One-Dimensional Motion
(Symon Chapter Two)

Physics A300*

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Contents

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Part I
Consequences of Newton's Second Law

Newton's second law, in its one-dimensional form, says

$$F = m\frac{d^2x}{dt^2} = \frac{dp}{dt} \tag{0.1}$$

In general, the force F experienced by the particle can depend on the location x of the particle, its velocity $v = \frac{dx}{dt}$, and the time t itself. So we write the equation of motion as

$$m\ddot{x} = F(x, v, t) \tag{0.2}$$

What we do with the equation of motion depends on the nature of the problem. Sometimes we are to assume the trajectory $x(t)$ as given, and make deductions using the form of $F(x(t), \dot{x}(t), t)$. Sometimes we need to derive the form of $x(t)$ itself by solving the differential equation (0.2). The method depends on the form of F. Following sections 2.1 and 2.3–2.5, we consider several different possibilities.

1 Momentum and Energy

First, we work from the perspective that the trajectory $x(t)$ is given, and ask about the implications of Newton's second law, which can be written

$$\frac{dp}{dt} = F(x(t), \dot{x}(t), t) \tag{1.1}$$

This is a first-order ordinary differential equation in one variable, and can be solved by integration

$$\int_{t_1}^{t_2} F(x(t), \dot{x}(t), t)\, dt = \int_{t_1}^{t_2} \frac{dp}{dt}\, dt = p(t_2) - p(t_1) = p_2 - p_1 \tag{1.2}$$

The integral $\int F\, dt$, which gives the change in momentum as a result of an applied force, is called the *impulse*. This is an especially useful quantity in cases where the detailed time dependence of the force is not known, but the change in momentum is, such as a ball bouncing elastically off a wall.

Another useful integral of the force is the *work* done between two times

$$W_{21} = \int_{t_1}^{t_2} F(x(t), \dot{x}(t), t)\, \dot{x}(t)\, dt \tag{1.3}$$

again applying Newton's Second Law we see

$$W_{21} = \int_{t_1}^{t_2} F\,\dot{x}\,dt = \int_{t_1}^{t_2} m\ddot{x}\dot{x}\,dt = \int_{t_1}^{t_2} \frac{d}{dt}\underbrace{\left(\frac{1}{2}m\dot{x}(t)^2\right)}_{T(t)}\,dt = T(t_2) - T(t_1) = T_2 - T_1 \tag{1.4}$$

The quantity

$$T(t) = \frac{1}{2}m\dot{x}^2 \tag{1.5}$$

is called the *kinetic energy*; we've just derived the *Work-Energy Theorem* (1.4) which shows that the work done on a particle over part of its trajectory is equal to the change in its kinetic energy.

If $x(t)$ is a monatonic function between two times, we can change variables in the integral defining the work from t to x:

$$W_{21} = \int_{t_1}^{t_2} F\,\dot{x}\,dt = \int_{x_1}^{x_2} F\,dx \tag{1.6}$$

If $x(t)$ is not a monatonic function, we can still break up the integral over time into intervals over which it is, and then write the contribution to the work for each interval as an integral over the position x.

2 Forces Depending only on Time (Symon Section 2.3)

In the case where the force depends only time and not the position or velocity of the particle, (0.2) becomes

$$m\ddot{x}(t) = F(t) \tag{2.1}$$

If we know the initial position $x(0)$ and velocity $\dot{x}(0)$, we can integrate twice to find the trajectory $x(t)$:

$$\dot{x}(t) = \dot{x}(0) + \int_0^t \frac{F(t')}{m}\,dt \tag{2.2}$$

and

$$\begin{aligned} x(t) &= x(0) + \int_0^t \dot{x}(t')\,dt' = x(0) + \int_0^t \left(\dot{x}(0) + \int_0^{t'} \frac{F(t'')}{m}\,dt''\right)dt' \\ &= x(0) + \dot{x}(0)t + \int_0^t \int_0^{t'} \frac{F(t'')}{m}\,dt''\,dt' \end{aligned} \tag{2.3}$$

Of course, in a sufficiently complicated problem, we may not have been able to choose 0 as the initial time, which gives the form in Symon with the lower limits of the integrals as t_0.

We have used this method already in the special case where F is a constant. In the text it's applied to a sinusoidal driving force, which we sketch here.

2.1 Example: Sinusoidal Driving Force

Suppose the force is $F(t) = F_0 \cos(\omega t + \theta_0)$. Then Newton's Second Law tells us

$$\ddot{x}(t) = \frac{F(t)}{m} = \frac{F_0}{m}\cos(\omega t + \theta_0) \tag{2.4}$$

If the initial conditions are $\dot{x}(0) = v_0$ and $x(0) = x_0$, we integrate once to get

$$\begin{aligned}\dot{x}(t) &= \dot{x}(0) + \int_0^t \frac{F(t')}{m}\,dt' = v_0 + \frac{F_0}{m}\int_0^t \cos(\omega t' + \theta_0)\,dt' = v_0 + \frac{F_0}{m}\left[\frac{1}{\omega}\sin(\omega t' + \theta_0)\right]_0^t \\ &= v_0 + \frac{F_0}{m\omega}\left(\sin(\omega t + \theta_0) - \sin\theta_0\right)\end{aligned} \tag{2.5}$$

Integrating again gives

$$\begin{aligned}x(t) &= x(0) + \int_0^t \dot{x}(t')\,dt' = x_0 + \int_0^t \left(v_0 - \frac{F_0 \sin\theta_0}{m\omega} + \frac{F_0}{m\omega}\sin(\omega t' + \theta_0)\right)\,dt' \\ &= x_0 + \left[\left(v_0 - \frac{F_0\sin\theta_0}{m\omega}\right)t' - \frac{F_0}{m\omega^2}\cos(\omega t' + \theta_0)\right]_0^t \\ &= x_0 + \left(v_0 - \frac{F_0\sin\theta_0}{m\omega}\right)t - \frac{F_0}{m\omega^2}[\cos(\omega t + \theta_0) - \cos\theta_0]\end{aligned} \tag{2.6}$$

3 Forces Depending only on Velocity (Symon Section 2.4)

The next simplest force is one which depends only on the position, but it's actually got a deeper meaning, so we postpone it for the moment and turn to a force depending only on the velocity. In that case, the key to integrating (0.2) is to write it as

$$m\frac{dv}{dt} = F(v) \tag{3.1}$$

To do the integral, we need to put everything with a v in it on one side of the equation and everything with a t on the other:

$$\frac{dv}{F(v)} = \frac{dt}{m} \tag{3.2}$$

so that when we integrate we get

$$\int_{v(0)}^{v(t)} \frac{dv'}{F(v')} = \int_0^t \frac{dt'}{m} = \frac{t}{m} \tag{3.3}$$

We then need to solve the resulting equation for v as a function of t so that we can ultimately find

$$x(t) = x(0) + \int_0^t v(t')\,dt' \tag{3.4}$$

The important velocity-dependent forces in mechanics are all damping forces, which are always counter to the direction of motion. In general, the velocity dependence is fairly complicated, but in some regimes, it can be reasonably approximated by a power law:

$$F = -b\,|v|^n \frac{v}{|v|} \tag{3.5}$$

Written this way, it's automatically in the opposite direction to the motion. Sliding friction, which has a constant magnitude, is just the $n = 0$ case of this. Note that for odd n, the absolute values are not necessary, for example if $n = 1$,

$$F(v) = -b\cancel{|v|}\frac{v}{\cancel{|v|}} = -bv \tag{3.6}$$

3.1 Example: Coasting to a Stop under Viscous Damping (supplemental)

As an example of how the method is applied, we consider the motion of an object which starts off with velocity v_0 and then is decelerated by the viscous damping force (3.6). Newton's Second Law is then

$$m\frac{dv}{dt} = -bv \tag{3.7}$$

which makes the integral

$$\int_{v_0}^{v} \frac{dv'}{v'} = -\frac{b}{m}\int_0^t dt' \tag{3.8}$$

i.e.,

$$\ln v'\Big|_{v_0}^{v} = -\left.\frac{b}{m}\right|_0^t \tag{3.9}$$

or

$$\ln v - \ln v_0 = \ln\frac{v}{v_0} = -\frac{bt}{m} \tag{3.10}$$

Before we proceed to solving this equation for v, we note that this is an example of how logarithms can confuse dimensional analysis if we're not careful. The function $\ln x$, like e^x, is transcendental, and therefore we should only take logarithms of dimensionless quantities. And sure enough, v/v_0 is dimensionless. On the other hand, there are intermediate expressions like $\ln v$ and $\ln v_0$ where we seem to be breaking our own rules by taking transcendental functions of numbers with units of velocity. Of course, what makes this okay is that we combine the two of them to produce the log of something dimensionless. Because of the relationship

$$\ln a - \ln b = \ln\frac{a}{b} \tag{3.11}$$

we can't rule out an expression on dimensional grounds just because it involves the logarithm of a dimensionful number. If there is another logarithmic term which can be combined with the first to cancel out the units, it's okay. It's especially important to be aware of this when doing indefinite integrals, since the "arbitrary constant" might just contain the logarithm we need.

Okay, so having defined $v(t)$ implicitly in (3.10), we can invert the relationship by solving for v:

$$\frac{v}{v_0} = e^{-bt/m} \tag{3.12}$$

so

$$v(t) = v_0 e^{-bt/m} \tag{3.13}$$

Now we can integrate this to get

$$\begin{aligned} x(t) &= x(0) + \int_0^t v_0 e^{-bt'/m}\, dt' = x(0) + \frac{mv_0}{-b} e^{-bt'/m} \Big|_0^t = x(0) - \frac{mv_0}{b}\left(e^{-bt/m} - 1\right) \\ &= x(0) + \frac{mv_0}{b}\left(1 - e^{-bt/m}\right) \end{aligned} \tag{3.14}$$

Note that in this case the term arising from the lower limit $t = 0$ was *not* zero, so it was very important that we treated the limits of integration properly.

Further implications of this solution are explored in the text.

