Physics A301: Classical Mechanics II

Problem Set 4

Assigned 2006 May 23
Due 2006 May 29

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Gauss's Law Redux

Consider an infinitely long right circular cylinder of constant density ρ and radius a.

a) Define a cylindrical coördinate system (q, ϕ, z) (chosen to fit the symmetries of the problem), and explain carefully on symmetry grounds what components of the gravitational field $\vec{g}(\vec{r})$ should be non-zero and upon which coördinates they should depend.

b) By applying Gauss's Law to suitably chosen surfaces [and using the result of part a)], find the gravitational field at any point inside or outside the cylinder.

c) By integrating the equation $\vec{g} = -\vec{\nabla}\varphi$, find the gravitational potential both inside and outside the cylinder. Choose your integration constants so that $\varphi = 0$ at the surface of the cylinder.

d) Why is it impossible to choose $\varphi = 0$ infinitely far from the cylinder? What would have happened if you had tried to find the gravitational potential by explicitly evaluating the integral

$$\varphi(\vec{r}) = -\iiint \frac{G\rho(\vec{r}\,')d^3V'}{|\vec{r} - \vec{r}\,'|} \;? \tag{1.1}$$

Note: you do **NOT** have to evaluate the integral to answer this question!

2 Coriolis Force and Conservation of Angular Momentum

[Note: this problem is not an exercise in the technical formalism developed in this chapter; you're actually intended to build up each step by thinking about the physics and geometry of the situation and then in the end verify that your physically-derived results agree with the more general formal ones.]

Consider a flat turntable rotating counter-clockwise at a fixed angular velocity ω_z. Let a particle of mass m be instantaneously a distance q from the center, seen by an observer rotating with the turntable to be moving radially outward at a speed v_q^*.

a) What is the tangential component v_ϕ of the particle's velocity as seen by an inertial observer not rotating with the turntable?

b) The angular momentum of the particle according to the inertial observer is $\ell_z = mqv_\phi$. (If this is not obvious, you should review the section on rotational motion in your Basic Physics text.) Write ℓ_z in terms of the parameters of the problem (ω_z, m, q, and v_q^*).

c) After an infinitesimal time dt, the particle is a distance $q + dq$ from the center; find dq in terms of the parameters of the problem and dt. (Throughout this problem, you should drop any terms which are second order and higher in infinitesimal quantities.)

d) As a result of non-inertial effects, the particle may have picked up a small transverse velocity dv^*_ϕ as seen by the co-rotating observer. In terms of this unknown dv^*_ϕ, what is the transverse component $v_\phi + dv_\phi$ of the particle's velocity as seen by the inertial observer? (This will be the transverse velocity of a fixed point on the turntable $q + dq$ from the center, plus dv^*_ϕ.)

e) Using the results of parts c) and d), work out the angular momentum $\ell_z + d\ell_z$ of the particle after an infinitesimal time dt, as seen by the inertial observer, in terms of dt, the unknown dv^*_ϕ, and the parameters of the problem.

f) If the particle is moving freely and not subject to any torques, the angular momentum as measured by an inertial observer will be conserved, i.e., $\ell_z + d\ell_z = \ell_z$. Use this condition to solve for dv^*_ϕ in terms of the parameters of the problem and dt.

g) Verify that this apparent acceleration in the ϕ direction seen by the rotating observer, needed to conserve angular momentum, has the same magnitude and direction as the acceleration which would be produced by the Coriolis force derived formally in class and in the text.

3 Lunar and Solar Tides

In class, we showed that the distortion of the surface of a nearly spherical, perfectly elastic planet of mass M_1 and radius R by a body of mass M_2 a distance a away, in the limit $\frac{M_2}{M_1}\frac{R^3}{a^3} \ll 1$, was given by

$$\delta R_{\text{tide}}(\hat{r}^*) = R\frac{M_2}{M_1}\frac{R^3}{a^3}\frac{3(\hat{r}^* \cdot \hat{r}^*_2)^2 - 1}{2} \tag{3.1}$$

where $\hat{r}^*$ and $\hat{r}^*_2$ are unit vectors from the center of the planet to the place at which δR_{tide} is measured and to the distant body, respectively.

a) According to this approximation, what is the height difference between the high and low tides caused on the Earth by the Moon, in meters?

b) According to this approximation, what is the height difference between the high and low tides caused on the Earth by the Sun, in meters?

c) In the perturbative approximation we've been using, the combined effect of the Sun and Moon can be obtained by adding the tidal distortions from each of them. Writing the masses of the Sun and Moon as $M_\odot$ and $M_{☽}$, the distances of each from the Earth as $a_\odot$ and $a_{☽}$, the directions to each as $\hat{r}^*_\odot$ and $\hat{r}^*_{☽}$, and the Mass and radius of the Earth and $M_\oplus$ and $R_\oplus$, write the tidal height in a direction $\hat{r}^*$ from the center. (You don't need to use any astronomical data in this part of the problem.)

d) Let the directions to the Sun and Moon be either the same or opposite, and define the x^* axis to lie in this direction. Write the tidal height as a function of the spherical coördinates θ^* and ϕ^* (in terms of the relevant masses and distances) and evaluate this height in meters at the following locations:
i) $\theta^* = 0$; ii) $\theta^* = \pi/2, \phi^* = 0$; iii) $\theta^* = \pi/2, \phi^* = \pi/2$

e) Let the directions to the Sun and Moon be at right angles to one another, and define the x^* axis to point moonward and the y^* axis to point sunward. Write the tidal height as a function of the spherical coördinates θ^* and ϕ^* (in terms of the relevant masses and distances) and evaluate this height in meters at the following locations:
i) $\theta^* = 0$; ii) $\theta^* = \pi/2, \phi^* = 0$; iii) $\theta^* = \pi/2, \phi^* = \pi/2$