Physics A300: Classical Mechanics I

Problem Set 10

Assigned 2006 April 10
Due 2006 April 17

1 Decomposition of Angular Momentum

a) Substitute Symon's (4.19), the definition of angular momentum of particle k about a point $\mathcal{Q}$, into Symon's (4.23), the total angular momentum of a system about $\mathcal{Q}$. Expand the cross product $(\vec{r}_k - \vec{r}_{\mathcal{Q}}) \times (\dot{\vec{r}}_k - \dot{\vec{r}}_{\mathcal{Q}})$ appearing inside the sum in the resulting expression for $\vec{L}_{\mathcal{Q}}$ and use the definitions of the total mass M, center of mass $\vec{R}$, and total momentum $\vec{P}$ of the system to simplify your expression for $\vec{L}_{\mathcal{Q}}$ so that only one of the four terms still explicitly contains the sum over k, and the rest only contain M, $\vec{R}$, $\vec{P}$, $\vec{r}_{\mathcal{Q}}$, and $\dot{\vec{r}}_{\mathcal{Q}}$.

b) Simplify the result of part a) in the special case where the point $\mathcal{Q}$ is the origin of coördinates ($\vec{r}_{\mathcal{Q}} = \vec{0}$). Call this the total angular momentum $\vec{L}$.

c) Simplify the result of part a) in the special case where the point $\mathcal{Q}$ is the center of mass ($\vec{r}_{\mathcal{Q}} = \vec{R}$). Call this the angular momentum $\vec{L}_{\text{com}}$ relative to the center of mass.

d) Use the results of parts b) and c) to find an expression for the total angular momentum $\vec{L}$ in terms of $\vec{R}$, $\vec{P}$, and $\vec{L}_{\text{com}}$.

2 Internal and External Potential Gravitational Forces

The gravitational potential energy of two point masses m_1 and m_2 moving in the external gravitational field of a point mass m_0 fixed at the origin is

$$V(\vec{r}_1, \vec{r}_2) = \underbrace{-\frac{Gm_0m_1}{r_1}}_{V_1^e(\vec{r}_1)} + \underbrace{-\frac{Gm_0m_2}{r_2}}_{V_2^e(\vec{r}_2)} + \underbrace{-\frac{Gm_1m_2}{r}}_{V^i(\vec{r}_1,\vec{r}_2)} \tag{2.1}$$

where

$$r_1 = |\vec{r}_1| = \sqrt{x_1^2 + y_1^2 + z_1^2} \tag{2.2a}$$

$$r_2 = |\vec{r}_2| = \sqrt{x_2^2 + y_2^2 + z_2^2} \tag{2.2b}$$

$$r = |\vec{r}_1 - \vec{r}_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \tag{2.2c}$$

a) Make a sketch of the locations of the three masses and the vectors $\vec{r}_1$, $\vec{r}_2$, and $\vec{r} = \vec{r}_1 - \vec{r}_2$. Indicate the distances r_1, r_2, r.

b) Calculate the gradients $\vec{\nabla}_1 r_1$, $\vec{\nabla}_1 r_2$, $\vec{\nabla}_1 r$, $\vec{\nabla}_2 r_1$, $\vec{\nabla}_2 r_2$, and $\vec{\nabla}_2 r$. Express your answers both in Cartesian coördinates and then in terms of the vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}$ and the magnitudes r_1, r_2, r.

c) Find the internal forces $\vec{F}_1^i = -\vec{\nabla}_1 V^i$ and $\vec{F}_2^i = -\vec{\nabla}_2 V^i$ and verify that the strong form of Newton's third law holds, i.e., that the vectors $\vec{F}_1^i$ and $\vec{F}_2^i$ are equal in magnitude, opposite in direction, and directed along the line connecting the locations of masses 1 and 2.

d) Find the external forces $\vec{F}_1^e = -\vec{\nabla}_1 V_1^e$ and $\vec{F}_2^e = -\vec{\nabla}_2 V_2^e$.

e) Using the formalism of the two-body problem, find an exact expression for $M\ddot{\vec{R}}$ in terms of m_0, m_1, m_2, $\vec{r}_1$, and $\vec{r}_2$, and their magnitudes. (Here $M = m_1 + m_2$ is the total mass and $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/M$ is the center of mass vector of the two freely-moving particles.)

f) Using the formalism of the two-body problem, find an exact expression for $\mu\ddot{\vec{r}}$, where $\mu = m_1 m_2/M$ is the reduced mass of the two freely-moving particles. Note that Symon's equation (4.96) does *not* hold, and you will need to retain a term

$$\vec{F}^t(\vec{r}_1, \vec{r}_2) = \mu\left(\frac{\vec{F}_1^e}{m_1} - \frac{\vec{F}_2^e}{m_2}\right) \tag{2.3}$$

Evaluate $\vec{F}^t$ explicitly in terms of $\vec{r}_1$ and $\vec{r}_2$.

g) (bonus) In the limit that particles 1 and 2 are much closer to each other than they are to the origin, $R = \left|\vec{R}\right| \gg r$, things simplify somewhat. Using Symon's (4.92–4.93), expand your result for $M\ddot{\vec{R}}$ to zeroth order and your result for $\vec{F}^t$ to first order in the small parameter $\xi = r/R$. Explicitly, this means

i) In your expression for $M\ddot{\vec{R}}$, just replace $\vec{r}_1$ and $\vec{r}_2$ with $\vec{R}$ (and similarly for their magnitudes) and you should get an answer which depends only on the properties of the 1-2 system as a whole (M and $\vec{R}$). What does this correspond to physically?

ii) In your expression for $\vec{F}^t$, you'll need to substitute in Symon's (4.92–4.93) with $\vec{r} = \xi R\hat{r}$ and $r = \xi R$ into the results of part f) and then Taylor expand the result and keep the terms linear in ξ. Then you should be able to replace ξ with r/R and end up with an expression which depends on M, μ, $\vec{r}$, and $\vec{R}$. This describes the effects of the tidal field of the mass m_0 on the two-body system of masses 1 and 2.

3 Properties of a Mass Distribution

Consider the right triangular pyramid defined by

$$x \geq 0 \tag{3.1a}$$
$$y \geq 0 \tag{3.1b}$$
$$z \geq 0 \tag{3.1c}$$
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1 \tag{3.1d}$$

with a constant density ρ.

a) Sketch a (two-dimensional) constant-z cross-section of the pyramid. Indicate the x and y axes and label the coördinates of any points on the edges of the pyramid. (Some of these will depend upon z.)

b) Calculate the total mass M of this pyramid by evaluating a triple integral over the volume of the pyramid. (Do *not* just use a formula for the volume of a pyramid.)

c) By performing triple integrals, calculate X, Y, and Z, the coördinates of the center of mass of the pyramid.