3.2 Dimensional Considerations and "Small" Damping Forces (supplemental)

In Sections 2.4 and 2.6, Symon considers two different problems with velocity-dependent damping forces. They can be summarized as follows

1. A boat pushed with an initial velocity v_0 is decelerated by a damping force $F = -bv$
2. An object is dropped from rest, and falls under the influence of gravity and air resistance, experiencing a net force $F = -mg - bv$

Or, mathematically, in problem #1

$$\ddot{x}(t) = -\frac{b}{m}\dot{x}(t) \tag{3.15a}$$

$$\dot{x}(0) = v_0 \tag{3.15b}$$

$$x(0) = 0 \tag{3.15c}$$

while in problem #2

$$\ddot{x}(t) = -g - \frac{b}{m}\dot{x}(t) \tag{3.16a}$$

$$\dot{x}(0) = 0 \tag{3.16b}$$

$$x(0) = 0 \tag{3.16c}$$

In both cases, the problem can be solved exactly, and the solution approximated for "small t" and "large t". But what is meant by something with dimensions being large or small? We can't be talking about $t \ll 1$ or $t \gg 1$ because our lessons in dimensional analysis have told us that it's meaningless to compare two quantities with different dimensions. Instead, we have to compare t to something with dimensions. Each problem has three dimensionful parameters associated with it:

1. m, b, and v_0 in problem 1

2. m, b, and g in problem 2

In fact, in the equations of motion, m and b only appear in the combination b/m, so the list of parameters can actually be cut to two in each case:

1. b/m and v_0 in problem 1

2. b/m and g in problem 2

The combination b/m has units of

$$\frac{(\text{acceleration})}{(\text{velocity})} = \frac{1}{(\text{time})} \tag{3.17}$$

so in each problem the *only* combination of parameters with units of time is m/b, and in each case we mean $\frac{bt}{m} \ll 1$ or $\frac{bt}{m} \gg 1$. So it's worth noting that in an absolute sense, we cannot say than b is small for either of these problems. Only the combination $\frac{bt}{m}$ can be large or small, which means whatever the value of b, t will eventually get large enough that the damping term will dominate. (This is the boat drifting to a near-stop in problem #1 and the falling object reaching terminal velocity in problem #2.)

Note that this would not necessarily be the situation in a problem with an additional dimensionful parameter. For example, if we fire a projectile into the air with initial speed v_0 subject to the equation of motion

$$\ddot{x}(t) = -g - \frac{b}{m}\dot{x}(t) \tag{3.18}$$

we now have three dimensionful parameters b/m, g, and v_0, and there are combinations of these which are dimensionless, allowing us to say "b is small" in a sense which applies for all times by constructing a dimensionless combination of b/m, g, and v_0. Similarly, we can construct a combination of t, g, and v_0 which is dimensionless and thus say "t is not too big" in a way which is independent of the size of b.

Friday, January 20, 2006

4 Forces Depending only on Position (Symon Section 2.5)

4.1 Potential Energy

If the force $F(x)$ experienced by a particle is a function only of the particle's location x and not the time it's there or how fast it's moving, the concept of work becomes even more useful. Recall that the change in kinetic energy $T = \frac{1}{2}mv^2$ between two instants in a particle's trajectory is given by

$$T_2 - T_1 = T(t_1) - T(t_2) = \int_{t_1}^{t_2} F(x)\frac{dx}{dt}\,dt = \int_{x_1}^{x_2} F(x)\,dx \tag{4.1}$$

This is even more simply written in terms of the *Potential Energy*

$$V(x) = -\int F(x)\,dx \tag{4.2}$$

Note that we've used an indefinite integral to define the potential energy. This means it's only defined up to a constant, and we need to fix that constant once per problem. An equivalent definition, which extends more easily to multiple dimensions, is

$$F(x) = -V'(x) \tag{4.3}$$

In terms of the potential energy, we have

$$T_2 - T_1 = -\left[V(x)\right]_{x_1}^{x_2} = -[V(x_2) - V(x_1)] = -(V_2 - V_1) \tag{4.4}$$

This means that

$$T_1 + V_1 = T_2 + V_2 \tag{4.5}$$

or that the total energy

$$E = T + V = \frac{1}{2}m\dot{x}^2 + V(x) \tag{4.6}$$

is a constant of the motion.

As an example, in the case of a particle in a constant gravitational field, where $F(x) = -mg$,

$$V(x) = -\int(-mg)\,dx = mgx + \text{constant} \tag{4.7}$$

Note that as in the case of the gravitational force, "$V = mgh$" is not universally true, just for those cases where the gravitational force is a constant $-mg$.

The potential energy can be used to gain a lot of insight into the motion of a particle, and in particular into the possible ways a particle can move in a given potential. An individual trajectory can be characterized, among other things, by its constant value of E. Solving for the velocity tells us

$$\dot{x}^2 = \frac{2[E - V(x)]}{m} \tag{4.8}$$

Note that if we know the value of the total energy E for a particular trajectory, we know the instantaneous velocity $\dot{x}$ at any position up to a sign. If we add to that the requirement that $\dot{x}$ not change sign discontinuously (i.e., that it can only go from positive to negative at in places where it is instantaneously zero), we can describe the motion qualitatively given a starting position and velocity.

Specific properties include.

1. The particle can only move in regions where $V(x) < E$.

   Values of x for which $V(x) > E$ represent locations which the particle can't reach because (4.8) would tell us that $\dot{x}^2$ was negative.

2. If the particle reaches a point where $V(x) = E$, the velocity is instantaneously zero.

   Likewise, at those values of x, (4.8) tells us that $\dot{x} = 0$. Seen another way, if at some instant the potential energy equals the total energy, the kinetic energy must vanish. For a given trajectory, and hence a given constant value of E, we note where a line drawn at that value of E crosses the $V(x)$ curve.

3. If this is not a local maximum, the particle must turn around and go the other way, since it can't move into the forbidden region.

   Such places, where $V(x) = E$, are called *turning points*. We can see that the sign of $\dot{x}$ must change there unless $V'(x) = \frac{dV}{dx}$ happens to vanish, either by looking at the $V(x)$ graph or by considering where the particle will be an instant later.

   Letting the time at which we reach the turning point be t_0, so that $\dot{x}(t_0) = 0$, we ask what $\dot{x}(t_0 + dt)$ is for infinitesimal dt. Writing a derivative as a ratio of inifinitesimal changes,

   $$\ddot{x}(t_0) = \frac{\dot{x}(t_0 + dt) - \dot{x}(t_0)}{dt} \tag{4.9}$$

   so

   $$\dot{x}(t_0 + dt) = \dot{x}(t_0) + \ddot{x}(t_0)\, dt = 0 + \frac{F(x(t_0))}{m}\, dt = -\frac{dt}{m} V'(x(t_0)) \tag{4.10}$$

   So

   - If the slope of the potential is negative at the turning point $[V'(x(t_0)) < 0]$, the velocity is negative an instant before the particle reaches the turning point (when $dt < 0$) and positive an instant after the particle reaches the turning point (when $dt > 0$).
   - If the slope of the potential is positive at the turning point $[V'(x(t_0)) > 0]$, the velocity is positive an instant before the particle reaches the turning point (when $dt < 0$) and negative an instant after the particle reaches the turning point (when $dt > 0$).

4. A minimum is a stable equilibrium

   If a particle is at a position corresponding to a local minimum of the potential, it stays there because any adjacent position would be forbidden by statement 1.

If a particle has a total energy just above the potential energy corresponding to a local minimum, it will oscillate about that minimum, trapped by the potential "walls" around it.

5. A maximum is an unstable equilibrium

Monday, January 23, 2006

4.2 Interlude: Taylor Series and the Euler Relation

Math department graffiti: "$e^{i\pi} = -1$. Yeah, right."

But it is true. Unfortunately, have to prove with Taylor Series:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \tag{4.11}$$

Apply this to three functions: $e^{i\theta}$, $\cos\theta$, and $\sin\theta$:

$f(\theta)$	$e^{i\theta}$	$\cos\theta$	$\sin\theta$
$f(0)$	1	1	0
$f'(\theta)$	$ie^{i\theta}$	$-\sin\theta$	$\cos\theta$
$f'(0)$	i	0	1
$f'''(\theta)$	$-e^{i\theta}$	$-\cos\theta$	$-\sin\theta$
$f'''(0)$	-1	-1	0
$f^{(4)}(\theta)$	$-ie^{i\theta}$	$\sin\theta$	$-\cos\theta$
$f^{(4)}(0)$	$-i$	0	-1
$f^{(n)}(0)(n \text{ even})$	$(-1)^{n/2} = i^n$	$(-1)^{n/2} = i^n$	0
$f^{(n)}(0)(n \text{ odd})$	$-1^{(n-1)/2}i = i^n$	0	$-1^{(n-1)/2} = i^{n-1}$

So the three Taylor series are

$$e^{i\theta} = 1 + i\theta - \frac{1}{2}\theta^2 - \frac{i}{3!}\theta^3 + \ldots \tag{4.12a}$$

$$\cos\theta = 1 - \frac{1}{2}\theta^2 + \ldots \tag{4.12b}$$

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \ldots \tag{4.12c}$$

from which, along with

$$i\sin\theta = i\theta - \frac{i}{3!}\theta^3 + \ldots \tag{4.13}$$

we see

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{4.14}$$

This is called the *Euler relation*.

