Rigid Bodies and Mass Distributions (Symon Chapter Five)

Physics A300*

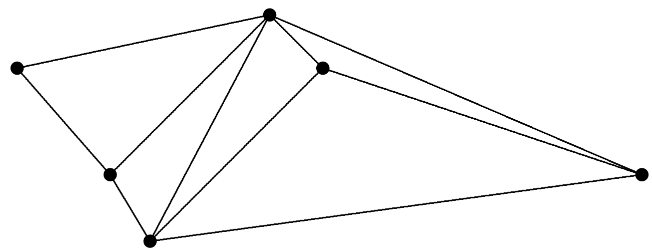
Spring 2006

Contents

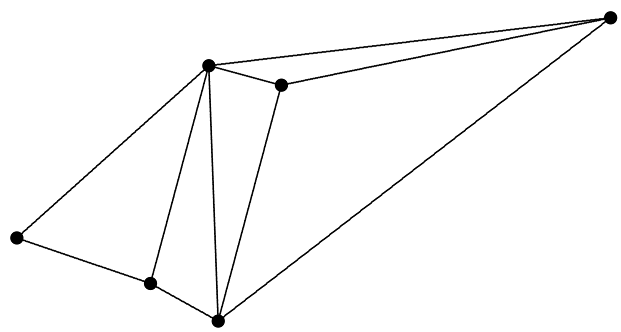
Thursday, April 13, 2006

1 Definition of a Rigid Body

A rigid body is a system of particles which all move together, like a bunch of point particles connected by massless rigid rods:



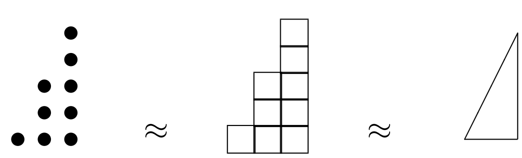
The rigidity is reflected by the fact that while each particle can move, and therefore change the position vectors, all the distances $|\vec{r}_k - \vec{r}_\ell|$ are constant. Only the orientation and overall location can change; there is no flexing or twisting:



Next semester, we will consider the general case; for now we will just consider rotation about a single axis.

2 Correspondence Between Collections of Point Masses and Continuous Mass Distributions

It is often useful to think about a continuous distribution of mass rather than a collection of discrete points. For example, one might want to know the properties of a slab of rock of a given density without having to count up all the individual atoms in the slab. We can still carry over our standard formulas by thinking of the mass distribution as a collection of infinitesimal little volume elements; the sum over particles is replaced by a sum over volume elements, which becomes a triple integral over the volume in the limit that the individual elements become small:



To make the correspondence, one replaces the particle of mass m_k at position $\vec{r}_k$ with a little block of volume d^3V at position $\vec{r}$. The solid has some density $\rho(\vec{r})$ at that position, in terms of which the mass of the little block, which we can call d^3M to emphasize its triply-infinitesimal nature, is[1]

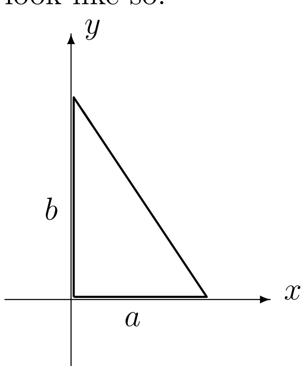
$$d^3M = \rho(\vec{r})\, d^3V \tag{2.1}$$

To translate any of our expressions from chapter four into the realm of continuous mass distributions, we just need to replace sums over k with triple integrals, and the factor of m_k which always appears in those sums is replaced with the infinitesimal volume d^3M from (2.1). This is easiest to see from the example of the total mass and the center of mass vector:

Point Masses	Mass Distribution
$M = \sum_{k=1}^{N} m_k$	$M = \iiint \rho(\vec{r})\, d^3V$
$\vec{R} = \frac{\sum_{k=1}^{N} m_k \vec{r}_k}{M}$	$\vec{R} = \frac{\iiint \rho(\vec{r})\, \vec{r}\, d^3V}{M}$

2.1 Example: Mass of a Prism

We'll worry about motion later; for now let's look at an example of how to calculate the mass for a solid body, This body is a right triangular prism of height h, which for simplicity we take to have uniform density ρ. The triangular faces have legs of length a and b, and look like so:



The prism is defined by

$$x \geq 0 \tag{2.2a}$$

$$y \geq 0 \tag{2.2b}$$

$$-\frac{h}{2} \leq z \leq \frac{h}{2} \tag{2.2c}$$

$$xb + ay \leq ab \tag{2.2d}$$

[1] Warning: Symon writes this relationship as $\rho = \frac{dM}{dV}$ which looks deceptively like a derivative, but doesn't make much sense if we try to interpret it as one.

