1060-710
Mathematical and Statistical Methods for Astrophysics

Problem Set 1

Assigned 2009 September 10
Due 2009 September 17

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1.1 A Product of Gamma Functions

Show that

$$\Gamma(k+1)\Gamma(k+1/2) = \frac{\Gamma(2k+1)\sqrt{\pi}}{2^{2k}} \tag{1.1}$$

for non-negative integer k. (It's true in general, but the demonstration is easier when k is an integer.)

1.2 The Error Function

Consider the *Complementary Error Function*

$$\operatorname{erfc}(a) = \frac{2}{\sqrt{\pi}} \int_a^\infty e^{-x^2}\,dx \tag{1.2}$$

a) Show that $\operatorname{erfc}(-\infty) = 2$, $\operatorname{erfc}(0) = 1$ and $\operatorname{erfc}(\infty) = 0$ using results derived in class.

b) Try to apply the coördinate transformation trick used to calculate $\Gamma(1/2)$ to the evaluation of

$$\frac{\pi}{4}[\operatorname{erfc}(a)]^2 = \int_a^\infty \int_a^\infty e^{-x^2-y^2}\,dx\,dy \tag{1.3}$$

for $a > 0$

   i) Sketch the region of integration in (x, y) coördinates; what are the corresponding limits of integration for r and ϕ if you put the r integral inside the ϕ integral? (Hint: it's easier if you break up the integral into two pieces, one with $\phi < \pi/4$ and one with $\phi > \pi/4$)

   ii) Do the r integral explicitly to obtain an expression for $\frac{\pi}{4}[\operatorname{erfc}(a)]^2$ as an integral over ϕ. Note that this integral cannot be done in closed form unless $a = 0$

2 Legendre's Equation

Consider the differential equation

$$(1-x^2)y''(x) - 2xy'(x) + \ell(\ell+1)y(x) = 0 \tag{2.1}$$

and look for a solution of the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^{s+k} \tag{2.2}$$

a) From the condition $a_0 \neq 0$ find the indicial equation giving the possible values for s.

b) Explain why it's okay to take $a_1 = 0$ in (2.2) for either value of s.

c) Obtain the recursion relation for a_{k+2} in terms of a_k, for each of the two allowed values of s.

d) Show that when $x^2 = 1$, the infinite series diverge for both values of s.

e) Show that an appropriate choice of ℓ can make one of the two series stop after a finite number of terms.

3 Neumann Functions

The Neumann function, or Bessel function of the second kind, is defined as

$$N_\nu(x) = \frac{\cos \pi\nu J_\nu(x) - J_{-\nu}(x)}{\sin \pi\nu} \tag{3.1}$$

a) Use L'Hôpital's rule to show that as $\nu \to n \in \mathbb{Z}$,

$$\lim_{\nu \to n} N_\nu(x) = \frac{1}{\pi} \left(\left. \frac{\partial J_\nu(x)}{\partial \nu} \right|_{\nu=n} - (-1)^n \left. \frac{\partial J_{-\nu}(x)}{\partial \nu} \right|_{\nu=n} \right) \tag{3.2}$$

b) Show that $N_n(x)$ for integer n satisfies the Bessel equation by noting that

$$\mathcal{B}(\nu, x) = x^2 \frac{\partial^2}{\partial x^2} J_\nu(x) + x \frac{\partial}{\partial x} J_\nu(x) + (x^2 - \nu^2) J_\nu(x) \tag{3.3}$$

vanishes for all ν and x, and considering an appropriate combination of $\frac{\partial \mathcal{B}(\nu,x)}{\partial \nu}$ and $\frac{\partial \mathcal{B}(-\nu,x)}{\partial \nu}$ at $\nu = n$.

c) Use the power series form of the Bessel function

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{\nu+2k} \tag{3.4}$$

and the definition (3.2) to study the form of $N_n(x)$ for integer n:

i) Write $N_n(x)$ as a sum of two pieces: a power series in $x/2$ and a term involving $\ln(x/2)$ and $J_n(x)$. It's convenient to write the coëfficients in the power series in terms of $L'(z)$ where $L(z) = 1/\Gamma(z)$ is the reciprocal gamma function.

ii) Check that $L'(\ell)$ is finite (and non-zero) for integer ℓ. (It's sufficient to do this anecdotally in a computing environment like SciPy or matlab, checking that it's true from around -5 to 5 or so.)

iii) Use your result to find the lowest-order behavior of $N_n(x)$ as $x \to 0$ for $n = 0$, and for positive integer n.

d) Use the asymptotic form

$$J_\nu(x) \overset{x\to\infty}{\longrightarrow} \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{(2\nu+1)\pi}{4}\right) \tag{3.5}$$

and the definition (3.1) to show that

$$N_\nu(x) \overset{x\to\infty}{\longrightarrow} \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{(2\nu+1)\pi}{4}\right) \tag{3.6}$$

4 Cylindrical Waves and Plane Waves

By separation of variables, we know that

$$\psi(s, \phi, z) = H_m^{(1)}(k_s s) e^{im\phi} e^{ik_z z} \tag{4.1}$$

is a solution to the Helmholtz equation when m is an integer and $k_s^+ k_z^2 = k^2$. Consider the behavior of this solution at a point with coördinates $\{s = s_0 + \delta s, \phi = \phi_0 + \delta\phi, z = z_0 + \delta z,$ where $s_0 \gg m/k_s$, and δs, $\delta\phi$ and δz are all small enough that we can neglect all corrections higher than first-order in them.

a) Work out $\psi(s, \phi, z) - \psi(s_0, \phi_0, z_0)$ to first order in δs, $\delta\phi$ and δz, using the asymptotic form

$$H_\nu^{(1)}(k_s s) \approx \sqrt{\frac{2}{\pi k_s s}} \exp\left(i\left[k_s s - \frac{(2\nu+1)\pi}{4}\right]\right) \tag{4.2}$$

b) Using the usual cylindrical-to-Cartesian coördinate transformations $x = s\cos\phi$, $y = s\sin\phi$ etc, work out the form of δs and $\delta\phi$ in terms of δx and δy (along with s_0, ϕ_0, and z_0).

c) Show that to this approximation (4.1) implies

$$\psi(s, \phi, z) \approx A e^{ik_x x} e^{ik_y y} e^{ik_z z} \tag{4.3}$$

and find A, k_x and k_y.