4.3 Small Oscillations about a Stable Equilibrium

Recall that a local minimum is a stable equilibrium point. If the equilibrium is at $x = x_e$, equilibrium tells us $V'(x_e) = 0$ and stability tells us $V''(x_e) > 0$. Expand in powers of $x - x_e$ to get

$$V(x) = V(x_e) + V'(x_e)(x-x_e) + \frac{1}{2}V''(x_e)(x-x_e)^2 + \frac{1}{3!}V^{(3)}(x_e)(x-x_e)^3 + \mathcal{O}([x-x_e]^4) \tag{4.15}$$

To make life easy, we can choose the origin of the coördinate so that the equilibrium point of interest lies at $x = 0$. Also exploit the arbitrary freedom to add a constant to the potential, and set $V(0) = 0$. Then if we do a Taylor expansion of $V(x)$ about $x = 0$, we'll find

$$V(x) = \overbrace{V(0)}^{\text{0 by def'n}} + x \overbrace{V'(0)}^{\text{0 for equilibrium}} + \frac{x^2}{2} \overbrace{V''(0)}^{>0 \text{ by stability}} + \frac{x^3}{6} V'''(0) + \ldots \tag{4.16}$$

So for small enough x, i.e.,

$$x \ll \frac{3V''(0)}{V'''(0)} \tag{4.17a}$$

$$x \ll \frac{12V''(0)}{V''''(0)} \tag{4.17b}$$

$$\text{etc}$$

the potential is well approximated by

$$V(x) \approx \frac{1}{2} k x^2 \tag{4.18}$$

where

$$k = V''(0) > 0 \tag{4.19}$$

This is called the *Harmonic Oscillator Potential*

Friday, January 27, 2006

Part II
The Harmonic Oscillator

So far we've looked at the special cases $F(t)$, $F(v)$ and $F(x)$. The rest of the chapter consists of building up step-by-step a classic problem involving a force which is a sum of all three:

- A linear position-dependent restoring force $F_{\text{restoring}}(x) = -kx$ arising from a potential
- A linear velocity-dependent damping force $F_{\text{damping}}(v) = -bv$
- A time-dependent external driving force $F_{\text{driving}}(t)$, both in general and of the specific sinusoidal form $F(t) = F_0 \cos(\omega t + \theta_0)$

We build this up by adding one force at a time.

5 The Simple Harmonic Oscillator

As we showed in section 4.3, a quite generic potential can be approximated near a local minimum, which for convenience we take to be at $x = 0$, as

$$V(x) = \frac{1}{2}kx^2 \tag{5.1}$$

where $k = V''(x) > 0$. (There are some exceptions, e.g., if $V''(0) = 0 = V'''(0)$ and $V^{(4)}(0) > 0$.) The force associated with this potential is

$$F(x) = -V'(x) = -kx \tag{5.2}$$

Now, when you've learned about harmonic oscillators, you've probably started with something called Hooke's Law:

$$F_{\text{Hooke}}(x) = -kx \tag{5.3}$$

which was probably described as a special property of springs. But actually, Hooke's Law is an approximation to just about any force field sufficiently close to a stable equilibrium, thanks to the Taylor expansion (4.16).

Returning to the equation of motion

$$m\ddot{x} = F(x) = -kx \tag{5.4}$$

we can write this as

$$m\ddot{x} + kx = 0 \tag{5.5}$$

or, dividing through so that the highest-order term has a coëfficient of unity,

$$\ddot{x} + \frac{k}{m}x = 0 \tag{5.6}$$

Since the constants k and m only appear in the combination k/m, it's useful to define a new constant with that value. When choosing a name for something like this, it's good to take into consideration its units. We could start with the units of k and m, and work it out, but if we note that $\ddot{x}$ and $\frac{k}{m}x$ have the same units, $\frac{k}{m}$ must have units of one-over-time-squared, so we define

$$\omega_0 = \sqrt{k/m} \tag{5.7}$$

and then write (5.6) as

$$\ddot{x} + \omega_0^2 x = 0 \tag{5.8}$$

To obtain the general solution to (5.8), we note that it's a second order homogeneous linear ordinary differential equation.

- second order because the highest number of time derivatives is two
- homogeneous because there are no terms that don't depend on $x(t)$ or its derivatives
- linear because each term has only one power of $x(t)$ or one of its derivatives, which allows us to write the equation as

  $$\left(\frac{d^2}{dt^2} + \omega_0^2\right) x = 0 \tag{5.9}$$

  where the quantity in parentheses is what's called a *linear differential operator*.
- ordinary (as opposed to partial) because it has only one independent variable t.

This means it has two important properties.

- If $x_1(t)$ and $x_2(t)$ are solutions, then the superposition $c_1x_1(t) + c_2x_2(t)$ is also, for any constants c_1 and c_2. (This is because the equation(5.8) is linear and homogeneous)
- If we have *two* linearly independent solutions $x_1(t)$ and $x_2(t)$, then any solution can be written as a superposition of those two. (This is because it's a second-order equation.)

So we need to find two independent solutions. We apply the time-honored differential equation technique of guessing the right answer. In all seriousness, if we come up with a parametrized candidate solution which is sufficiently general, we'll find that it satisfies the differential equation for some choice of parameters. Since the differential equation has a term with a time derivative of x and another term where x is multiplied by a constant, it's reasonable to guess an exponential form for the solution, since that's a function whose time derivative is the original function times a constant. We could guess e^{pt}, but note that that would not have units of length, so we need to throw in a multiplicative constant to get the units right:

$$x(t) = ce^{pt} \ . \tag{5.10}$$

Differentiating gives

$$\dot{x}(t) = cpe^{pt} \tag{5.11a}$$

$$\ddot{x}(t) = cp^2e^{pt} \tag{5.11b}$$

so (5.8) becomes, for this candidate solution,

$$cp^2e^{pt} + \omega_0^2 ce^{pt} = 0 \tag{5.12}$$

Dividing by ce^{pt}, we have

$$p^2 + \omega_0^2 = 0 \tag{5.13}$$

now, the two solutions to this are

$$p = \pm i\omega_0 \tag{5.14}$$

which would make the general solution[1]

$$x(t) = c_+ e^{i\omega_0 t} + c_- e^{-i\omega_0 t} \tag{5.15}$$

Now, this looks funny, since it involves complex numbers, and our physical $x(t)$ will have to be real. But if we choose $c_\pm$ carefully, we can ensure that $x(t)$ is indeed real. The key is the Euler relation (4.14) which allows us to write

$$x(t) = c_+(\cos\omega_0 t + i\sin\omega_0 t) + c_-(\cos\omega_0 t - i\sin\omega_0 t) = (c_+ + c_-)\cos\omega_0 t + i(c_+ - c_-)\sin\omega_0 t \tag{5.16}$$

Now, if we define new constants

$$A_c = c_+ + c_- \tag{5.17a}$$

$$A_s = i(c_+ - c_-) \tag{5.17b}$$

and require that A_c and A_s be real, we can give the general real solution

$$x(t) = A_c \cos\omega_0 t + A_s \sin\omega_0 t \tag{5.18}$$

in terms of arbitrary real constants A_c and A_s. (Symon calls these B_1 and B_2, respectively.) A slightly more useful set of constants can be obtained by defining $A \geq 0$ and θ so that

$$A_c = A\cos\theta \tag{5.19a}$$

$$A_s = -A\sin\theta \tag{5.19b}$$

with this definition, we have

$$x(t) = A\cos\omega_0 t\cos\theta - A\sin\omega_0 t\sin\theta = A\cos(\omega_0 t + \theta) \tag{5.20}$$

Note that this is at a maximum when $\omega_0 t = -\theta$, and the maximum value is A. A is called the amplitude and θ the phase of the oscillation. (It's fairly common to use $\phi = -\theta$ as the phase instead.)

Excercise for the student: take one of the two forms of the solution and choose the constants A_c and A_s, or A and θ, as appropriate, to match the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$.

[1]Note that this has two independent parameters, which could be set by matching the initial conditions for a physical problem.

Monday, January 30, 2006

6 The Damped Harmonic Oscillator

The next level of complexity we introduce into the system is a retarding force. We considered a whole family of these in section 3, but the most convenient one is the viscous damping force (3.6), which we write as

$$F_{\text{damping}} = -b\dot{x} \tag{6.1}$$

This is the kind of resisting force you get when moving through a viscous medium like oil or honey, and the usual physical image is to have the mass on a spring attached to some sort of apparatus which moves an object through a sealed pot of oil (this was for some reason described as chicken fat when I was a student), generating the damping force described by (6.1).

This is not a terribly common sort of damping in what you'd think of as traditional mechanical systems, but it does come up in more complicated oscillations, and it's the natural resisting term in the analogous electric circuit. Of course, the real reason we talk about it here is that it's linear in the velocity, so it keeps the differential equation linear.

Putting together the restoring force and the damping force, the one-dimensional equation of motion becomes

$$m\ddot{x} = -b\dot{x} - kx \tag{6.2}$$

As before, we divide by m and define the natural frequency

$$\omega_0 = \sqrt{k/m} \tag{6.3}$$

We also define a damping parameter with units of inverse time

$$\gamma = \frac{b}{2m} \tag{6.4}$$

Note the factor of two, which will make things more convenient later. The ODE becomes

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \tag{6.5}$$

This is another second order homogeneous linear differential equation, so we apply the same strategy to find two independent solutions, guessing the form

$$x(t) = c\, e^{pt} \tag{6.6}$$

This makes the differential equation

$$c\, p^2 e^{pt} + 2\gamma\, c\, p\, e^{pt} + \omega_0^2 c\, e^{pt} = 0 \tag{6.7}$$

Again, we can divide by ce^{pt} to get

$$p^2 + 2\gamma p + \omega_0^2 = 0 \tag{6.8}$$

Applying the quadratic equation gives the solutions

$$p_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \tag{6.9}$$

If $\gamma \neq \omega_0$ this will give us our two independent solutions; we'll handle the $\gamma = \omega_0$ case later. Clearly, the nature of the two solutions will depend on the sign of $\gamma^2 - \omega_0^2$. We give names to the three situations as follows

$\gamma^2 - \omega_0^2 < 0$	Underdamped
$\gamma^2 - \omega_0^2 = 0$	Critically damped
$\gamma^2 - \omega_0^2 > 0$	Overdamped

6.1 Underdamped Oscillations

In this case, $\sqrt{\gamma^2 - \omega_0^2}$ is an imaginary number, so we define

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2} \tag{6.10}$$

so that

$$p_\pm = -\gamma \pm i\omega_1 \tag{6.11}$$

Now the general solution is

$$x(t) = c_+ e^{p_+ t} + c_- e^{p_- t} = c_+ e^{-\gamma t} e^{i\omega_1 t} + c_- e^{-\gamma t} e^{-i\omega_1 t} \tag{6.12}$$

Again, we use the Euler relation to write

$$\begin{aligned} x(t) =& c_+ e^{-\gamma t}(\cos\omega_1 t + i\sin\omega_1 t) + c_- e^{-\gamma t}(\cos\omega_1 t - i\sin\omega_1 t) \\ =& (c_+ + c_-)e^{-\gamma t}\cos\omega_1 t + i(c_+ - c_-)e^{-\gamma t}\sin\omega_1 t \end{aligned} \tag{6.13}$$

and define new real constants

$$A_c = c_+ + c_- \tag{6.14a}$$

$$A_s = i(c_+ - c_-) \tag{6.14b}$$

(again, Symon calls these B_1 and B_2) which gives us a solution

$$x(t) = A_c e^{-\gamma t}\cos\omega_1 t + A_s e^{-\gamma t}\sin\omega_1 t \tag{6.15}$$

And as before we can define A and θ by

$$A_c = A\cos\theta \tag{6.16a}$$

$$A_s = -A\sin\theta \tag{6.16b}$$

which gives us a general solution

$$x(t) = Ae^{-\gamma t}\cos(\omega_1 t + \theta) \tag{6.17}$$

Note that this differs from the solution from the undamped oscillator only in that the oscillations are multiplied by the decaying exponential $e^{-\gamma t}$, and in that the oscillation frequency is

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2} \tag{6.18}$$

In the limit $\gamma \to 0$, $\omega_1 \to \omega_0$ and we get back the undamped solution, as we must.