To calculate the total mass, we have to integrate $\rho\, d^3V$ over the entire prism:

$$M = \iiint_{\text{prism}} \rho\, d^3V = \int_{-h/2}^{h/2} \iint_{\text{triangle}} \rho\, \overbrace{dx\, dy\, dz}^{d^3V} = \rho\, h \underbrace{\iint_{\text{triangle}} dx\, dy}_{\text{area}} \tag{2.3}$$

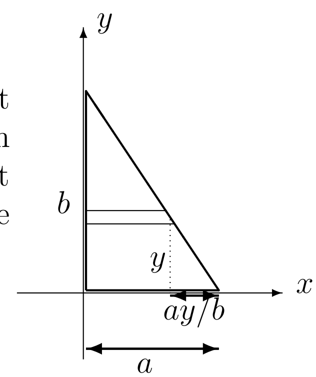
where the double integral over x and y is the area of the triangle.

This integral has to cover every point in the triangle once. As a review of how one does area integrals (which extends naturally to volume integrals), we follow through in detail how one does the integral in two different ways, either one of which is correct, and both of which (naturally) give the right answer.

2.1.1 x integral inside y integral

Here we integrate over x first, then over y. We need to choose the limits of those two integrals to cover each point in the triangle once. Taking the integrals from the outside in, the range of y values included in the prism is

$$0 \le y \le b \tag{2.4}$$



Now, the x integral is inside the y integral, which means that the limits need to cover the possible x values only for *that* given y. We can look at the triangle to see that the lower limit of that integral is still $x = 0$, but the upper limit is cut off by the edge of the triangle at $x = a\left(1 - \frac{y}{b}\right)$:

That then makes the integral for the area of the triangle[2]

$$\begin{aligned} \text{Area} = \frac{M}{\rho h} &= \int_0^b \left(\int_0^{a\left(1-\frac{y}{b}\right)} dx \right) dy = \frac{M}{\rho h} = \int_0^b a \left(1 - \frac{y}{b}\right) dy \\ &= a\left(y - \frac{y^2}{2b}\right)\bigg|_0^b = a\left(b - \frac{b}{2}\right) = \frac{ab}{2} \end{aligned} \tag{2.5}$$

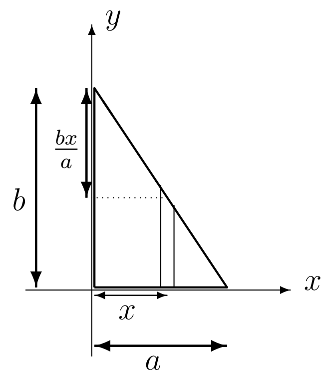
2.1.2 y integral inside x integral

Here we integrate over y first, then over x. We need to choose the limits of those two integrals to cover each point in the triangle once. Taking the integrals from the outside in, the range of x values included in the prism is

$$0 \le x \le a \tag{2.6}$$

[2]Note that the limit of the y integral does not depend on x, as it must not, since it's outside the x integral, while the limit of the x integral does depend on y, which it may, since it's inside the y integral.

