handled by reduced Lagrangian with only f coördinates, but then you don't get the constraining forces.

3) Conservation Laws

a) If Lagrangian is independent of a coördinate, the corresponding conjugate momentum is a constant of the motion.

i.e., if $\dfrac{\partial L}{\partial q_k} = 0$, then $\dfrac{d}{dt} p_k = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_k}\right) = 0$ as a result of Lagrange's eqns (7.8)

b) If the Lagrangian is independent of time (no explicit time dependence, i.e., $\left(\frac{\partial L}{\partial t}\right)_{\{q_k\},\{\dot{q}_k\}} = 0$, then

$$H = \sum_{k=1}^{f} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \tag{7.9}$$

is a constant, i.e., $\frac{dH}{dt} = 0$ (this is the total derivative along the trajectory which satisfies Lagrange's equations).

H is "usually" the total energy. specifically, if

i. V depends only on $\{q_k\}$ and t, not $\{\dot{q}_k\}$

ii. T is purely quadratic in the $\{\dot{q}_k\}$, i.e.,

$$T = \sum_{k=1}^{f} \sum_{k'=1}^{f} \frac{1}{2} A_{kk'}(\{q_{k''}\}, t)\, \dot{q}_k\, \dot{q}_{k'} \tag{7.10}$$

(the most general form also has $\sum_{k=1}^{f} B_k(\{q_{k'}\}, t)\, \dot{q}_k$ and $T_0(\{q_k\}, t)$ terms) then $\sum_{k=1}^{f} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} = 2T$ and $H = T + V = E$.

4) Hamiltonian Mechanics

Change of variables (Legendre transform) from $L(\{q_k\}, \{\dot{q}_k\}, t)$ to $H(\{q_k\}, \{p_k\}, t)$.

Hamilton's equations are

$$\dot{q}_k = \frac{dq_k}{dt} = \left(\frac{\partial H}{\partial p_k}\right) \tag{7.11a}$$

$$\dot{p}_k = \frac{dp_k}{dt} = -\left(\frac{\partial H}{\partial q_k}\right) \tag{7.11b}$$

(7.11a) is a derivative at constant $\{q_{k'}\}$, t, and $\{p_{k'}|k' \neq k\}$; (7.11b) is a derivative at constant $\{q_{k'}|k' \neq k\}$, t, and $\{p_{k'}\}$, *not* at constant $\{\dot{q}_{k'}\}$ because H is not written as a function of velocities.

There are $2k$ first-order equations which are equivalent to the k second-order Lagrange's equations.

a) The Hamiltonian is defined by

$$H = \sum_{k=1}^{f} p_k \dot{q}_k - L \tag{7.12}$$

with a transformation of arguments via

$$p_k(\{q_{k'}\}, \{\dot{q}_{k'}\}, t) = \frac{\partial L}{\partial \dot{q}_k} \tag{7.13}$$

which can be inverted to get

$$\dot{q}_k(\{q_{k'}\}, \{p_{k'}\}, t) \tag{7.14}$$

b) The method is most easily derived by implicit differentiation:

$$\begin{aligned} dH &= \sum_{k=1}^{f} \left(\frac{\partial H}{\partial q_k}\, dq_k + \frac{\partial H}{\partial p_k}\, dp_k \right) + \frac{\partial H}{\partial t}\, dt \\ &= \sum_{k=1}^{f} \left(p_k\, d\dot{q}_k \dot{q}_k\, dp_k - \frac{\partial L}{\partial q_k}\, dq_k - \frac{\partial L}{\partial \dot{q}_k}\, d\dot{q}_k \right) - \frac{\partial L}{\partial t}\, dt \\ &= \sum_{k=1}^{f} \Bigg[\underbrace{-\frac{\partial L}{\partial q_k}}_{\substack{=-\dot{p}_k \text{ by} \\ \text{Lagrange} \\ \text{eqns}}}\, dq_k + \dot{q}_k\, dp_k + \underbrace{\left(p_k - \frac{\partial L}{\partial \dot{q}_k} \right)}_{0 \text{ by defn of } p_k} d\dot{q}_k \Bigg] - \frac{\partial L}{\partial t}\, dt \end{aligned} \tag{7.15}$$

A Appendix: Correspondence to Class Lectures

Date	Sections	Pages	Topics
2004 February 12	1–1.3.4	3–8	Motivation and Formalism
2004 February 17	1.3.5–1.4	8–14	Derivation and Application
2004 February 19	Prelim One		
2004 March 2	Exam Recap		
2004 March 4	2.0–2.1	14–18	Lagrangian with Constraints: Pendulum Example
2004 March 9	2.2	18–20	Lagrange Multiplier Method
2004 March 11	2.3–3	20–24	Atwood Machine; Velocity-Dependent Potentials
2004 March 16	4	24–27	Conservation Laws
2004 March 18	Class Cancelled		
2004 March 23	5–5.1	27–30	Hamiltonian Mechanics
2004 March 25	5.2–6	30–34	Hamiltonian Mechanics, conclusion
2004 April 13	7	34–36	Review