6.1.1 Example: Choosing a Particular Solution to Match Initial Conditions

To give an example of how to set the values of the constants A_c and A_s if we're given initial conditions for the problem, suppose we're told to find $x(t)$ an underdamped oscillator with spring constant $m\omega_0^2$ and damping parameter $2m\gamma$, which is released from rest at a position x_0 away from equilibrium. We'll handle this by starting with the form (6.15) and finding the values of A_c and A_s. (Alternatively, we could find the values of A and θ in the general solution (6.17). You should try this as an exercise, checking with the Fall 2003 notes if you get tripped up. Either method works, but this one is usually more convenient for dealing with conditions at $t = 0$.)

We know the general solution is given by (6.15), and its derivative is

$$\dot{x}(t) = -\gamma A_c\, e^{-\gamma t}\cos\omega_1 t - \omega_1 A_c\, e^{-\gamma t}\sin\omega_1 t - \gamma A_s\, e^{-\gamma t}\sin\omega_1 t + \omega_1 A_s\, e^{-\gamma t}\cos\omega_1 t \tag{6.19}$$

That means that the general solution has

$$x(0) = A_c \tag{6.20a}$$
$$\dot{x}(0) = -\gamma A_c + \omega_1 A_s \tag{6.20b}$$

so we need to determine A_c and A_s from the inital conditions

$$x_0 = x(0) = A_c \tag{6.21a}$$
$$0 = \dot{x}(0) = -\gamma A_c + \omega_1 A_s \tag{6.21b}$$

Equation (6.21a) tells us

$$A_c = x_0 \tag{6.22}$$

which we can substitute back into (6.21b) to get

$$-\gamma x_0 + \omega_1 A_s = 0 \tag{6.23}$$

which can be solved for

$$A_s = \frac{\gamma x_0}{\omega_1} \tag{6.24}$$

$$x(t) = x_0\, e^{-\gamma t}\cos\omega_1 t + \frac{\gamma x_0}{\omega_1}\, e^{-\gamma t}\sin\omega_1 t = x_0 e^{-\gamma t}\left(\cos\omega_1 t + \frac{\gamma}{\omega_1}\sin\omega_1 t\right)\ . \tag{6.25}$$

Finally, since the statement of the problem talked about ω_0 and γ, but not ω_1, we should be sure to define what we mean by ω_1, when presenting the answer, so we say:

$$x(t) = x_0 e^{-\gamma t}\left(\cos\omega_1 t + \frac{\gamma}{\omega_1}\sin\omega_1 t\right) \qquad \text{where } \omega_1 = \sqrt{\omega_0^2 - \gamma^2} \tag{6.26}$$

6.2 Overdamped Oscillations

Turning to another general class of solution, consider the case when $\gamma^2 - \omega_0^2 > 0$. In this case the two roots

$$p_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \tag{6.27}$$

are real. This makes the general solution

$$x(t) = c_1 e^{-\gamma_1 t} + c_2 e^{-\gamma_2 t} \tag{6.28}$$

where

$$\gamma_1 = \gamma + \sqrt{\gamma^2 - \omega_0^2} \tag{6.29a}$$

$$\gamma_2 = \gamma - \sqrt{\gamma^2 - \omega_0^2} \tag{6.29b}$$

This does not oscillate, but has two terms which go to zero at different rates. (It can, however, change sign once or twice if c_1 and c_2 have different signs.)

An overdamped oscillator has "too much" damping in the sense that if it starts off out of equilibrium, the damping force can resist the motion so much that it takes a long time to get back to equilibrium. An example would be a door that "takes forever to close" because the pneumatic cylinder provides too much resistance.

6.3 Critically Damped Oscillations

We turn at last to the special case $\gamma = \omega_0$. In this case our attempt to find two independent solutions of the form

$$x(t) = ce^{pt} \tag{6.30}$$

fails, because p solves the quadratic equation

$$0 = p^2 + 2\gamma p + \gamma^2 = (p + \gamma)^2 \tag{6.31}$$

which has only one solution, $p = -\gamma$

It's thus necessary to cast a wider net in looking for a pair of independent solutions, and it turns out that what works is

$$x(t) = (C_1 + C_2 t)e^{-\gamma t} \tag{6.32}$$

We can verify that this is a solution for all C_1 and C_2; differentiating gives

$$\dot{x}(t) = (-\gamma C_1 + C_2 - C_2 \gamma t)e^{-\gamma t} \tag{6.33}$$

and differentiating again gives

$$\ddot{x}(t) = (\gamma^2 C_1 - \gamma C_2 - C_2 \gamma + C_2 \gamma^2 t)e^{-\gamma t} \tag{6.34}$$

So that

$$\ddot{x} + 2\gamma\dot{x} + \gamma^2 = \big(C_1(\gamma^2 - 2\gamma^2 + \gamma^2) + C_2(\gamma^2 t + 2\gamma - 2\gamma^2 t - 2\gamma + \gamma^2 t)\big) = 0 \tag{6.35}$$

Note that in all three cases, the general solution dies off exponentially as $t \to 0$, so long as $\gamma > 0$.

Friday, February 3, 2006

7 The Forced, Damped Harmonic Oscillator

Finally, we combine all three forces:

- A restoring force $F_{\text{restoring}}(x) = -kx = V'(x)$
- A damping force $F_{\text{damping}}(v) = -b\dot{x}$
- A driving force $F_{\text{driving}}(t) = F(t)$

(We write the restoring force as simply $F(t)$, even though it's not the net force by itself, because it will save us some writing.)

Newton's second law tells us

$$m\ddot{x} = -kx - b\dot{x} + F(t) \tag{7.1}$$

which gives us the differential equation

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F(t)}{m} \tag{7.2}$$

This is a slightly different sort of differential equation we've seen before; it's a second order inhomogeneous linear ordinary differential equation:

- second order because the highest number of time derivatives is two
- inhomogeneous because of the driving force $F(t)$
- linear because each term has only one power of $x(t)$ or one of its derivatives
- ordinary (as opposed to partial) because it has only one independent variable t.

It's conventional physically to think of the oscillator itself as defined by the damping and restoring forces, and then apply various different external driving forces to the same oscillator. This also has a mathematical parallel, in that we can write (7.2) as

$$\mathcal{L}x = \frac{F(t)}{m} \tag{7.3}$$

where $\mathcal{L}$ is the *linear differential operator*

$$\mathcal{L} = \frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2 \tag{7.4}$$

The linearity of the differential operator $\mathcal{L}$ tells us two very useful things:

1. If $x_h(t)$ is a solution to the homogenous equation $\mathcal{L}x_h = 0$ and $x_p(t)$ is a solution to the inhomogeneous equation $\mathcal{L}x_p = F/m$, then

$$\mathcal{L}(x_h + x_p) = \mathcal{L}x_h + \mathcal{L}x_p = 0 + \frac{F(t)}{m} \tag{7.5}$$

   so $x_h + x_p$ is also a solution of the inhomogeneous equation. But this makes it easy to find the general solution to the inhomogeneous equation, which needs to have two arbitrary constants. We can simply add together the *general* solution $x_h(t)$ to the homogeneous equation and *any* solution $x_p(t)$ to the inhomogeneous equation, and the sum $x_h(t) + x_p(t)$ will have two arbitrary constants and solve the inhomogeneous equation. This means we've already done much of the work of solving the forced damped harmonic oscillator problem by solving the unforced problem for the same oscillator.

2. The other useful fact is that if we have two different forces $F_1(t)$ and $F_2(t)$, and $x_1(t)$ and $x_2(t)$ are the solutions to the respective differential equations:

$$\mathcal{L}x_1 = F_1/m \tag{7.6a}$$
$$\mathcal{L}x_2 = F_2/m \tag{7.6b}$$

   (which means they are the responses of the oscillator to the respective forces) then $a_1x_1(t) + a_2x_2(t)$ is the oscillator's response to the combined force $aF_1(t) + bF_2(t)$:

$$\mathcal{L}(a_1x_1 + a_2x_2) = \frac{aF_1 + bF_2}{m} \tag{7.7}$$

7.1 Sinusoidal Driving Forces

One very useful driving force is a general sinusoid

$$F(t) = F_0 \cos(\omega t + \theta_0) \tag{7.8}$$

where F_0, ω, and θ_0 are all parameters of the force. Note that in general, ω, ω_0, and γ are all different.

