Physics A300: Classical Mechanics I

Supplemental Exercises on Fourier Series

Fall 2002

1 Trigonometric Fourier Series

Consider a function $h(t)$ defined for $-\frac{T}{2} < t < \frac{T}{2}$. Assume it can be described by the expansion

$$h(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{T}\right)$$

1. Write the complex conjugate $h^*(t)$.[1] If $h(t)$ is a real function ($h^*(t) = h(t)$), what conditions must the coëfficients $\{a_n | n = 0, \ldots \infty\}$ and $\{b_n | n = 1, \ldots \infty\}$ satisfy? What if it's an imaginary function ($h^*(t) = -h(t)$)?

2. Write $h(-t)$, using the symmetry properties of the sine and cosine to express it as a Fourier series with different coëfficients. If $h(t)$ is an even function ($h(-t) = h(t)$), which coëfficients must vanish? Are there any restrictions on the others? What if it's an odd function ($h(-t) = -h(t)$)?

3. Using the identities

$$\begin{aligned}
\cos(\theta_1 + \theta_2) &= \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \\
\cos(\theta_1 - \theta_2) &= \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \\
\sin(\theta_1 + \theta_2) &= \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2 \\
\sin(\theta_1 - \theta_2) &= \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2
\end{aligned}$$

find expressions for

$$\begin{gathered}
\cos\left(\frac{2\pi m t}{T}\right) \cos\left(\frac{2\pi n t}{T}\right) \\
\sin\left(\frac{2\pi m t}{T}\right) \sin\left(\frac{2\pi n t}{T}\right) \\
\cos\left(\frac{2\pi m t}{T}\right) \sin\left(\frac{2\pi n t}{T}\right) \\
\sin\left(\frac{2\pi m t}{T}\right) \cos\left(\frac{2\pi n t}{T}\right)
\end{gathered}$$

as sums and differences of trig functions.

[1]If $z = x + iy$ is a complex number (x and y are both real numbers), the complex conjugate is defined as $z^* = x - iy$; a real number ($y = 0$ in this representation) is thus unchanged by complex conjugation (thus $x^* = x$); a pure imaginary number ($x = 0$) changes sign under complex conjugation ($(iy)^* = -iy$).

4. Using these expressions, calculate the integrals

$$\int_{-T/2}^{T/2} \cos\left(\frac{2\pi mt}{T}\right)\cos\left(\frac{2\pi nt}{T}\right) dt \qquad m \neq n$$
$$\int_{-T/2}^{T/2} \sin\left(\frac{2\pi mt}{T}\right)\sin\left(\frac{2\pi nt}{T}\right) dt \qquad m \neq n$$
$$\int_{-T/2}^{T/2} \cos\left(\frac{2\pi mt}{T}\right)\sin\left(\frac{2\pi nt}{T}\right) dt \qquad m \neq n$$
$$\int_{-T/2}^{T/2} \cos^2\left(\frac{2\pi nt}{T}\right) dt$$
$$\int_{-T/2}^{T/2} \sin^2\left(\frac{2\pi nt}{T}\right) dt$$
$$\int_{-T/2}^{T/2} \cos\left(\frac{2\pi nt}{T}\right)\sin\left(\frac{2\pi nt}{T}\right) dt$$

The results can be expressed more compactly using the "Kronecker delta"

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Notice that setting $m = 0$ in the above expressions also gives expressions for

$$\int_{-T/2}^{T/2} \cos\left(\frac{2\pi nt}{T}\right) dt$$
$$\int_{-T/2}^{T/2} \sin\left(\frac{2\pi nt}{T}\right) dt$$

5. Use the previous results to calculate the integrals

$$\int_{-T/2}^{T/2} h(t)\cos\left(\frac{2\pi nt}{T}\right) dt$$
$$\int_{-T/2}^{T/2} h(t)\sin\left(\frac{2\pi nt}{T}\right) dt$$
$$\int_{-T/2}^{T/2} h(t) dt$$

and thus obtain expressions for $\{a_n | n = 0, \dots \infty\}$ and $\{b_n | n = 1, \dots \infty\}$.

1.1 Answers

1.

$$h^*(t) = \frac{a_0^*}{2} + \sum_{n=1}^{\infty} a_n^* \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} b_n^* \sin\left(\frac{2\pi nt}{T}\right)$$

If $h(t)$ is real, all the $\{a_n\}$ and $\{b_n\}$ are real. If $h(t)$ is imaginary, all the $\{a_n\}$ and $\{b_n\}$ are imaginary.

2.

$$h(-t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right) - \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

If $h(t)$ is even, all the b_n vanish, and the a_n are unrestricted. If $h(t)$ is odd, all the a_n vanish, and the b_n are unrestricted.

