1060-710
Mathematical and Statistical Methods for Astrophysics

Problem Set 3

Assigned 2009 September 24
Due 2009 October 1

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Finite differencing

In class we showed that if we define a function on a uniform grid, $f(x_0 + nh)$, where x_0 is a particular grid point, n an integer, and h the grid spacing, we can define various differencing operators D by assigning constant coefficients a_k as follows:

$$[Df](x_0) = \sum_{k=n_{\min}}^{n_{\max}} a_k f(x_0 + kh) \tag{1.1}$$

where the particular values of $n_{\min}$ and $n_{\max}$ are chosen to produce a result accurate to a given order. For the second-order centered first derivative operator, $a_1 = 1/(2h)$; $a_{-1} = -1/(2h)$, while for the second-order centered second derivative, we find $a_1 = 1/h^2$; $a_0 = -2/h^2$; $a_{-1} = 1/h^2$.

To evaluate the order of the operator, we assume that the function near x_0 can be expanded as a Taylor polynomial,

$$f(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + c_3(x - x_0)^3 + ... \tag{1.2}$$

and we can simplify further by letting $X = x - x_0$, finding

$$f(x) = c_0 + c_1 X + c_2 X^2 + c_3 X^3 + ... \tag{1.3}$$

When we insert the Taylor series expansion of the function into the differencing operator, we end up with a set of equations for the coefficients a_k. To fix N coefficients, we require that the differencing operator return the proper value for the terms c_0 up to c_{N-1}. For the fourth-order first derivative we discussed in class, we required that $[D_2 f](0) = 0c_0 + 1c_1 + 0c_2 + 0c_3 + 0c_4 + \text{h.o.t.}$, finding that the higher-order terms introduced an error that scaled like h^4.

a) Differencing operators need not be centered, as we discussed briefly for the cases where calculated the derivative using a point and its neighbor. What are the coefficients a_0, a_1, a_2 for the second-order forward differenced first derivative operator? What are a_{-2}, a_{-1}, a_0 for the second-order backward differenced first derivative? What are the coefficients of the second derivative operator for those two stencils?

b) While even orders of accuracy are inevitable for centered differences, it is straightforward to construct off-centered differencing operators with odd orders of accuracy. What are the rarely discussed third-order coefficients a_{-1}, a_0, a_1, a_2 for the third-order forward differenced first derivative operator?

c) In two or more dimensions, differencing proceeds in an analogous fashion, though we need to keep track of an additional index. To do a mixed second derivative of a function $f(x, y)$, we first do one derivative, and then do the other to the result (order has been proven in class not to matter. As an example, $\partial^2/(\partial x \partial y)$ can be done as follows

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \approx D_{1y}[D_{1x} f(x_0, y_0)] = D_1 y \left[\frac{f(x_0 + h, y_0) - f(x_0 - h, y_0)}{2h} \right]$$
$$\approx \frac{f(x_0 + h, y_0 + h) - f(x_0 + h, y_0 - h) - f(x_0 - h, y_0 + h) + f(x_0 - h, y_0 - h)}{4h^2} \tag{1.4}$$

and we find coefficients $a_{1,1} = a_{-1,-1} = 1/(4h^2)$ and $a_{1,-1} = a_{-1,1} = -1/(4h^2)$.

If we wish to calculate a higher order mixed second-derivative, we can use the same procedure with the fourth-order centered first derivative, applying it first in the x direction and then in the y, but as each involves 4 nonzero components, this requires $4 \times 4 = 16$ non-zero coefficients. A Taylor series in two variables only includes 10 terms up to third order: $f(x, y) = c_0 + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{30}x^3 + c_{21}x^2y + c_{12}xy^2 + c_{03}y^3 + \text{h.o.t.}$, so 16 coefficients seems rather excessive. Can you work out a version requiring only 8 nonzero coefficients along the diagonals of a 5×5 stencil box?

2 Multipoles

If we assume that the Earth is a uniform-density oblate spheroid with equatorial radius 6378km and polar radius 6356km, then what is its quadrupole moment

$$Q_{zz} = \int \rho(3z^2 - r^2) dV \ ? \tag{2.1}$$

What is the approximate magnitude, neglecting the angular terms, of the quadrupole component of the gravitational potential at roughly the moon's distance, 400,000km, expressed as a fraction of the monopole contribution to the potential?