Physics A301: Classical Mechanics II

Problem Set 10

Assigned 2004 April 22
Due 2004 April 29

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Inertia Tensor Relative to Different Origins

Consider a rectangular prism (also known as a right parallelepiped) with sides of length a, b, and c, of uniform density and mass M.

a) Calculate the density ρ in terms of M, a, b, and c. Use this relationship to remove ρ from the answers to all subsequent parts of this problem, and express them in terms of M, a, b, and c.

b) Define a coördinate system with its origin $\mathcal{O}$ at one vertex of the prism, so that the prism is defined by

$$0 \leq x \leq a \tag{1.1a}$$
$$0 \leq y \leq b \tag{1.1b}$$
$$0 \leq z \leq c \tag{1.1c}$$

and work out the components $\{I^{\mathcal{O}}_{ij}\}$ of the inertia tensor $\overleftrightarrow{I_{\mathcal{O}}}$ relative to the origin $\mathcal{O}$, in the basis $(\hat{x}, \hat{y}, \hat{z})$ associated with the specified Cartesian coördinate system. (In this problem, it's okay to work out $I^{\mathcal{O}}_{xx}$ and $I^{\mathcal{O}}_{xy}$ directly, and then explain the forms of the other components by analogy.)

c) The center of mass $\mathcal{G}$ of this prism has coördinates $x_{\mathcal{G}} = a/2$, $y_{\mathcal{G}} = b/2$, $z_{\mathcal{G}} = c/2$. Define a new set of coördinate axes, parallel to the first ones and centered at $\mathcal{G}$, so that the prism is defined by

$$-a/2 \leq x' \leq a/2 \tag{1.2a}$$
$$-b/2 \leq y' \leq b/2 \tag{1.2b}$$
$$-c/2 \leq z' \leq c/2 \tag{1.2c}$$

and calculate directly the components $\{I^{\mathcal{G}}_{ij}\}$ of the inertia tensor $\overleftrightarrow{I_{\mathcal{G}}}$ relative to the origin $\mathcal{G}$, in the basis $(\hat{x}, \hat{y}, \hat{z})$. (Note that since we have only translated the origin and not rotated the axes, there is no need to define a different set of basis vectors.)

d) Verify that the relationship between the two inertia tensors is that predicted by Symon's equation (10.147) (the analogue of the parallel axis theorem).

2 Principal Axes of Inertia

Consider a rigid body which consists of two point masses, each of mass $M/2$, separated by a massless rigid rod of length $2a$.

a) Let the body coördinates (x, y, z) be chosen so that the coördinates of the masses are $(x_\mathcal{P}, y_\mathcal{P}, z_\mathcal{P}) = (a/2, a\sqrt{3}/2, 0)$ and $(x_\mathcal{Q}, y_\mathcal{Q}, z_\mathcal{Q}) = (-a/2, -a\sqrt{3}/2, 0)$. Work out (by direct calculation) the components of the inertia tensor $\overleftrightarrow{I}$ and write them as a matrix

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \tag{2.1}$$

b) Define an alternate set of coördinates (x', y', z') according to

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2.2}$$

i) What are the coördinates $(x'_\mathcal{P}, y'_\mathcal{P}, z'_\mathcal{P})$ and $(x'_\mathcal{Q}, y'_\mathcal{Q}, z'_\mathcal{Q})$ of the two point masses?

ii) Using the results of part i), calculate directly the components $\{I'_{k\ell}\}$ of the inertia tensor in this new basis and write them as a matrix

$$\mathbf{I}' = \begin{pmatrix} I'_{xx} & I'_{xy} & I'_{xz} \\ I'_{yx} & I'_{yy} & I'_{yz} \\ I'_{zx} & I'_{zy} & I'_{zz} \end{pmatrix} \tag{2.3}$$

c) Calculate $\mathbf{A}\mathbf{I}\mathbf{A}^t$ and verify that it is equal to $\mathbf{I}'$.

3 Angular Momentum and Rotational Energy

Consider a rigid body with body axes $\hat{x}$, $\hat{y}$, and $\hat{z}$ chosen to lie along the principal axes of inertia so that the inertia tensor is diagonal.

a) Write the angular velocity components ω_x, ω_y, and ω_z in terms of the angular momentum components L_x, L_y, and L_z and the moments of inertia I_{xx}, I_{yy}, and I_{zz}.

b) Use the results of part a) to write the rotational kinetic energy (which we'll call E since it's the only form of energy we're worrying about in this problem) in terms of L_x, L_y, L_z, I_{xx}, I_{yy}, and I_{zz}, without reference to any of the angular velocity components.

c) Limit attention to the case where two of the moments of inertia are equal, $I_{xx} = I_{yy} = I_1 \neq I_{zz} = I_3$. Let α be the angle between the angular momentum vector $\vec{L}$ and the symmetry axis $\hat{z}$, so that $L_z = L\cos\alpha$ where $L = \sqrt{L_x^2 + L_y^2 + L_z^2}$ is the magnitude of the angular velocity vector. Use this to write E as a function of L and α (and the body parameters I_1 and I_3), eliminating all references to L_x, L_y, and L_z.

d) Calculate $\left(\frac{\partial E}{\partial \alpha}\right)_L$ and indicate for what values of $\alpha \in (0, \pi/2)$ it is positive, negative, or zero[1]

   i) for an oblate object ($I_3 > I_1$)

   ii) for a prolate object ($I_1 > I_3$)

e) Many interactions with the outside world and/or small corrections to the assumption of rigid body motion will reduce the rotational energy of an object while leaving its angular momentum unchanged. Assuming $0 < \alpha < \pi/2$, use your results from the previous part to determine whether such interactions will cause the angle between the symmetry axis and the angular momentum vector to increase or decrease

   i) for an oblate object ($I_3 > I_1$)

   ii) for a prolate object ($I_1 > I_3$)

[1] $\alpha \in (0, \pi/2)$ means α in the open interval defined by $0 < \alpha < \pi/2$