# Physics A300: Classical Mechanics I

Problem Set 4

Assigned 2006 February 3
Due 2006 February 10

**Show your work on all problems!** Note that the answers to the some problems may be in the appendix of Symon, but they should only be used to check your work, since the answers alone are an insufficient solution to the problem.

Instructions to find $y$ as a function of or in terms of $a$, $b$, and $c$ should *not* be taken as an assertion that $a$, $b$, and $c$ will all appear in the correct answer.

## 1 Overdamped Oscillations

By analogy to the formulas

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

which can be derived from the Euler relation, one defines the *hyperbolic trigonometric functions*

$$\sinh\eta = \frac{e^{\eta} - e^{-\eta}}{2} \qquad \cosh\eta = \frac{e^{\eta} + e^{-\eta}}{2}$$

a) Using the definitions above, *derive* (i.e., do not just look up) expressions for $\sin i\eta$ and $\cos i\eta$ in terms of $\sinh\eta$ and $\cosh\eta$.

b) Show that the general overdamped solution given in Symon's equation (2.140) can be rewritten as
$$x(t) = e^{-\gamma t}(B_c \cosh\omega_h t + B_s \sinh\omega_h t) \tag{1.1}$$
where $\omega_h = \sqrt{\gamma^2 - \omega_0^2}$. Find the values of $B_c$, $B_s$ in terms of $C_1$, and $C_2$.

c) By extending the definition in Symon's equation (2.129) to the case where $\gamma > \omega_0$ as $\omega_1 = i\omega_h$ (where again $\omega_h = \sqrt{\gamma^2 - \omega_0^2}$) show that the general underdamped solution in Symon's equation (2.134) is equivalent to (1.1) above and determine the resulting $B_c$ and $B_s$ in terms of $B_1$ and $B_2$. What conditions must $B_1$ and $B_2$ satisfy for $x(t)$ to be real in this case?

## 2 Effects of Signs of Force Terms

Symon Chapter Two, Problem 36. Note that $m$, $b$, and $k$ are all supposed to be positive in each case.

# 3 Damped Oscillator Driven by Decaying Driving Force

Consider an underdamped harmonic oscillator with spring constant $m\omega_0^2$ and damping parameter $2m\gamma$ (with $\gamma < \omega$) driven by the force

$$F(t) = F_0 \, e^{-\alpha t} \cos(\omega t + \theta_0) \tag{3.1}$$

Find a solution of the form

$$x(t) = A_s \, e^{-\alpha t} \cos(\omega t + \theta_s) \tag{3.2}$$

and work out the values of $A_s$ and $\theta_s$ in terms of $\gamma$, $\omega$, $\omega_0$, $\alpha$, and $F_0$. [Hint: write $F(t)$ as a superposition of two terms, using $\cos \omega t = (e^{i\omega t} + e^{-i\omega t})/2$, as we did in class.]

# 4 Initial Conditions and Transients

A force $F_0 \cos(\omega t + \theta_0)$ acts on an underdamped harmonic oscillator with spring constant $m\omega_0^2$, damping constant $2m\gamma$, and mass $m$, beginning at $t = 0$. As we showed in class, the general solution for $t > 0$ can be written

$$x(t) = A_s \sin(\omega t + \theta_0 + \beta) + A \cos\theta \, e^{-\gamma t} \cos \omega_1 t - A \sin\theta \, e^{-\gamma t} \sin \omega_1 t \tag{4.1}$$

where

$$\beta = \tan^{-1} \frac{\omega_0^2 - \omega^2}{2\gamma\omega} \tag{4.2}$$

and

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2} \tag{4.3}$$

a) What must the initial position and velocity be in order for there to be no transient?

b) If instead $x(0) = \dot{x}(0) = 0$, find the coëfficients $A \cos\theta$ and $A \sin\theta$ of the transient in terms of $F_0$ and $\theta_0$.