Physics A301: Classical Mechanics II

Problem Set 9

Assigned 2004 April 15
Due 2004 April 22

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Practice with Tensors

Consider the vectors $\vec{V} = 2\hat{x} + \hat{y}$ and $\vec{W} = -\hat{y} + 2\hat{z}$. Write the following tensors i) in terms of basis tensors such as $\hat{x} \otimes \hat{x}$, $\hat{x} \otimes \hat{y}$, etc., and ii) as a matrix

$$\mathbf{T} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

where

$$\overleftrightarrow{T} = T_{xx}\hat{x}\otimes\hat{x} + T_{xy}\hat{x}\otimes\hat{y} + T_{xz}\hat{x}\otimes\hat{z} + T_{yx}\hat{y}\otimes\hat{x} + T_{yy}\hat{y}\otimes\hat{y} + T_{yz}\hat{y}\otimes\hat{z} + T_{zx}\hat{z}\otimes\hat{x} + T_{zy}\hat{z}\otimes\hat{y} + T_{zz}\hat{z}\otimes\hat{z}$$

(As an example, if $\vec{a} = 2\hat{y}$ and $\vec{b} = 3\hat{x}$, $\overleftrightarrow{Q} = \vec{a} \otimes \vec{b} = 6(\hat{y} \otimes \hat{x})$, which means the only non-zero component of $\overleftrightarrow{Q}$ is $Q_{21} = 6$, and thus $\mathbf{Q} = \begin{pmatrix} 0 & 0 & 0 \\ 6 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.)

a) $\overleftrightarrow{A} = \vec{V} \otimes \vec{V}$; b) $\overleftrightarrow{B} = \vec{W} \otimes \vec{W}$; c) $\overleftrightarrow{C} = \vec{V} \otimes \vec{W}$;

d) $\overleftrightarrow{D} = \vec{W} \otimes \vec{V}$; e) $\overleftrightarrow{E} = \frac{\overleftrightarrow{C} + \overleftrightarrow{D}}{2}$; f) $\overleftrightarrow{F} = \frac{\overleftrightarrow{C} - \overleftrightarrow{D}}{2}$;

2 Tensors and Rotating Coördinates

a) Show that the centrifugal force can be written as a tensor dotted into the position vector:

$$-m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \overleftrightarrow{M} \cdot \vec{r}$$

First write the tensor $\overleftrightarrow{M}$ in terms of tensor products of vectors, along with the unit tensor $\overleftrightarrow{1}$, then write its components in terms of the (Cartesian) components of $\vec{\omega}$.

b) Do Symon Chapter 10, Problem 3.

3 Kinetic Energy

Consider a distribution of N point particles all rotating with angular velocity $\vec{\omega}$ relative to a fixed origin.

a) Write the velocity $\dot{\vec{r}}_k$ of the kth particle in terms of the angular velocity $\vec{\omega}$ and its position vector $\vec{r}_k$.

b) Write the total kinetic energy

$$T = \sum_{k=1}^{N} \frac{1}{2} m_k (\dot{\vec{r}}_k \cdot \dot{\vec{r}}_k)$$

in terms of $\vec{\omega}$ and the positions of the particles.

c) Prove the vector identity

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}) \tag{3.1}$$

by applying the identities

$$\vec{U} \cdot (\vec{V} \times \vec{W}) = (\vec{U} \times \vec{V}) \cdot \vec{W} \tag{3.2}$$

and

$$\vec{U} \times (\vec{V} \times \vec{W}) = \vec{V}(\vec{U} \cdot \vec{W}) - \vec{W}(\vec{U} \cdot \vec{V}) \tag{3.3}$$

with suitable choices for $\vec{U}$, $\vec{V}$, and $\vec{W}$ each time.

d) Use the identity (3.1) to rewrite T in a form containing no cross products.

e) Use tensor notation to pull $\vec{\omega}$ outside the sum as we did in the derivation of $\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$, and write the kinetic energy in the form

$$T = \vec{\omega} \cdot \overleftrightarrow{A} \cdot \vec{\omega} \tag{3.4}$$

writing the tensor $\overleftrightarrow{A}$ in terms of the masses and position vectors of the particles.

f) How is $\overleftrightarrow{A}$ related to the moment of inertia $\overleftrightarrow{I}$?