Physics A300: Classical Mechanics I

Problem Set 8

Assigned 2006 March 16
Due 2006 March 24

1 Logarithmic Spiral Orbit

Consider a particle of mass m following the trajectory

$$q(t) = r(t) = r_0\sqrt{at+b} \tag{1.1a}$$
$$\phi(t) = \phi_0 \ln(at+b) \tag{1.1b}$$
$$z(t) = 0 \tag{1.1c}$$

where a, b, r_0, and ϕ_0 are all constants.

a) Calculate the angular momentum L_z about the z axis and verify that it is a constant.

b) Assuming this trajectory is an orbit in a central force field $\vec{F} = F(r)\hat{r}$, find the form of $F(r)$. [Hint: use the trajectory (1.1) to write the radial component of the acceleration vector as a function of t, then use (1.1a) to replace the t dependence with r dependence.]

c) Integrate your result from part b) to obtain an expression for the potential energy $V(r)$.

d) Use the explicit form of the trajectory to work out the kinetic energy T and potential energy V as functions of time for this trajectory, calculate the total energy E, and verify that it is a constant.

2 Central Force with Quadratic Potential

Consider a particle of mass m moving with angular momentum L in a potential $V(r) = \frac{1}{2}kr^2$.

a) Construct the following combinations of k, L, and m: i) E_u, with units of energy and ii) r_u, with units of length.

b) Construct the effective potential $V_{\text{eff}}(r)$, write V_{eff}/E_u as a function of r/r_u, and use a computer plotting program to plot V_{eff}/E_u versus r/r_u. Be sure to include the commands used as well as the plot itself. (Hint: consider the combinations E_u/r_u^2 and $E_u r_u^2$.)

c) For what values of total energy are there two turning points r_{min} and r_{max}? Find r_{min} and r_{max} in terms of the energy E.

d) Use the function $V_{\text{eff}}(r)$ to find the radius r_{circ} of a circular orbit with angular momentum L. What is the total energy E_{circ} of this orbit?

e) For an energy only slightly larger than E_{circ}, calculate the frequency ω_R of the small radial oscillations about r_{circ}. Calculate the angular frequency ω_Φ of the angular oscillations when $r \approx r_{\text{circ}}$ and compare the two frequencies *quantitatively*. (Both frequencies should be expressed in terms of the parameters k, m, and L, and not in terms of e.g., r_{circ} or E_{circ}.)

3 Circular Orbits in a Gravitational Field

Note: None of your answers to this problem should involve the constant K; you should use the relationship $K = -GMm$ to express them in terms of the masses of the attracting body and the test particle.

Consider a test particle of mass m moving in a circular orbit of radius R under the gravitational attraction of a body of mass M fixed at the center of the circle.

a) Use Kepler's third law [see, e.g., Symon's Eq. (3.267)] to calculate the orbital speed v as a function of R.

b) Use the fact that this orbit has semimajor axis $a = R$ and eccentricity $\varepsilon = 0$, and the expressions for L and E in terms of the orbital parameters to express the total energy E and angular momentum L as functions of the radius R of the orbit (and not of each other or v).

c) Use the result of part a) to find the kinetic energy T as a function of R.

d) Write the potential energy $V(R)$ and verify that $T + V = E$.

e) Suppose we reduce the orbital energy from a satellite in such a way that it changes from one circular orbit to another. Do the following quantities increase or decrease?

i) orbital radius; ii) orbital speed; iii) orbital period

iv) kinetic energy; v) potential energy; vi) orbital angular momentum