3.

$$\begin{aligned}
\cos\left(\frac{2\pi mt}{T}\right)\cos\left(\frac{2\pi nt}{T}\right) &= \frac{1}{2}\left[\cos\left(\frac{2\pi(m-n)t}{T}\right) + \cos\left(\frac{2\pi(m+n)t}{T}\right)\right] \\
\sin\left(\frac{2\pi mt}{T}\right)\sin\left(\frac{2\pi nt}{T}\right) &= \frac{1}{2}\left[\cos\left(\frac{2\pi(m-n)t}{T}\right) - \cos\left(\frac{2\pi(m+n)t}{T}\right)\right] \\
\cos\left(\frac{2\pi mt}{T}\right)\sin\left(\frac{2\pi nt}{T}\right) &= \frac{1}{2}\left[\sin\left(\frac{2\pi(m+n)t}{T}\right) - \sin\left(\frac{2\pi(m-n)t}{T}\right)\right] \\
\sin\left(\frac{2\pi mt}{T}\right)\cos\left(\frac{2\pi nt}{T}\right) &= \frac{1}{2}\left[\sin\left(\frac{2\pi(m+n)t}{T}\right) + \sin\left(\frac{2\pi(m-n)t}{T}\right)\right]
\end{aligned}$$

4.

$$\begin{aligned}
\int_{-T/2}^{T/2} \cos\left(\frac{2\pi mt}{T}\right)\cos\left(\frac{2\pi nt}{T}\right) dt &= \delta_{mn}\frac{T}{2} \\
\int_{-T/2}^{T/2} \sin\left(\frac{2\pi mt}{T}\right)\sin\left(\frac{2\pi nt}{T}\right) dt &= \delta_{mn}\frac{T}{2} \\
\int_{-T/2}^{T/2} \cos\left(\frac{2\pi mt}{T}\right)\sin\left(\frac{2\pi nt}{T}\right) dt &= 0 \\
\int_{-T/2}^{T/2} \sin\left(\frac{2\pi mt}{T}\right)\cos\left(\frac{2\pi nt}{T}\right) dt &= 0
\end{aligned}$$

5.

$$\begin{aligned}
\int_{-T/2}^{T/2} h(t)\cos\left(\frac{2\pi nt}{T}\right) dt &= a_n\frac{T}{2} \\
\int_{-T/2}^{T/2} h(t)\sin\left(\frac{2\pi nt}{T}\right) dt &= b_n\frac{T}{2}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{T}\int_{-T/2}^{T/2} h(t)\cos\left(\frac{2\pi nt}{T}\right) dt \\
b_n &= \frac{2}{T}\int_{-T/2}^{T/2} h(t)\sin\left(\frac{2\pi nt}{T}\right) dt
\end{aligned}$$

2 Complex Fourier Series

Putting aside for the moment your previous results with trigonometric Fourier series, consider once again a function $h(t)$ defined for $-\frac{T}{2} < t < \frac{T}{2}$. Now assume it can be described by the expansion

$$h(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(-i\frac{2\pi n t}{T}\right)$$

($\exp(x)$ is just another way of writing e^x which is more legible if x is a complicated expression.) Note that now n ranges over negative as well as positive integers.

1. Write the complex conjugate $h^*(t)$.[2] By redefining the summation index (e.g., to be $m = -n$), write $h^*(t)$ as a complex Fourier series with the same exponentials and different coëfficients. If $h(t)$ is a real function ($h^*(t) = h(t)$), what conditions must the coëfficients $\{c_n | n = -\infty, \ldots \infty\}$ satisfy? What if it's an imaginary function ($h^*(t) = -h(t)$)? What can you say about c_0 in each case?

2. Write $h(-t)$, once again changing the index to express it as a Fourier series with different coëfficients. If $h(t)$ is an even function ($h(-t) = h(t)$), what conditions does this set on the coëfficients $\{c_n | n = -\infty, \ldots \infty\}$? What if it's an odd function ($h(-t) = -h(t)$)? How does this differ from the conditions in the previous exercise?

3. Apply the rule $e^\alpha e^\beta = e^{\alpha+\beta}$ for products of exponentials to obtain an expression for

$$\exp\left(i\frac{2\pi m t}{T}\right) \exp\left(-i\frac{2\pi n t}{T}\right)$$

4. Using the Euler relation $e^{i\theta} = \cos\theta + i\sin\theta$, calculate $e^{i2\pi}$. Use this to obtain an expression for $e^{i(\theta+2\pi N)}$, where N is any integer.