Now, the y integral is inside the x integral, which means that the limits need to cover the possible y values only for *that* given x. We can look at the triangle to see that the lower limit of that integral is still $y = 0$, but the upper limit is cut off by the edge of the triangle at $y = b\left(1 - \frac{y}{a}\right)$:



That then makes the integral for the area of the triangle[3]

$$\begin{aligned} \text{Area} = \frac{M}{\rho h} &= \int_0^a \left(\int_0^{b\left(1-\frac{x}{a}\right)} dy \right) dx = \frac{M}{\rho h} = \int_0^a b\left(1 - \frac{x}{a}\right) dx \\ &= b\left(x - \frac{x^2}{2a}\right)\Big|_0^a = b\left(a - \frac{a}{2}\right) = \frac{ab}{2} \end{aligned} \tag{2.7}$$

which is the same answer as before

2.2 Excercise: Center-of-Mass Coördinates

As an exercise, you should do the corresponding integrals for the center of mass coördinates. These are the Cartesian compoments of the center-of-mass vector

$$\vec{R} = \frac{\iiint_{\text{prism}} \vec{r}\, d^3V}{M} = X\hat{x} + Y\hat{y} + Z\hat{z} \tag{2.8}$$

and the limits on the integrals over the prism are just the same ones we found when calculating the mass. This means, explicitly,

$$X = \frac{1}{M} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^b \int_0^{a\left(1-\frac{y}{b}\right)} \rho\, x\, dx\, dy\, dz \tag{2.9a}$$

$$Y = \frac{1}{M} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^b \int_0^{a\left(1-\frac{y}{b}\right)} \rho\, y\, dx\, dy\, dz \tag{2.9b}$$

$$Z = \frac{1}{M} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^b \int_0^{a\left(1-\frac{y}{b}\right)} \rho\, z\, dx\, dy\, dz \tag{2.9c}$$

Note that this only works because the Cartesian basis vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ are constants, and can be pulled out of the triple integrals.

[3]Note that the limit of the x integral does not depend on y, as it must not, since it's outside the y integral, while the limit of the y integral does depend on x, which it may, since it's inside the x integral.

Monday, April 17, 2006

3 Rotation about an Axis

We close with a brief consideration of rigid body rotation. Next semester we'll consider the general problem of rotation about an arbitrary axis, but for now consider the rotation of a solid of revolution about its axis of symmetry, I.e., consider a body whose density $\rho(q,z)$ is independent of the angle ϕ, rotating about the z axis with angular velocity ω. (ω here is a signed quantity, like the velocity in one dimension, which is positive if the rotation is counter-clockwise and negative if it's clockwise.) That way, even as the body rotates, the density at a given point in space remains the same.

If we call the body $\mathcal{V}$ we would like to calculate not only the total mass

$$M = \iiint\limits_{\mathcal{V}} \rho(\vec{r})\, d^3V \tag{3.1}$$

and center of mass position

$$\vec{R} = \frac{1}{M} \iiint\limits_{\mathcal{V}} \vec{r}\, d^3V \tag{3.2}$$

but also the angular momentum about the origin

$$\vec{L} = \iiint\limits_{\mathcal{V}} \rho(\vec{r})\, \vec{r} \times \vec{v}(\vec{r})\, d^3V \tag{3.3}$$

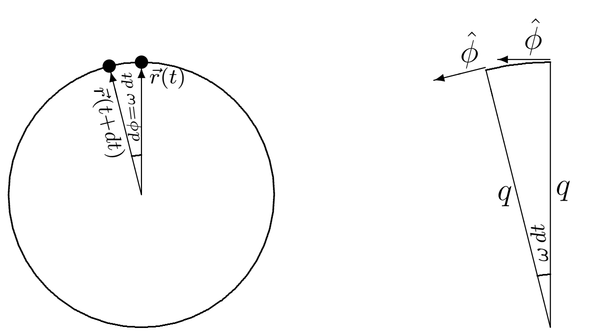
To do this we need the velocity $\vec{v}(\vec{r})$ at a point with position vector $\vec{r}$.w