In the usual tradition of Physicists solving differential equations, we guess an answer and see if it works. (Keep in mind that we only need one solution, and then we can add the general solution to the corresponding homogenous equation to obtain the whole family.) A reasonable thing to try is an oscillating solution; we know that if the driving force keeps going forever, the oscillator will never settle down to zero displacement, so we should try a solution which just oscillates and doesn't decay; and since the driving force is what's keeping it going, let's assume it oscillates at that frequency, but not necessarily in phase. The solution we try is thus

$$x(t) = A_s \cos(\omega t + \theta_s) \tag{7.9}$$

The output amplitude A_s and phase θ_s are *not* arbitrary; we need to figure out which values are needed for (7.9) to be a Solution to (7.2). Now, we could calculate $\dot{x}$ and $\ddot{x}$ and substitute those into (7.2) to find out θ_s and A_s, but it turns out the math is easier if we us the superposition property, noting that

$$\cos(\omega t + \theta_0) = \frac{e^{i(\omega t+\theta_0)} + e^{-i(\omega t+\theta_0)}}{2} \tag{7.10}$$

If we define

$$F_\pm = F_0 e^{\pm i(\omega t+\theta_0)} \tag{7.11}$$

and

$$x_\pm = A_s e^{\pm i(\omega t+\theta_s)} \tag{7.12}$$

then the actual force is

$$F(t) = \frac{1}{2}F_+(t) + \frac{1}{2}F_-(t) \tag{7.13}$$

which means the oscillator response will be

$$x(t) = \frac{1}{2}x_+(t) + \frac{1}{2}x_-(t) \tag{7.14}$$

We pause to make contact with Symon's notation, which uses boldfaced letters to represent complex quantities. There are a couple of reasons why this is a bad idea:

1. It's easy to confuse boldfaced quantities with non-boldfaced ones, and to make matters worse, the relationship between, for example, $\mathbf{x}(t)$ and $x(t)$ is not the same as that between $\mathbf{F}_0$ and F_0.

2. Symon will use boldface to indicate vectors in the next chapter (we will use arrows instead).

But at any rate, here's the correspondence:

$$F_+(t) = \mathbf{F}(t) \tag{7.15a}$$

$$F_-(t) = \mathbf{F}^*(t) \tag{7.15b}$$

$$x_+(t) = \mathbf{x}(t) \tag{7.15c}$$

$$x_-(t) = \mathbf{x}^*(t) \tag{7.15d}$$

$$F_0 e^{i\theta_0} = \mathbf{F}_0 \tag{7.15e}$$

$$A_s e^{i\theta_s} = \mathbf{x}_0 \tag{7.15f}$$

Note that the parameters F_0, ω, θ_0, A_s, and θ_s in the original expressions (7.8) and (7.9) are real. This didn't seem worth mentioning when we defined them (of course we were only working with real numbers) but now that we've introduced complex quantities like $x_+(t)$ it's important. If we consider that

$$F_\pm = F_0 e^{\pm i(\omega t+\theta_0)} = F_0\cos(\omega t + \theta_0) \pm iF_0\sin(\omega t + \theta_0) \tag{7.16}$$

we'll see that taking the complex conjugate, changing the sign of the imaginary part, tells us that

$$F_+(t)^* = F_-(t) \tag{7.17}$$

and likewise[2]

$$x_+(t)^* = x_-(t) \tag{7.18}$$

This means that if we can choose A_s and θ_s so that x_+ solves $\mathcal{L}x_+ = F_+/m$, then x_- will automatically satisfy $\mathcal{L}x_- = F_-/m$ (since the differential equations are just complex conjugates of each other).

To do this, we note that since

$$x_+ = A_s e^{i\theta_s} e^{i\omega t} \tag{7.19}$$

we have

$$\dot{x}_+ = i\omega x_+ \tag{7.20}$$

and

$$\ddot{x}_+ = -\omega^2 x_+ \tag{7.21}$$

so

$$\mathcal{L}x_+ = (-\omega^2 + 2i\gamma\omega + \omega_0^2) A_s e^{i\theta_s} e^{i\omega t} = \frac{F_0 e^{i\theta_0} e^{i\omega t}}{m} \tag{7.22}$$

or, cancelling out the $e^{i\omega t}$ and rearranging,

$$\frac{F_0}{m} = A_s e^{i(\theta_s - \theta_0)} \left[(\omega_0^2 - \omega^2) + 2i\gamma\omega \right] \tag{7.23}$$

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[2]Making contact with Symon, note that

$$F(t) = \frac{F_+(t) + F_-(t)}{2} = \frac{F_+(t) + F_+^*(t)}{2} = \operatorname{Re} F_+(t)$$

and

$$x_p(t) = \frac{x_+(t) + x_-(t)}{2} = \frac{x_+(t) + x_+^*(t)}{2} = \operatorname{Re} x_+(t)$$

Monday, February 6, 2006

7.1.1 Determination of the Amplitude

Now, (7.23) is a complex equation, which states that one complex number is equal to another. Recall that if $z = x + iy$ and $w = u + iv$ are two complex numbers, the complex equation

$$z = w \tag{7.24}$$

is equivalent to the two real equations

$$x = u \tag{7.25a}$$

$$y = v \tag{7.25b}$$

i.e., if two complex expressions are equal, then the real parts are equal and the imaginary parts are equal. Another real equation, which is not independent of the other two, is that the magnitude-squared of the two equations is equal:

$$z^*z = x^2 + y^2 = u^2 + v^2 = w^*w \tag{7.26}$$

In this case, the most useful pair of equations is the equality of the imaginary parts (which A_s drops out of) and of the squared magnitudes (which doesn't involve θ_s).

The most direct way to find the amplitude A_s is to take the square of the magnitude of each side of (7.23) by multiplying it by its complex conjugate:

$$\left\{\left[(\omega_0^2 - \omega^2) + 2i\gamma\omega\right] A_s e^{i\theta_s}\right\}\left\{\left[(\omega_0^2 - \omega^2) - 2i\gamma\omega\right] A_s e^{-i\theta_s}\right\} = \frac{F_0}{m}\frac{F_0}{m} \tag{7.27}$$

or

$$\left[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2\right] {A_s}^2 = \frac{F_0^2}{m^2} \tag{7.28}$$

which tells us

$$A_s = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \tag{7.29}$$

We have chosen to take the positive square root, but this is a reasonable thing to do, since it just means we choose A_s to have the same sign as F_0. We'll see that there is always a choice of θ_s which makes this work.

7.1.2 Determination of the Phase

To find θ_s, we concentrate on the imaginary part of (7.23), which can be extracted by writing it as

$$\begin{aligned}\frac{F_0}{m} =& A_s[\cos(\theta_s - \theta_0) + i\sin(\theta_s - \theta_0)]\left[(\omega_0^2 - \omega^2) + 2i\gamma\omega\right] \\ =& A_s\big\{(\omega_0^2 - \omega^2)\cos(\theta_s - \theta_0) - 2\gamma\omega\sin(\theta_s - \theta_0) \\ &+ i\left[2\gamma\omega\cos(\theta_s - \theta_0) + (\omega_0^2 - \omega^2)\sin(\theta_s - \theta_0)\right]\big\}\end{aligned} \tag{7.30}$$

Now, we could set the imaginary part to zero to find out the tangent of $\theta_s - \theta_0$, but despite having initially defined θ_s in his equation (2.149), Symon proceeds to find the solution in terms of a different phase angle β, a wacky convention which he says comes from electrical engineering and is related to θ_s by

$$\beta = \theta_s - \theta_0 + \frac{\pi}{2} \tag{7.31}$$

so that

$$\cos(\theta_s - \theta_0) = \sin\beta \tag{7.32a}$$

$$\sin(\theta_s - \theta_0) = -\cos\beta \tag{7.32b}$$

and the candidate solution (7.9) is

$$x_p(t) = A_s \sin(\omega t + \theta_0 + \beta) \tag{7.33}$$

In terms of this new angle, (7.30) becomes

$$\begin{aligned}\frac{F_0}{m} =& A_s\big[(\omega_0^2 - \omega^2)\sin\beta + 2\gamma\omega\cos\beta \\ &+ i\left(2\gamma\omega\sin\beta - (\omega_0^2 - \omega^2)\cos\beta\right)\big]\end{aligned} \tag{7.34}$$

The imaginary part of the left-hand side vanishes, so setting the imaginary part of the right-hand side to zero gives

$$\tan\beta = \frac{\omega_0^2 - \omega^2}{2\gamma\omega} \tag{7.35}$$

Now, knowing the tangent of an angle only tells us that angle modulo π (because $\tan(\theta + \pi) = \cos(\theta + \pi)/\sin(\theta + \pi) = (-\cos\theta)/(-\sin\theta) = \tan\theta$) so we should pause and make sure we know which branch of the arctangent we want to take when we calculate β. Put another way, given the tangent, we can calculate

$$\sin\beta = \pm\frac{\tan\beta}{1 + \tan^2\beta} = \pm\frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \tag{7.36a}$$

$$\cos\beta = \pm\frac{1}{1 + \tan^2\beta} = \pm\frac{2\gamma\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \tag{7.36b}$$

where all of the $\pm$ signs are the same, but we need to decide whether they're all plus or all minus. Fortunately, once we figure out the value of β for $\omega = 0$, we can vary ω smoothly, and since we can see from (7.29) that the amplitude A_s doesn't go through zero for any ω, we don't have to worry about β suddenly jumping by π.

At $\omega = 0$, the real part of (7.34) tells us that

$$\left.\frac{F_0}{m}\right|_{\omega=0} = A_s\omega_0^2\sin\beta \tag{7.37}$$

but we've chosen A_s so that it has the same sign as F_0. This means $\sin\beta > 0$ at $\omega = 0$ and we want to take the plus sign to get

$$\sin\beta = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \tag{7.38a}$$

$$\cos\beta = \frac{2\gamma\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \tag{7.38b}$$

Armed with this knowledge, we can follow the behavior of β as a function of ω:

$\omega = 0$	$\sin\beta = 1$	$\cos\beta = 0$	$\beta = \pi/2$
$0 < \omega < \omega_0$	$0 < \sin\beta < 1$	$0 < \cos\beta < 1$	$0 < \beta < \pi/2$
$\omega = \omega_0$	$\sin\beta = 0$	$\cos\beta = 1$	$\beta = 0$
$\omega > \omega_0$	$-1 < \sin\beta < 0$	$0 < \cos\beta < 1$	$-\pi/2 < \beta < 0$
$\omega \to \infty$	$\sin\beta \to -1$	$\cos\beta \to 0$	$\beta \to -\pi/2$

Note that this behavior is consistent with Figure 2.6 of Symon, but his discussion in the paragraph after equation (2.159) is completely wrong. (He must have been thinking of a different angle.)

