1060-710
Mathematical and Statistical Methods for Astrophysics

Problem Set 2

Assigned 2009 September 17
Due 2009 September 24

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Solutions to the Boundary Value Problem

In class we showed that the wave equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \tag{1.1}$$

with the boundary condition $\psi(a,\phi,t) = 0$ and initial conditions $\psi(r,\phi,0) = f(r,\phi)$ and $\dot{\psi}(r,\phi,0) = 0$ had a solution of the form

$$\psi(r,\phi,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\frac{\gamma_{mn}}{a}r\right)\left(A_{mn}\cos m\phi + B_{mn}\sin m\phi\right)\cos\frac{\gamma_{mn}}{a}ct \tag{1.2}$$

where γ_{mn} is the nth non-trivial zero of the mth bessel function so $J_m(\gamma_{mn}) = 0$.

a) Use your favorite plotting program to plot $J_m\left(\frac{\gamma_{mn}}{a}r\right)$ versus r/a for various choices of m and n as follows:

   i) Plot $J_0\left(\frac{\gamma_{0n}}{a}r\right)$ for $n = 1, 2, 3$ on one set of axes.

   ii) Plot $J_1\left(\frac{\gamma_{1n}}{a}r\right)$ for $n = 1, 2, 3$ on one set of axes.

   iii) Plot $J_2\left(\frac{\gamma_{2n}}{a}r\right)$ for $n = 1, 2, 3$ on one set of axes.

   In maplotlib, it's convenient to import from `scipy.special` the functions `jn` and `jn_zeros`. Note that e.g. `jn_zeros(2,3)` returns an array containing γ_{21}, γ_{22} and γ_{23}.

b) We knew already from the power series expansions that $J_m(0) = 0$ for $m > 0$; explain why that is necessary to ensure $\psi(r,\phi,t)$ is well-defined.

c) Suppose now that the boundary condition is $\left.\frac{\partial\psi}{\partial r}\right|_{r=a} \equiv \psi_{,r}(a,\phi,t) = 0$. Work out the solution to the wave equation in a form similar to (1.2) using the definition that ν_{mn} is the nth zero of the derivative $J'_m(x)$ of the mth Bessel function.

2 Orthogonality of Spherical Bessel Functions

Recall that the radial part of the Helmholtz equation, separated in spherical coördinates, produced an ODE which was solved by the spherical Bessel functions

$$r^2 \frac{d^2}{dr^2} j_\ell(kr) + 2r \frac{d}{dr} j_\ell(kr) + \left[k^2 r^2 - \ell(\ell+1)\right] j_\ell(kr) = 0 \tag{2.1}$$

a) Convert (2.1) into an eigenvalue equation of the form

$$\mathcal{L} j_\ell(kr) = \frac{1}{b(r)} \left[\frac{d}{dr}\left(p(r)\frac{d}{dr}\right) + q(r)\right] j_\ell(kr) = \lambda j_\ell(kr) \tag{2.2}$$

with explicit forms for the $b(r)$, $p(r)$ and $q(r)$ (which may depend on ℓ), and λ (which will depend on k).

b) Choose a physically-motivated weighting function $w(r)$ by considering the radial part $w(r)\,dr$ of the measure for volume integrals in spherical coördinates.

c) Show that $\mathcal{L}$ is self-adjoint under the inner product

$$\langle u, v\rangle = \int_0^a u(r)\, v(r)\, w(r)\, dr \tag{2.3}$$

where $u(r)$ and $v(r)$ are regular at the origin and vanish at $r = a$.

d) Use these results to write an orthogonality relation between $j_\ell(k_1 r)$ and $j_\ell(k_2 r)$ where $k_1 a$ and $k_2 a$ are zeros of the spherical Bessel function $j_\ell(x)$.

3 Associated Legendre Functions

Verify that if $P_\ell(x)$ is a solution to the Legendre equation

$$(1-x^2)P_\ell''(x) - 2xP_\ell'(x) + \ell(\ell+1)P_\ell(x) = 0 \tag{3.1}$$

then

$$P_\ell^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x) \tag{3.2}$$

is a solution to

$$(1-x^2)P_\ell^{m\prime\prime}(x) - 2xP_\ell^{m\prime}(x) + \left(\ell(\ell+1) + \frac{m^2}{1-x^2}\right) P_\ell^m(x) = 0 \tag{3.3}$$

4 Eigenfunction Expansion

Consider the expansion of $\sin\theta$ in Legendre polynomials

$$\sin\theta = \sqrt{1-\mu^2} = \sum_{\ell=0}^{\infty} c_\ell P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(\mu) \tag{4.1}$$

a) Use the orthogonality condition

$$\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta\, d\theta = \int_{-1}^1 P_\ell(\mu)P_{\ell'}(\mu)d\mu = \frac{2}{2\ell+1}\delta_{\ell\ell'} \tag{4.2}$$

to write c_ℓ in terms of an integral over μ and an equivalent integral over θ.

b) Show that $c_\ell = 0$ for odd ℓ.

c) Using the explicit forms $P_0(\cos\theta) = 1$ and $P_2(\cos\theta) = (3\cos^2\theta - 1)/2$, find the values of c_0 and c_2 explicitly.