First we ask what the speed is at a point with coördinates (x,y,z) or equivalently with cylindrical coördinates (q,ϕ,z), where

$$x = q\cos\phi \tag{3.4a}$$

$$y = q\sin\phi\ . \tag{3.4b}$$

Since the object is just rotating about the z axis, a point a distance q from that axis is moving in a circle of radius q. In an infinitesimal time dt, a piece of the object moves through an angle $d\phi = \omega\, dt$ and therefore moves a distance $q\,\omega\, dt$:



which means (remembering that ω can be positive or negative as we've defined it)

$$|\vec{v}(q,\phi,z)| = q\ |\omega| \tag{3.5}$$

Now, as for the direction, looking at the figure, we see the velocity is parallel to $\hat{\phi}$ for positive ω, and it's not hard to work out that it's antiparallel to $\hat{\phi}$ for negative ω, which means

$$\vec{v} = q\,\omega\,\hat{\phi} \tag{3.6}$$

and thus

$$\vec{r} \times \vec{v} = (q\,\hat{q} + z\,\hat{z}) \times q\,\omega\,\hat{\phi} = q^2\omega\hat{z} - qz\omega\,\hat{q} \tag{3.7}$$

which means

$$\vec{L} = \iiint\limits_{\mathcal{V}} \rho(q,z)\left(q^2\omega\hat{z} - qz\omega\,\hat{q}\right) q\,dq\,d\phi\,dz \tag{3.8}$$

Now, it turns out the contribution from the second term in parentheses is zero because

$$\int_0^{2\pi} \hat{q}\,d\phi = \int_0^{2\pi} (\hat{x}\cos\phi + \hat{y}\sin\phi)\,d\phi = \vec{0} \tag{3.9}$$

at any rate, all we're interested for a body rotating about the z axis is

$$L_z = \hat{z}\cdot\vec{L} = \iiint\limits_{\mathcal{V}} \rho(q,z)\,q^2\omega\,d^3V \tag{3.10}$$

Since ω is a constant, we can factor it out of the integral and get

$$L_z = \underbrace{\left[\iiint\limits_{\mathcal{V}} q^2\,\rho(\vec{r})\,d^3V\right]}_{I_z}\omega \tag{3.11}$$

The quantity in square brackets is called the *moment of inertia* about the z axis. Note that although we write it as I_z, it is *not* the z component of a vector. It is part of a more complicated geometrical object which we will study next semester.

3.1 Example: Constant-Density Cylinder

Consider a right circular cylinder of height h, radius a, and constant density and total mass M. Calculate the moment of inertia for rotations about its axis of symmetry.

Since we're told the mass and not the density (other than that it is a constant), we first need to calculate the mass in terms of the unknown density ρ:

$$M = \int_{-h/2}^{h/2}\int_0^{2\pi}\int_0^a \rho\,q\,dq\,d\phi\,dz = \rho\underbrace{\int_{-h/2}^{h/2} dz}_{h}\underbrace{\int_0^{2\pi} d\phi}_{2\pi}\underbrace{\int_0^a q\,dq}_{a^2/2} = \pi a^2 h\,\rho \tag{3.12}$$

Thus the density is $\frac{M}{\pi a^2 h}$ and the moment of inertia is

$$\begin{aligned} I_z &= \int_{-h/2}^{h/2}\int_0^{2\pi}\int_0^a \frac{M}{\pi a^2 h}\,q^2\,q\,dq\,d\phi\,dz \\ &= \frac{M}{\pi a^2 h}\underbrace{\int_{-h/2}^{h/2} dz}_{h}\underbrace{\int_0^{2\pi} d\phi}_{2\pi}\underbrace{\int_0^a q^3\,dq}_{a^4/4} = \frac{M}{\pi a^2 h}\frac{\pi a^4 h}{2} = \frac{1}{2}Ma^2 \end{aligned} \tag{3.13}$$

As an exercise, repeat this calculation for a constant-density sphere. (It's easiest to do the integral in spherical coördinates, in which case you'll need to convert q^2 into spherical coördinates as well.)

3.2 Example: Non-Uniform Mass Distribution

Consider a mass distribution

$$\rho(q, \phi, z) = \rho_0 e^{-z^2/b^2} e^{-q^2/a^2} \tag{3.14}$$

Calculate the mass M and moment of inertia I_z in terms of ρ_0, a, and b, then write I_z in terms of M, a, and b without reference to ρ_0.