So when all is said and done, we can recover the desired solution in the presence of the sinusoidal force (7.8). Since

$$x_+ = A_s e^{i(\omega t + \theta_0 + \beta - \pi/2)} = \frac{A_s}{i} e^{i(\omega t + \theta_0 + \beta)} \tag{7.39}$$

and

$$x_- = -\frac{A_s}{i} e^{-i(\omega t + \theta_0 + \beta)} \tag{7.40}$$

then

$$x_p(t) = \frac{x_+ + x_-}{2} = A_s \frac{e^{i(\omega t + \theta_0 + \beta)} - e^{-i(\omega t + \theta_0 + \beta)}}{2i} = A_s \sin(\omega t + \theta_0 + \beta) \tag{7.41}$$

where A_s is given by (7.29) and β is given by (7.38).

Now that we've found a particular solution to the differential equation (7.2), we can add the general solution to the homogenous equation (6.5) to obtain the general solution

$$x(t) = Ae^{-\gamma t} \cos(\omega t + \theta) + A_s \sin(\omega t + \theta_0 + \beta) \tag{7.42}$$

where A_s is given by (7.29) and β is given by (7.38).

Friday, February 10, 2006

7.1.3 Amplitude Resonance

Analytical Properties Consider the steady-state amplitude

$$A_s = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \tag{7.29}$$

as a function of the driving frequency ω for fixed ω_0 and γ (and F_0 and m).

Let's start by looking at the behavior as ω goes from 0 to ∞, noting any maxima or minima.

- At $\omega = 0$, $A_s = F_0/m\omega_0^2$, This basically says that if we drive the oscillator slowly enough, the displacement at any time will be just enough so the restoring force from the spring cancels the driving force at that instant. (Note that at $\omega = 0$, $\beta = \pi/2$ and thus $\theta_s = \theta_0$ and the displacement is just in phase with the driving force.)

- As $\omega \to \infty$, $A_s \to F_0/m\omega^2 \to 0$.

To see what happens in between, we take the derivative $\frac{\partial A_s}{\partial \omega}$ at constant F_0, ω_0, γ, etc. to see if there is a maximum or minimum steady-state amplitude at some frequency:

$$\frac{\partial A_s}{\partial \omega} = -\frac{1}{2}\frac{F_0}{m\left[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2\right]^{3/2}}\left[-4\omega(\omega_0^2 - \omega^2) + 8\gamma^2\omega\right] \tag{7.43}$$

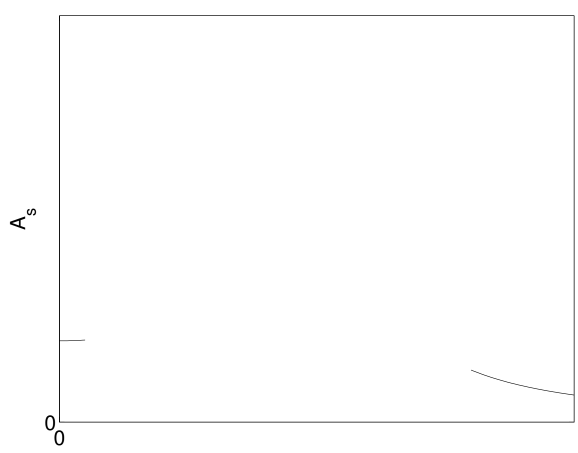
This vanishes when

$$\omega_0^2 - \omega^2 = 2\gamma^2 \tag{7.44}$$

i.e.,

$$\omega^2 = \omega_0^2 - 2\gamma^2 \tag{7.45}$$

So for positive ω there is at most one extremum. We can see that it is a maximum either by checking the sign of $\frac{\partial^2 A_s}{\partial \omega^2}$ (ugh!) or noting that A_s starts out at a positive value for $\omega = 0$, and ends up approaching 0 from above for large ω:



If there's only one extremum in between those two limits, it has to be a maximum. (Think about it!)

If $2\gamma^2 \geq \omega_0^2$, there is no maximum for positive ω, and the output amplitude simply decreases with increasing frequency.

If $2\gamma^2 < \omega_0^2$, the amplitude increases with frequency until it reaches a maximum, then decreases from there on. The maximum occurs at a frequency known as the *resonant frequency*.

$$\omega_R := \sqrt{\omega_0^2 - 2\gamma^2} \tag{7.46}$$

If we expand out the contents of the square root in the denominator of (7.29), we see that

$$(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2 = \omega_0^4 - 2\omega^2(\omega_0^2 - 2\beta^2) + \omega^4 = \omega_0^4 - 2\omega^2\omega_R^2 + \omega^4 \tag{7.47}$$

and so

$$A_s = \frac{F_0}{m\sqrt{\omega_0^4 - 2\omega^2\omega_R^2 + \omega^4}} \tag{7.48}$$

The maximum value of A_s, which occurs when $\omega = \omega_R$, is

$$(A_s)_{\max} = \frac{m\omega_0^4}{\sqrt{\omega_0^4 - \omega_R^4}} = \frac{m\omega_0^4}{\sqrt{\omega_0^4 - (\omega_0^4 - 4\beta^2\omega_0^2 + 4\beta^4)}} = \frac{m\omega_0^4}{2\beta\sqrt{\omega_0^2 - \beta^2}} = \frac{\omega_0^2}{2\beta\omega_1}m\omega_0^2 \tag{7.49}$$

Numerical Plot and Quality Factor So we'd like to plot the behaviour of A_s as a function of ω, using a computer. As usual, we start by trying to rewrite the functional relationship in terms of dimensionless quantities:

- $\frac{A_s}{F_0/m\omega_0^2}$ is dimensionless
- $\frac{\omega}{\omega_0}$ is dimensionless

However, when we try to write $\frac{A_s}{F_0/m\omega_0^2}$ in terms of $\frac{\omega}{\omega_0}$, we get

$$\frac{A_s}{F_0/m\omega_0^2} = \left[\left(1 - (\omega/\omega_0)^2\right) + 4\left(\gamma/\omega_0\right)^2\left(\omega/\omega_0\right)^2\right]^{-1/2} \tag{7.50}$$

This time not all of the other parameters have gone away. We still have the (dimensionless) quantity γ/ω_0, which describes the strength of the damping, floating around. This means that the shape of A_s vs ω is different for different values of the other parameters, and we cannot make the graphs for all sets of parameters look the same by simply scaling the axes.

The usual convention for specifying how strong the damping is is to define the dimensionless *quality factor*[3]

$$Q := \frac{\omega_0}{2\gamma} \tag{7.51}$$

so that an oscillator with relatively little damping (γ small) has a high Q factor.

[3]There actually doesn't seem to be a uniform convention on the definition of Q outside of the small-damping regime. Other definitions are $\omega_1/2\gamma$ and $\omega_R/2\gamma$, but since ω_1 and ω_R both go to ω as $\gamma \to 0$, all the definitions agree in the limit of high quality factor

In terms of this Q, (7.50) becomes

$$\frac{A_s}{F_0/m\omega_0^2} = \left[\left(1-(\omega/\omega_0)^2\right) + \frac{1}{Q^2}\left(\omega/\omega_0\right)^2\right]^{-1/2} \tag{7.52}$$

which we can now explore with a computer for different values of Q.

In class we defined the matlab function `ampfcn` by

```
function As_x0 = ampfcn(om_om0,Q);
  As_x0 = 1./sqrt( (1-om_om0.^2).^2 + om_om0.^2 / Q^2  );
return;
```

and the matlab function `phsfcn` by

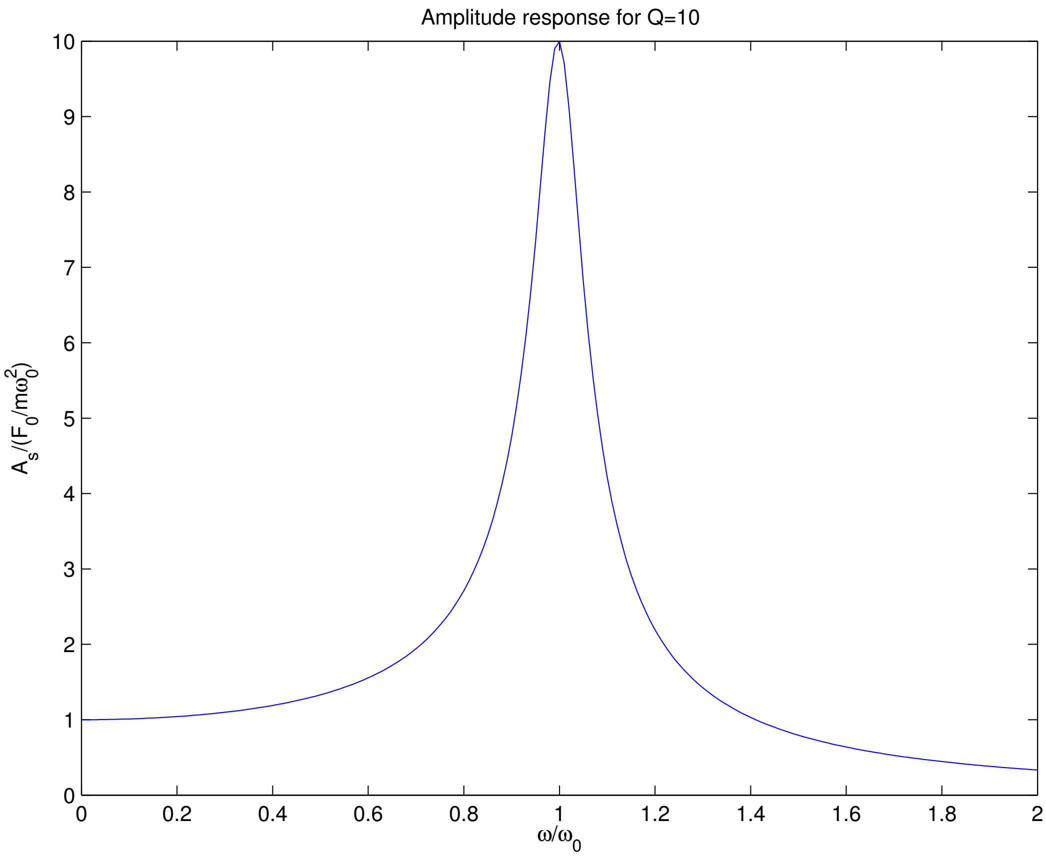
```
function beta = phsfcn(om_om0,Q);
  beta = atan(Q*(1-om_om0.^2)./om_om0);
return;
```

