Physics A301: Classical Mechanics II

Problem Set 1

Assigned 2006 May 8
Due 2006 May 15

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Position-Dependent Mass Distribution

Consider a spherically symmetric distribution of mass with density

$$\rho(\vec{r}') = \frac{Ma^2}{2\pi r'(r'^2 + a^2)^2} \qquad 0 \le r' < \infty \tag{1.1}$$

a) Show by explicit calculation that the parameter M appearing in (1.1) is indeed the total mass of the distribution.

b) Calculate (by direct evaluation of the integral over source points) the gravitational potential $\varphi(\vec{r})$ due to this mass distribution. (Hint: do the integral over the source point $\vec{r}'$ in spherical coördinates oriented so that θ' is the angle between $\vec{r}$ and $\vec{r}'$. You'll also need to break up the integral over r' into a piece where $r' < r$ and a piece where $r' > r$.)

c) From your result to part b), calculate the gravitational field $\vec{g}(\vec{r})$ of this mass distribution, taking the gradient ($\vec{g} = -\vec{\nabla}\varphi$) in spherical coördinates.

2 Flat Earth Society

Consider an infinite sheet of mass of thickness L and uniform density ρ. Use symmetry properties of the mass distribution, along with Gauss's law for gravitation (which says that the flux of the gravitational field through any closed surface is equal to $-4\pi G$ times the mass enclosed within the surface) to deduce the gravitational field as follows:

a) By considering the flux through a surface lying entirely above or below the sheet, show that the gravitational field is constant outside the sheet. Be sure to explain clearly and carefully your reasoning in each step.

b) By considering the flux through a surface whose top is above and whose bottom is below the sheet, find the magnitude of the gravitational field outside the sheet.

c) If the Earth were an infinite sheet of density $5.5\,\mathrm{g/cm^3}$, how thick would it need to be to produce the observed gravitational acceleration of $9.8\,\mathrm{m/s^2}$?

3 Dig a Hole to China

Assume the Earth is a uniform-density sphere of mass M and radius R.

a) What is the density ρ inside the Earth in terms of M and R? For the rest of the problem, express your final answers in terms of M and R, without referring to ρ.

b) Defining spherical coördinates with their center at the center of the Earth, find the gravitational field $\vec{g}(\vec{r})$ at any point, inside or outside the Earth. (To make this as easy as possible, you should use the physical result concerning the gravitational influence of a spherical shell of mass from section 6.2 of Symon, also derived in class May 11th.[1])

c) By solving the differential equation $\vec{g} = -\vec{\nabla}\varphi$, find the gravitational potential $\varphi(\vec{r})$ both inside and outside the Earth.

d) Consider a tunnel of negligible width (so we can ignore the mass removed to drill it) through the Earth along the z axis.

   i) Show that the equation of motion for a particle dropped into the tunnel is that of a simple harmonic oscillator and write the oscillation frequency in terms of M and R (and G).

   ii) Supposing the particle is dropped from rest at the surface of the Earth (so that $\vec{r}(0) = R\hat{z}$ and $\vec{v}(0) = \vec{0}$), what is its speed as it passes the center of the Earth?

   iii) Look up the actual mass and radius of the Earth and express the results to parts i) and ii) in reasonable physical units for those values of M and R. Make sure your results are expressed to an appropriate number of significant figures for the values you quote.

[1] Succinctly stated, this is that the gravitational field due to a spherical shell vanishes if you're inside the shell and is the same as that of a point mass at the center if you're outside the shell.