The total mass is

$$M = \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \overbrace{\rho_0 e^{-z^2/b^2} e^{-q^2/a^2}}^{\rho(q,\phi,z)} q\, dq\, d\phi\, dz = \rho_0 \int_{-\infty}^{\infty} e^{-z^2/b^2} dz \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_0^{\infty} q\, e^{-q^2/a^2}\, dq \tag{3.15}$$

The q integral is relatively straightforward:

$$\int_0^{\infty} q\, e^{-q^2/a^2}\, dq = -\frac{a^2}{2}\, e^{-q^2/a^2}\Big|_0^{\infty} = \frac{a^2}{2} \tag{3.16}$$

The z integral is also a common one in physics, the integral of a Gaussian, but trickier to evaluate. We could look up the standard result, but the calculation is actually kind of clever, so we repeat it here. First, note that since the integrand is positive definite and the upper limit is greater than the lower limit, the result must be positive:

$$\int_{-\infty}^{\infty} e^{-z^2/b^2} dz > 0 \tag{3.17}$$

That means

$$\int_{-\infty}^{\infty} e^{-z^2/b^2} dz = \left| \int_{-\infty}^{\infty} e^{-z^2/b^2} dz \right| = \sqrt{\left(\int_{-\infty}^{\infty} e^{-z^2/b^2} dz \right)^2} \tag{3.18}$$

But now there are two copies of this integral. Since it's a definite integral, we can call the integration variable in each whatever we want, so let's rename the z in the first integral x and the one in the second y:

$$\int_{-\infty}^{\infty} e^{-z^2/b^2} dz = \sqrt{\left(\int_{-\infty}^{\infty} e^{-x^2/b^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2/b^2} dy \right)} = \sqrt{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/b^2}\, dx\, dy} \tag{3.19}$$

But now if we make the change of variables in this double integral from (x, y) to (r, ϕ), this is mathematically equivalent to changing from Cartesian to polar coördinates, so

$$\int_{-\infty}^{\infty} e^{-z^2/b^2} dz = \sqrt{\int_0^{2\pi} \int_{-\infty}^{\infty} e^{-r^2/b^2}\, r\, dr\, d\phi} = \sqrt{2\pi \int_{-\infty}^{\infty} r\, e^{-r^2/b^2}\, dr} = b\sqrt{\pi} \tag{3.20}$$

where the calculation of the r integral is identical to the one we've already done in (3.16).

Putting it all together, the mass is

$$M = \rho_0\, a\sqrt{\pi}\, 2\pi\, \frac{a^2}{2} = (2\pi)^{3/2}\rho_0 a^2 b \tag{3.21}$$

To calculate the moment of inertia, we take

$$I_z = \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} \overbrace{\rho_0 e^{-z^2/b^2} e^{-q^2/a^2}}^{\rho(q,\phi,z)} q^2\, q\, dq\, d\phi\, dz = \rho_0 \underbrace{\int_{-\infty}^{\infty} e^{-z^2/b^2} dz}_{b\sqrt{\pi}} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_0^{\infty} q^3\, e^{-q^2/a^2}\, dq \tag{3.22}$$

The q integral can be done via integration by parts with

$$u = q^2 \qquad \Longrightarrow \qquad du = 2q\, dq \tag{3.23a}$$

$$dv = e^{-q^2/a^2}\, q\, dq \qquad \Longrightarrow \qquad v = -\frac{a^2}{2} e^{-q^2/a^2} \tag{3.23b}$$

to get

$$\int_0^{\infty} q^3\, e^{-q^2/a^2}\, dq = -\frac{a^2}{2} q^2 e^{-q^2/a^2}\Big|_0^{\infty} \nearrow 0 + a^2 \int_0^{\infty} q\, e^{-q^2/a^2}\, dq = \frac{a^4}{2} \tag{3.24}$$

which means

$$I_z = \rho_0 b\sqrt{\pi}\, 2\pi\, \frac{a^4}{2} = (2\pi)^{3/2}\rho_0 a^4 b \tag{3.25}$$

or, in terms of the mass,

$$I_z = M a^2 \tag{3.26}$$

A Appendix: Correspondence to Class Lectures

Date	Sections	Pages	Topics
2006 April 13	1–2	2–5	Def'n of rigid body; discrete→continuous transition
2006 April 17	3	6–9	Rotation about an axis; moment of inertia
2006 April 21	(Review)		