[2] A shortcut to taking complex conjugates is to change every i you see to $-i$ and to put a star on every complex parameter or variable you see. So for instance, if z is complex and θ is real, $(ze^{i\theta})^* = z^* e^{-i\theta}$

5. Using the last two results, calculate the integrals

$$\int_{-T/2}^{T/2} \exp\left(i\frac{2\pi mt}{T}\right)\exp\left(-i\frac{2\pi nt}{T}\right)dt \qquad m \neq n$$
$$\int_{-T/2}^{T/2} \exp\left(i\frac{2\pi mt}{T}\right)\exp\left(-i\frac{2\pi nt}{T}\right)dt \qquad m = n$$

where m and n are integers.

Combine both results into a single expression for

$$\int_{-T/2}^{T/2} \exp\left(i\frac{2\pi mt}{T}\right)\exp\left(-i\frac{2\pi nt}{T}\right)dt$$

using the "Kronecker delta"

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

6. Consider the sum

$$\sum_{n=-\infty}^{\infty} A_n \delta_{mn}$$

If m is an integer, there must be one term in the sum where $n = m$. What is the value of this term? What is the value of any term with $n \neq m$? Use the results to obtain a simple expression for the entire sum, assuming m is an integer.

7. Use the previous results to calculate the integral

$$\int_{-T/2}^{T/2} h(t)\,\exp\left(i\frac{2\pi mt}{T}\right)\,dt$$

where m is any integer. Use this result to obtain an expression for c_m.

8. Suppose the same function $h(t)$ can be written as a complex exponential Fourier series and a trigonometric Fourier series:

$$\begin{aligned} h(t) &= \sum_{n=-\infty}^{\infty} c_n \exp\left(-i\frac{2\pi nt}{T}\right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right) \end{aligned}$$

Using the Euler relation $e^{i\theta} = \cos\theta + i\sin\theta$, rewrite the first expression as a sum of sines and cosines, and by matching up coëfficients obtain expressions for a_0, $\{a_n|n = 0, \ldots \infty\}$ and $\{b_n|n = 1, \ldots \infty\}$ in terms of $\{c_n|n = -\infty, \ldots \infty\}$.

2.1 Answers

1.

$$\begin{aligned} h^*(t) &= \sum_{n=-\infty}^{\infty} c_n^* \exp\left(i\frac{2\pi nt}{T}\right) \\ &= \sum_{n=-\infty}^{\infty} c_{-n}^* \exp\left(-i\frac{2\pi nt}{T}\right) \end{aligned}$$

If $h(t)$ is real, we must have $c_n = c_{-n}^*$ for all integer n. In particular, this means $c_0 = c_0^*$, or c_0 is real. If $h(t)$ is imaginary, we must have $c_n = -c_{-n}^*$ for all integer n. In particular, this means $c_0 = -c_0^*$, or c_0 is imaginary.

2.

$$\begin{aligned} h(-t) &= \sum_{n=-\infty}^{\infty} c_n \exp\left(i\frac{2\pi nt}{T}\right) \\ &= \sum_{n=-\infty}^{\infty} c_{-n} \exp\left(-i\frac{2\pi nt}{T}\right) \end{aligned}$$

If $h(t)$ is even, we must have $c_n = c_{-n}$ for all integer n. If $h(t)$ is odd, we must have $c_n = -c_{-n}$ for all integer n. These conditions differ from the previous one in that they don't involve the complex conjugate.

3.

$$\exp\left(i\frac{2\pi mt}{T}\right)\exp\left(-i\frac{2\pi nt}{T}\right) = \exp\left(i\frac{2\pi(m-n)t}{T}\right)$$

4. $e^{i2\pi} = 1$, so $e^{i(\theta+2\pi N)} = e^{i\theta}$.

5.

$$\int_{-T/2}^{T/2} \exp\left(i\frac{2\pi mt}{T}\right)\exp\left(-i\frac{2\pi nt}{T}\right) dt = T\,\delta_{mn}$$

6. When $n = m$, $A_n\delta_{mn} = A_m$, and when $n \neq m$, $A_n\delta_{mn} = 0$, so

$$\sum_{n=-\infty}^{\infty} A_n\delta_{mn} = A_m$$

7.

$$\int_{-T/2}^{T/2} h(t)\,\exp\left(i\frac{2\pi mt}{T}\right)\,dt = T\,c_m$$

so

$$c_m = \frac{1}{T}\int_{-T/2}^{T/2} h(t)\,\exp\left(i\frac{2\pi mt}{T}\right)\,dt$$

8.

$$\begin{aligned} a_0 &= 2\,c_0 \\ a_n &= c_n + c_{-n} \qquad n = 1,\ldots,\infty \\ b_n &= \frac{c_n - c_{-n}}{i} \qquad n = 1,\ldots,\infty \end{aligned}$$