Physics A300: Classical Mechanics I

Problem Set 6

Assigned 2006 March 3
Due 2006 March 10

Show your work on all problems!

1 Work Done Along a Path

Consider the path parametrized by

$$x(s) = s\,x_0 \tag{1.1a}$$
$$y(s) = s\,y_0 \tag{1.1b}$$
$$z(s) = s\,z_0 \tag{1.1c}$$

where s ranges from 0 to 1. The position vector associated with this path is

$$\vec{r}(s) = \hat{x}\,x(s) + \hat{y}\,y(s) + \hat{z}\,z(s) \tag{1.2}$$

a) What are the position vectors $\vec{r}(0)$ and $\vec{r}(1)$ of the endpoints of this path?

b) Describe the path succinctly in words.

c) Calculate the derivative $\frac{d\vec{r}}{ds}$ of the position vector with respect to the parameter.

d) Suppose that a particle moves along this path while being acted on by a force field $\vec{F}(\vec{r})$ with components

$$F_x(x, y, z) = ay \tag{1.3a}$$
$$F_y(x, y, z) = ax + by^3 + cyz \tag{1.3b}$$
$$F_z(x, y, z) = bz^3 + cy^2 z \tag{1.3c}$$

i) Write the dot product $\vec{F}(\vec{r}(s)) \cdot \frac{d\vec{r}}{ds}$, using the trajectory (1.1) to substitute for x, y, and z and write your answer only as a function of s (and the constants a, b, c, x_0, y_0, and z_0).

ii) Calculate the work done by the force (1.3) on the particle as it moves along the path $\vec{r}(s)$ from $s = 0$ to $s = 1$. (Note that there must be other forces involved in the problem to keep the particle on this path, so Newton's second law is not really useful here.)

2 Conversion to Polar Coördinates

Converting a two-dimensional vector field from Cartesian to polar coördinates requires application not only of the coördinate transformations

$$x = r\cos\phi \tag{2.1a}$$
$$y = r\sin\phi \tag{2.1b}$$

but also the definitions of the basis vectors adapted to the two coördinate systems:

$$\hat{x} = \hat{r}\,\cos\phi - \hat{\phi}\,\sin\phi \tag{2.2a}$$
$$\hat{y} = \hat{r}\,\sin\phi + \hat{\phi}\,\cos\phi \tag{2.2b}$$

a) Show explicitly starting from (2.2) that

$$\hat{x}\,\cos\phi + \hat{y}\,\sin\phi = \hat{r} \tag{2.3a}$$
$$-\hat{x}\,\sin\phi + \hat{y}\,\cos\phi = \hat{\phi} \tag{2.3b}$$

b) Convert the following vector fields into polar coördinates. As an example, the vector field $\vec{F} = -kx\,\hat{x} - ky\,\hat{y}$ would be written $\vec{F} = -kr\,\hat{r}$.

   i) $\vec{F} = -kx\,\hat{x}$

   ii) $\vec{F} = -kx\,\hat{y} + ky\,\hat{x}$

   iii) $\vec{F} = -\frac{\alpha}{x^2+y^2}\,(x\,\hat{x} + y\,\hat{y})$

3 Spherical Coördinates

Consider the unit vectors

$$\hat{r} = \sin\theta\cos\phi\;\hat{x} + \sin\theta\sin\phi\;\hat{y} + \cos\theta\;\hat{z} \tag{3.1a}$$
$$\hat{\theta} = \cos\theta\cos\phi\;\hat{x} + \cos\theta\sin\phi\;\hat{y} - \sin\theta\;\hat{z} \tag{3.1b}$$
$$\hat{\phi} = -\sin\phi\;\hat{x} + \cos\phi\;\hat{y} \tag{3.1c}$$

a) Using the usual expression for the dot product in terms of Cartesian components [e.g., Symon's Eq. (3.23)], calculate explicitly the six independent inner products $\hat{r}\cdot\hat{r}$, $\hat{r}\cdot\hat{\theta}$, $\hat{r}\cdot\hat{\phi}$, $\hat{\theta}\cdot\hat{\theta}$, $\hat{\theta}\cdot\hat{\phi}$ and $\hat{\phi}\cdot\hat{\phi}$, and thereby show that the unit vectors defined in (3.1) are themselves an orthonormal basis.

b) Using the usual expression for the dot product in terms of Cartesian components [e.g., Symon's Eq. (3.33)], calculate $\hat{r}\times\hat{\theta}$, $\hat{\theta}\times\hat{\phi}$, and $\hat{\phi}\times\hat{r}$.

c) By differentiating the form (3.1), calculate the nine partial derivatives $\frac{\partial\hat{r}}{\partial r}$, $\frac{\partial\hat{r}}{\partial\theta}$, $\frac{\partial\hat{r}}{\partial\phi}$, $\frac{\partial\hat{\theta}}{\partial r}$, $\frac{\partial\hat{\theta}}{\partial\theta}$, $\frac{\partial\hat{\theta}}{\partial\phi}$, $\frac{\partial\hat{\phi}}{\partial r}$, $\frac{\partial\hat{\phi}}{\partial\theta}$ and $\frac{\partial\hat{\phi}}{\partial\phi}$. First express your results in terms of the Cartesian basis vectors (with components written in terms of the spherical coördinates r, θ, and ϕ). Then use your results along with (3.1) to verify Symon's Eq. (3.99) for the derivatives written purely in terms of the spherical coördinates and the corresponding basis.