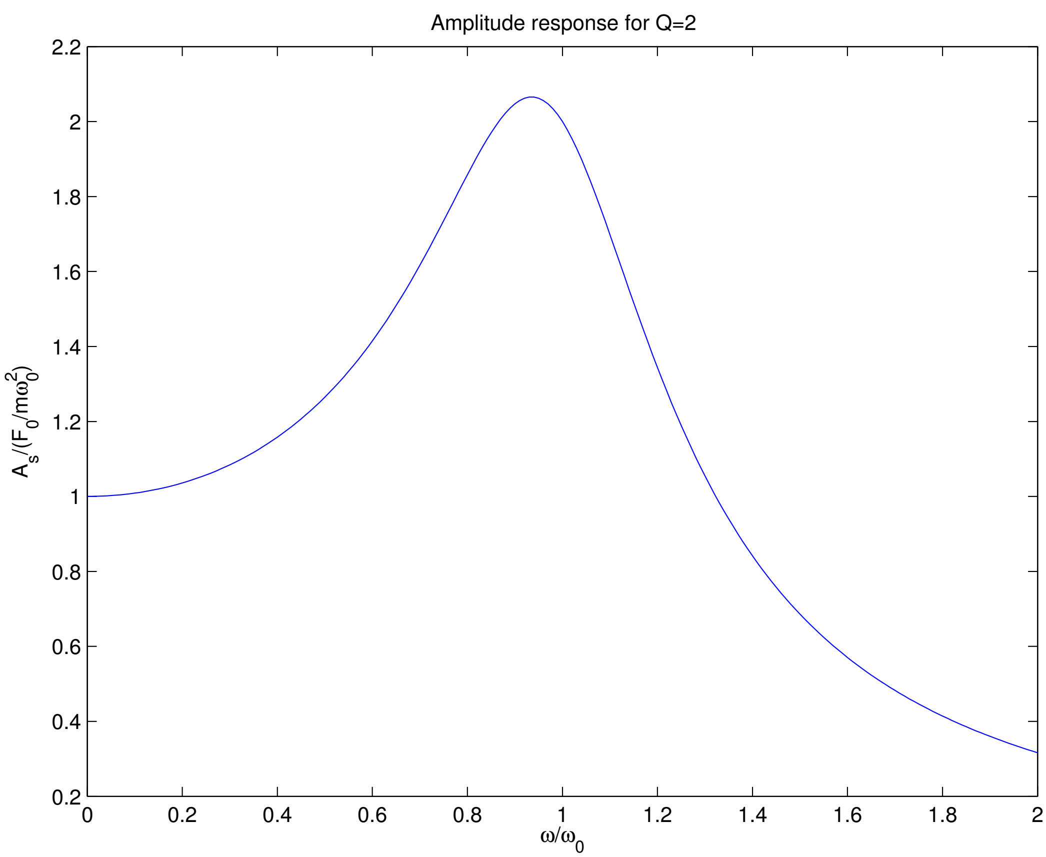
and then explored the functional dependence with the commands

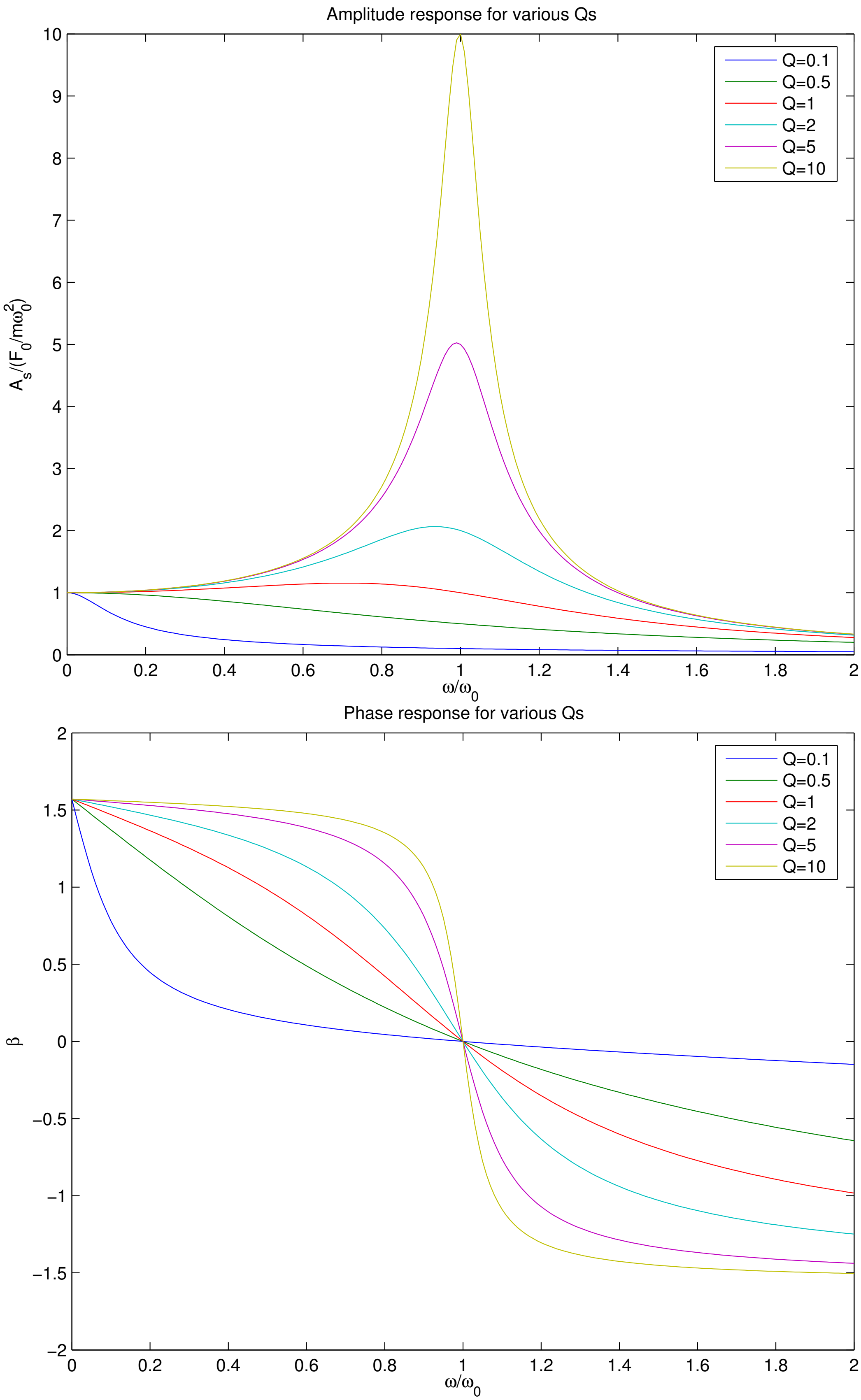
```
%-- 2/10/06  2:41 PM --%
om_om0 = (0:2)
om_om0 = (0:0.01:2);
Q = 10;
As_x0 = 1./sqrt( (1-om_om0.^2).^2 + om_om0.^2 / Q^2  )
As_x0 = 1./sqrt( (1-om_om0.^2).^2 + om_om0.^2 / Q^2  );
figure(1)
plot(om_om0,As_x0)
sqrt(1-1/(2*Q^2))
Q=2;
figure(2)
As_x0 = 1./sqrt( (1-om_om0.^2).^2 + om_om0.^2 / Q^2  );
plot(om_om0,As_x0)
sqrt(1-1/(2*Q^2))
xlabel('\omega/\omega_0')
ylabel('A_s/(F_0/m\omge_0^2)')
ylabel('A_s/(F_0/m\omega_0^2)')
title('Amplitude response for Q=10')
figure(1)
xlabel('\omega/\omega_0')
ylabel('A_s/(F_0/m\omega_0^2)')
title('Amplitude response for Q=10')
figure(2)
title('Amplitude response for Q=2')
% ampfn(0,10)
ampfcn(0,10)
A = ampfcn(0,10)
ampfcn(0,10)
```

```
figure(3)
plot(om_om0,ampfcn(om_om0,.1),om_om0,ampfcn(om_om0,.5),om_om0,ampfcn(om_om0,1),...
     om_om0,ampfcn(om_om0,.2),om_om0,ampfcn(om_om0,.10))
plot(om_om0,ampfcn(om_om0,.1),om_om0,ampfcn(om_om0,.5),om_om0,ampfcn(om_om0,1),...
     om_om0,ampfcn(om_om0,2),om_om0,ampfcn(om_om0,10))
plot(om_om0,ampfcn(om_om0,.1),om_om0,ampfcn(om_om0,.5),om_om0,ampfcn(om_om0,1),...
     om_om0,ampfcn(om_om0,2),om_om0,ampfcn(om_om0,5),om_om0,ampfcn(om_om0,10))
xlabel('\omega/\omega_0')
% help arctan
% help atan
figure(4)
plot(om_om0,phsfcn(om_om0,.1),om_om0,phsfcn(om_om0,.5),om_om0,phsfcn(om_om0,1),...
     om_om0,phsfcn(om_om0,2),om_om0,phsfcn(om_om0,5),om_om0,phsfcn(om_om0,10))
plot(om_om0,phsfcn(om_om0,.1))
phsfcn(om_om0,.1)
plot(om_om0,phsfcn(om_om0,.1))
plot(om_om0,phsfcn(om_om0,.1),om_om0,phsfcn(om_om0,.5),om_om0,phsfcn(om_om0,1),...
     om_om0,phsfcn(om_om0,2),om_om0,phsfcn(om_om0,5),om_om0,phsfcn(om_om0,10))
xlabel('\omega/\omega_0')
ylabel('\beta')
title('Phase response for various Qs')
% legend(.1,.5,1,2,5,10)
legend('0.1','0.5','1','2','5','10')
legend('Q=0.1','Q=0.5','Q=1','Q=2','Q=5','Q=10')
print -depsc2 -f4 phaseresp
ylabel('A_s/(F_0/m\omega_0^2)')
figure(3)
ylabel('A_s/(F_0/m\omega_0^2)')
title('Amplitude response for various Qs')
legend('Q=0.1','Q=0.5','Q=1','Q=2','Q=5','Q=10')
print -depsc2 -f3 ampresp
figure(2)
print -depsc2 -f2 ampresp_2
figure(1)
print -depsc2 -f1 ampresp_10
figure(4)
ylabel('\beta')
```

