# Physics A301: Classical Mechanics II

Problem Set 2

Assigned 2004 January 22
Due 2004 January 29

**Show your work on all problems!** Be sure to give credit to any collaborators, or outside sources used in solving the problems.

## 1 Deriving Gauss's Law

The point of this problem is to sketch out a demonstration that Gauss's Law holds for a point mass of mass $M$, which generates the gravitational field

$$\vec{g}(\vec{r}) = -\frac{GM}{r^2}\hat{r} \tag{1.1}$$

a) Using the spherical coördinate expression [see Symon eq. (3.124)]

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \ , \tag{1.2}$$

and the partial derivatives [see Symon's (3.99)] of $\hat{r}$ which can be summarized as

$$d\hat{r} = \hat{\theta}\, d\theta + \hat{\phi}\sin\theta\, d\phi \ , \tag{1.3}$$

along with the product rule, **DERIVE** the formula for the divergence

$$\vec{\nabla}\cdot[A_r(r)\hat{r}] \tag{1.4}$$

in terms of $A_r$ and $\frac{dA_r}{dr}$. (You can use the expression on page 103 of Symon to check your result, but should *not* assume it.)

b) Show that $\vec{\nabla}\cdot\vec{g}$ vanishes for the field (1.1), as long as $r \neq 0$. Use this to show that the flux of $\vec{g}$ through any closed surface *not* enclosing the point mass at the origin is zero.

c) Consider a sphere of radius $a$ centered at the origin. Calculate directly the flux of the vector field (1.1) through this sphere, and show it is equal to $-4\pi GM$.

## 2 Galilean Transformations

Consider a system of two particles, of masses $m_1 = m$ and $m_2 = 2m$, connected by a spring of spring constant $k = 2m\omega^2/3$. If there are no other forces, the equations of motion should be

$$m_1\ddot{\vec{r}}_1 = -k(\vec{r}_1 - \vec{r}_2) \tag{2.1a}$$
$$m_2\ddot{\vec{r}}_2 = -k(\vec{r}_2 - \vec{r}_1) \tag{2.1b}$$

a) Show that

$$x_1(t) = \ell\sin\omega t \qquad x_2(t) = -\frac{\ell}{2}\sin\omega t \tag{2.2a}$$
$$y_1(t) = -2\ell\cos\omega t \qquad y_2(t) = \ell\cos\omega t \tag{2.2b}$$
$$z_1(t) = 0 \qquad z_2(t) = 0 \tag{2.2c}$$

is a solution the to the equations of motion (2.1).

b) Now consider the same system viewed in a moving coördinate system whose origin $\mathcal{O}^*$ is offset from the old origin $\mathcal{O}$ by a displacement vector $\vec{h} = \beta t\hat{x}$ where $\beta$ is a constant speed. Transform the trajectory (2.2) into these new coördinates to obtain $x_1^*(t)$, $x_2^*(t)$, $y_1^*(t)$, etc.

c) Show that the trajectory you got in part b) satisfies the equations of motion as well, i.e., that

$$m_1\ddot{\vec{r}}_1^* = -k(\vec{r}_1^* - \vec{r}_2^*) \tag{2.3a}$$
$$m_2\ddot{\vec{r}}_2^* = -k(\vec{r}_2^* - \vec{r}_1^*) \tag{2.3b}$$

## 3 Infinitesimal Rotations

In class and in the book, we considered geometrically the changes $d\hat{x}^*$, $d\hat{y}^*$, and $d\hat{z}^*$ in the basis vectors $\hat{x}^*$, $\hat{y}^*$, and $\hat{z}^*$ that occur when the axes are rotated through an infinitesimal angle $\vec{d\theta} = \vec{\omega}\,dt$. In this problem, you'll show that $\vec{d\theta}$ really is a vector, in that it can be constructed out of three components along the basis vectors.

In this problem, we'll make use of rotation matrices

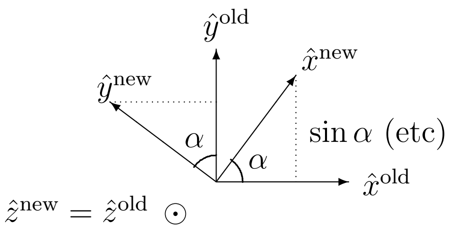
$$\mathbf{R}_1(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}; \quad \mathbf{R}_2(\alpha) = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}; \quad \mathbf{R}_3(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which describe the effects of rotating the basis vectors through an angle $\alpha$ (counter-clockwise) about the $x$, $y$, and $z$ axes, respectively.

a) The figure below verifies that if the "new" basis is obtained from the "old" basis by rotating through an angle of $\alpha$ about the $z$ axis,

$$\begin{pmatrix} \hat{x}^{\text{new}} \\ \hat{y}^{\text{new}} \\ \hat{z}^{\text{new}} \end{pmatrix} = \begin{pmatrix} \cos\alpha\,\hat{x}^{\text{old}} + \sin\alpha\,\hat{y}^{\text{old}} \\ -\sin\alpha\,\hat{x}^{\text{old}} + \cos\alpha\,\hat{y}^{\text{old}} \\ \hat{z}^{\text{old}} \end{pmatrix} = \mathbf{R}_3(\alpha)\begin{pmatrix} \hat{x}^{\text{old}} \\ \hat{y}^{\text{old}} \\ \hat{z}^{\text{old}} \end{pmatrix} \tag{3.1}$$

By drawing similar pictures, show that similar equations hold for rotations about the $x$ and $y$ axes.




b) Consider the infinitesimal rotation matrices $\mathbf{R}_1(d\theta_x)$, $\mathbf{R}_2(d\theta_y)$, and $\mathbf{R}_3(d\theta_z)$. By using the Taylor expansions for sine and cosine (or equivalently the small angle formula), write each of these matrices to first order in the infinitesimal angles (so discard all terms proportional to the second or higher powers in $d\theta_x$, $d\theta_y$, and $d\theta_z$).

c) If we apply the three infinitesimal rotations in succession, rotating through and angle $d\theta_x$ about the $x$ axis, then $d\theta_y$ about the $y$ axis, then $d\theta_z$ about the $z$ axis, the rotation matrix which will take our initial axes to the final ones is

$$\mathbf{R}(\vec{d\theta}) = \mathbf{R}_3(d\theta_z)\mathbf{R}_2(d\theta_y)\mathbf{R}_1(d\theta_x) \tag{3.2}$$

Calculate the elements of the matrix $\mathbf{R}(\vec{d\theta})$, again discarding any expression which is the product of two infinitesimal quantities. Would your answer have been different if you had multiplied the three matrices in a different order?

d) Show that the change in the basis vectors

$$\begin{pmatrix} d\hat{x} \\ d\hat{y} \\ d\hat{z} \end{pmatrix} = \mathbf{R}(\vec{d\theta}) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} - \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \tag{3.3}$$

due to this combined rotation is the same as we derived geometrically for an infinitesimal rotation through $\vec{d\theta} = \hat{x}d\theta_x + \hat{y}d\theta_y + \hat{z}d\theta_z$, i.e., $d\hat{x} = \vec{d\theta} \times \hat{x}$, etc.