We produced the following figures:


Amplitude response for Q=10
$A_s/(F_0/m\omega_0^2)$
ω/ω_0
0
1
2
3
4
5
6
7
8
9
10
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2


Amplitude response for Q=2
$A_s/(F_0/m\omega_0^2)$
ω/ω_0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2


Amplitude response for various Qs
$A_s/(F_0/m\omega_0^2)$
ω/ω_0
Q=0.1
Q=0.5
Q=1
Q=2
Q=5
Q=10
0
1
2
3
4
5
6
7
8
9
10
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Phase response for various Qs
β
ω/ω_0
Q=0.1
Q=0.5
Q=1
Q=2
Q=5
Q=10
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Monday, February 13, 2006

7.2 Linear Superposition Methods for the Forced, Damped Harmonic Oscillator

We've solved the problem of a forced damped harmonic oscillator where the driving force is a sinusoid of a fixed frequency. You might worry that this is not terribly general, and we'd have to solve the equation

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F(t)/m \tag{7.53}$$

over and over again for different external driving forces.

Fortunately, the linearity of the differential operator means we only have to solve the problem for a few basic types of forces, then build up the solution for a more general force as a superposition of these solutions.

Symon considers two different ways to build up general forces out of component functions, and thus to build up the oscillator response to those forces out of the responses to the component functions.

1. We can find the response of a damped harmonic oscillator to an impulsive force, i.e., one which is very large for a very small stretch of time. An arbitrary force can be built up as a series of impulses, allowing the solution to a general problem by what's known as a Green's function method. This is useful and interesting, but we won't go into it in depth.

2. We've already found the response of an oscillator to a sinusoidal driving force; In fact an arbitrary periodic force can be written as a sum of such sinusoidal forces.

7.2.1 Periodic Forces and Fourier Methods

Recall that the driven, damped harmonic oscillator is governed by the differential equation

$$\mathcal{L}x = F(t)/m \tag{7.54}$$

where

$$\mathcal{L} = \frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2 \tag{7.55}$$

and that one solution when $F(t) = F_0\cos(\omega t + \theta_0)$ is

$$x_p(t) = \frac{F_0\sin(\omega t + \theta_0 + \beta)}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \tag{7.56}$$

where

$$\beta = \tan^{-1}\left(\frac{\omega_0^2 - \omega^2}{2\gamma\omega}\right) \tag{7.57}$$

(The general solution is $x(t) = x_h(t) + x_p(t)$ where the transient $x_h(t)$ is a solution to the homogeneous differential equation.)

The linearity of the differential operator means that if the force $F(t)$ is a sum of two different sinusoids at different frequencies, like so

$$F(t) = F_{01}\cos(\omega_1 t + \theta_{01}) + F_{02}\cos(\omega_2 t + \theta_{02}) \tag{7.58}$$

then the general solution will be

$$x(t) = x_h(t) + x_{p1}(t) + x_{p2}(t) \tag{7.59}$$

where

$$x_{pn}(t) = \frac{F_{0n}\sin(\omega_n t + \theta_{0n} + \beta_n)}{m\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\gamma^2\omega_n^2}} \tag{7.60}$$

and

$$\beta_n = \tan^{-1}\left(\frac{\omega_0^2 - \omega_n^2}{2\gamma\omega_n}\right) \tag{7.61}$$

and for that matter the same thing works if we add together any number of sinusoidal driving forces:

$$F(t) = \sum_{n=1}^{N} F_{0n}\cos(\omega_n t + \theta_{0n}) \tag{7.62}$$

produces the solution

$$x(t) = x_h(t) + \sum_{n=1}^{N} \frac{F_{0n}\sin(\omega_n t + \theta_{0n} + \beta_n)}{m\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\gamma^2\omega_n^2}} \tag{7.63}$$

But in fact *any* periodic function can be written as the sum of a bunch of sines and cosines by the method of Fourier series (see the supplemental exercises on Fourier series handout): if $F(t)$ is periodic (not necessarily sinusoidal) with period T (so that $F(t+T) = F(t)$), we can write

$$F(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\omega_n t + \sum_{n=1}^{\infty} B_n \sin\omega_n t \tag{7.64}$$

where

$$\omega_n = \frac{2\pi n}{T}\ ; \tag{7.65}$$

the coëfficients are given by

$$A_n = \frac{2}{T}\int_0^T F(t)\cos\omega_n t\ dt \tag{7.66a}$$

$$B_n = \frac{2}{T}\int_0^T F(t)\sin\omega_n t\ dt \tag{7.66b}$$

Given the Fourier series (7.64) we can quickly obtain the response to each term. First, note that the constant term $A_0/2$ is a cosine with zero frequency and zero phase shift; we saw in section 7.1.2 that this corresponds to a β of $\pi/2$. Armed with that knowledge we can write the force as

$$F(t) = \frac{A_0}{2}\cos(0t + 0) + \sum_{n=1}^{\infty} A_n\cos(\omega_n t + 0) + \sum_{n=1}^{\infty} B_n\cos(\omega_n t - \pi/2) \tag{7.67}$$

and obtain the response

$$
\begin{aligned}
x(t) &= x_h(t) + \frac{A_0}{2}\frac{\sin(0t+0+\pi/2)}{m\omega_0^2} \\
&\quad + \sum_{n=1}^{\infty}\frac{A_n\sin(\omega_n t+0+\beta_n)}{m\sqrt{(\omega_0^2-\omega_n^2)^2+4\gamma^2\omega_n^2}} + \sum_{n=1}^{\infty}\frac{B_n\sin(\omega_n t-\pi/2+\beta_n)}{m\sqrt{(\omega_0^2-\omega_n^2)^2+4\gamma^2\omega_n^2}} \\
&= x_h(t) + \frac{A_0}{2m\omega_0^2} + \sum_{n=1}^{\infty}\frac{A_n\sin(\omega_n t+\beta_n)}{m\sqrt{(\omega_0^2-\omega_n^2)^2+4\gamma^2\omega_n^2}} - \sum_{n=1}^{\infty}\frac{B_n\cos(\omega_n t+\beta_n)}{m\sqrt{(\omega_0^2-\omega_n^2)^2+4\gamma^2\omega_n^2}}
\end{aligned}
\tag{7.68}
$$

Since the amplitude of the response depends on the driving frequency, you can find that the response has a rather different time dependence than the driving force. For example, if one of the frequencies ω_n is close to the resonant frequency ω_R of the oscillator, the output look a lot like a sinusoid at that frequency, even if the input was dominated by other frequencies.

It's also worth mentioning that by considering the limit $T \to \infty$, *any* function can be written as a superposition of sinusoids. This is the method of Fourier transforms, and while we won't delve further into it in this course, it's outlined in the following section of the notes.

7.2.2 Fourier Transforms: Expressing a General Function as a Superposition of Periodic Terms (supplemental)

The complex Fourier series for a function with period T can be written as

$$x(t) = \sum_{n=-\infty}^{\infty} \hat{x}_n\, e^{-i\omega_n t} \tag{7.69}$$

$$\hat{x}_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{i\omega_n t}\, dt \tag{7.70}$$

frequency spacing

$$\delta\omega = \frac{2\pi}{T} \tag{7.71}$$

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{T}{2\pi}\hat{x}_n\, e^{-in\,\delta\omega\, t}\,\delta\omega \tag{7.72}$$

If we define

$$\tilde{x}(\omega_n) = \frac{T}{\sqrt{2\pi}}\hat{x}_n \tag{7.73}$$

then

$$x(t) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} \tilde{x}(\omega)\, e^{-in\,\delta\omega\, t}\,\delta\omega \tag{7.74}$$

and

$$\tilde{x}(\omega_n) = \frac{1}{\sqrt{2\pi}}\int_{-T/2}^{T/2} x(t)\, e^{i\omega_n t}\, dt \tag{7.75}$$

In the limit $T \to \infty$, $\delta\omega \to 0$ and the sum becomes an integral:

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{x}(\omega)\, e^{-i\omega t}\, d\omega \tag{7.76}$$

and

$$\tilde{x}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)\, e^{i\omega t}\, dt \tag{7.77}$$

$\tilde{x}(\omega)$ is called the Fourier transform of $x(t)$.

A Appendix: Correspondence to Class Lectures

Date	Sections	Pages	Topics
2006 January 16	I–3	3–6	Energy and Momentum; Forces Depending on Time; Forces Depending on Velocity
2006 January 20	4.1	9–11	Forces Depending on Distance; Potential Energy; Motion in a Potential
2006 January 23	4.2–4.3	12–13	Taylor Series and the Euler Relation; Approximating a Potential near a Stable Equilibrium
2006 January 27	5	14–16	The Simple Harmonic Oscillator
2006 January 30	6	17–20	The Damped Harmonic Oscillator
2006 February 3	7–7.1	21–24	The Forced Harmonic Oscillator: Setup & Linearity; Sinusoidal Driving Force
2006 February 6	7.1.1–7.1.2	25–27	The Forced Harmonic Oscillator: Determination of Amplitude & Phase
2006 February 10	7.1.3	28–33	The Forced Harmonic Oscillator: Amplitude Resonance
2006 February 13	7.2	34–37	Superposition and Fourier Series